Integral Solutions of Non-Homogeneous Biquadratic Equation With Four Unknowns

M.A.Gopalan[1], G.Sumathi[2], S.Vidhyalakshmi[3]
[1]. Professor of Mathematics, SIGC,Trichy
[2]. Lecturer of Mathematics, SIGC,Trichy,
[3]. Professor of Mathematics, SIGC,Trichy

Abstract

The non-homogeneous biquadratic equation with four unknowns represented by the diophantine equation

$$x^3 + y^3 = (k^2 + 3)^n z^3 w$$

is analyzed for its patterns of non-zero distinct integral solutions and three different methods of integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas number,Pronic numbers, Stella octangular numbers, Octahedral numbers, Gnomonic numbers, Centered traingular numbers,Generalized Fibonacci and Lucas sequences are exhibited.

Keywords: *Integral solutions, Generalized Fibonacci and Lucas sequences, biquadratic non-homogeneous equation with four unknowns*

M.Sc 2000 mathematics subject classification: 11D25

NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

S_n : Star number of rank n

$\Pr_n$: Pronic number of rank n

So_n : Stella octangular number of rank n

j_n : Jacobsthal lucas number of rank n

J_n : Jacobsthal number of rank n

OH_n : Octahedral number of rank n

$Gnomic_n$: Gnomonic number of rank n

GF_n : Generalized Fibonacci sequence number of rank n

GL_n : Generalized Lucas sequence number of rank n

$Ct_{m,n}$: Centered Polygonal number of rank n with size m

I. INTRODUCTION

The biquadratic diophantine (homogeneous or non-homogeneous) equations offer an unlimited field for research due to their variety Dickson. L. E. [1],Mordell.L.J. [2],Carmichael R.D[3]. In particular, one may refer Gopalan.M.A.et.al [4-13] for ternary non-homogeneous biquadratic equations. This communication concerns with an interesting non-homogeneous biquadratic equation with four unknowns represented by

$x^3 + y^3 = (k^2 + 3)^n z^3 w$

for determining its infinitely many non-zero integral points. Three different methods are illustrated. In method 1, the solutions are obtained through the method of factorization. In method 2, the binomial expansion is introduced to obtain the integral solutions. In method 3, the integral solutions are expressed in terms of Generalized Fibonacci and Lucas sequences along with a few properties in terms of the above integer sequences Also, a few interesting relations among the solutions are presented.

II. METHOD OF ANALYSIS

The Diophantine equation representing a non-homogeneous biquadratic equation with four unknowns is

$$x^3 + y^3 = (k^2 + 3)^n z^3 w \quad (1)$$

Introducing the linear transformations

$$x = u + v, y = u - v, w = 2u \quad (2)$$

in (1), it leads to

$$u^2 + 3v^2 = (k^2 + 3)^n z^3 \quad (3)$$

The above equation (3) is solved through three different methods and thus, one obtains three distinct sets of solutions to (1).

2.1. Method:1

Let $$z = a^2 + 3b^2 \quad (4)$$

Substituting (4) in (3) and using the method of factorization,define

$$(u + i\sqrt{3}v) = (k + i\sqrt{3})^n (a + i\sqrt{3}b)^3 \quad (5)$$

$$= r^n \exp(in\theta)(a + i\sqrt{3}b)^3$$

where $$r = \sqrt{k^2 + 3}, \qquad \theta = \tan^{-1}\frac{\sqrt{3}}{k} \quad (6)$$

Equating real and imaginary parts in (5) we get

$$u = (k^2 + 3)^{\frac{n}{2}} \left\{ \cos n\theta(a^3 - 9ab^2) - \sqrt{3}\sin\theta(3a^2b - 3b^3) \right\}$$

$$v = (k^2 + 3)^{\frac{n}{2}} \left\{ \cos n\theta(3a^2b - 3b^3) + \frac{1}{\sqrt{3}}\sin\theta(a^3 - 9ab^2) \right\}$$

Substituting the values of u and v in (2), the corresponding values of x, y, z and w are represented by

$$x(a,b,k) = (k^2 + 3)^{\frac{n}{2}} \left\{ \cos n\theta(a^3 - 9ab^2 + 3a^2b - 3b^3) + \frac{\sin\theta}{\sqrt{3}}(a^3 - 9ab^2 - 9a^2b + 9b^3) \right\}$$

$$y(a,b,k) = (k^2 + 3)^{\frac{n}{2}} \left\{ \cos n\theta(a^3 - 9ab^2 - 3a^2b + 3b^3) - \frac{\sin\theta}{\sqrt{3}}(9a^2b - 9b^3 + a^3 - 9ab^2) \right\}$$

$$z(a,b) = a^2 + 3b^2$$

$$w(a,b,k) = 2(k^2 + 3)^{\frac{n}{2}} \left\{ \cos n\theta(a^3 - 9ab^2) - \sqrt{3}(3a^2b - 3b^3) \right\}$$

Properties:

1. $$x(a,b,k) - y(a,b,k) = (k^2 + 3)^{\frac{n}{2}}$$

$$\left\{ \cos n\theta(Ct_{12,a} + S_a + 2Gnomic_a - 4\Pr_a - 2t_{4,a}) + \frac{\sin\theta}{\sqrt{3}}(So_a - t_{4,a} + t_{38,a}) \right\}$$

2. $x(b,b,k)+y(b,b,k)+(k^2+3)^{\frac{n}{2}}(\cos n\theta(32P_b^5-16t_{4,b})=0$

4.Each of the following is a nasty number

(a) $6\cos n\theta\left[\frac{32(k^2+3)^{\frac{n}{2}}P_b^5\cos n\theta+x(b,b,k)+y(b,b,k)}{(k^2+3)^{\frac{n}{2}}}\right]$

(b) $6\cos(4\alpha\theta)\left[32(k^2+3)^{2\alpha}P_b^5\cos(4\alpha\theta)+x(b,b,k)+y(b,b,k)\right]$

3. $4\left[z(a^2,a^2)\right]$ is a biquadratic integer.

$$4.w(a,1,k)=2(k^2+3)^{\frac{n}{2}}\left[\cos n\theta(2P_a^5-9\Pr_a+8t_{4,a})-\sqrt{3}\sin\theta(t_{8,a}+Gnomic_a-2Ct_{6,a}+6t_{3,a}\right]$$

5. $x(a,a,k)=\frac{w(a,a,k)}{2}\left[1+\frac{\tan\theta}{\sqrt{3}}\right]$

2.2. Method 2:

Using the binomial expansion of $(k+i\sqrt{3})^n$ in (5) and equating real and imaginary parts, we have

$$u=f(\alpha)(a^3-9ab^2)-3g(\alpha)(3a^2b-3b^3)$$
$$v=f(\alpha)(3a^2b-3b^3)+g(\alpha)(a^3-9ab^2)$$

Where

$$\left.\begin{aligned} f(\alpha)&=\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r n_{C_{2r}}k^{n-2r}(3)^r\\ g(\alpha)&=\sum_{r=1}^{\left[\frac{n+1}{2}\right]}(-1)^{r-1}n_{C_{2r-1}}k^{n-2r+1}(3)^{r-1}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\quad(7)$$

In view of (2) and (7) the corresponding integer solution to (1) is obtained as

$$x=(f(\alpha)+g(\alpha))(a^3-9ab^2)+(f(\alpha)-3g(\alpha))(3a^2b-3b^3)$$
$$y=(f(\alpha)-g(\alpha))(a^3-9ab^2)-(f(\alpha)+3g(\alpha))(3a^2b-3b^3)$$
$$z=a^2+3b^2$$
$$w=2f(\alpha)(a^3-9ab^2)-6g(\alpha)(3a^2b-3b^3)$$

2.3. Method 3:

Taking $n=0$ in (3), we have,

$$u^2+3v^2=z^3 \qquad (8)$$

Substituting (4) in (8), we get

$$u^2+3v^2=(a^2+3b^2)^3 \qquad (9)$$

whose solution is given by

$$u_0=(a^3-9ab^2)$$
$$v_0=(3a^2b-3b^3)$$

Again taking $n=1$ in (3), we have

$$u^2+3v^2=(k^2+3)(a^2+3b^2)^3 \qquad (10)$$

whose solution is represented by

$u_1 = ku_0 - 3v_0$

$v_1 = u_0 + kv_0$

The general form of integral solutions to (1) is given by

$$\begin{pmatrix} u_s \\ v_s \end{pmatrix} = \begin{pmatrix} \frac{A_s}{2} & -\frac{\sqrt{3}B_s}{2i} \\ \frac{B_s}{2i\sqrt{3}} & \frac{A_s}{2} \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, s = 1,2,3.....$$

Where $A_s = (k + i\sqrt{3})^s + (k - i\sqrt{3})^s$

$B_s = (k + i\sqrt{3})^s - (k - i\sqrt{3})^s$

Thus, in view of (2), the following triples of integers x_s, y_s, w_s interms of Generalized Lucas and fiboanacci sequence satisfy (1) are as follows:

Triples:

$$x_s = \frac{1}{2}GL_n(2k,-k^2-3)(a^3-9ab^2+3a^2b-3b^3) + GF_n(2k,-k^2-3)(a^3-9ab^2-9a^2b+9b^3)$$

$$y_s = \frac{1}{2}GL_n(2k,-k^2-3)(a^3-9ab^2-3a^2b+3b^3) - GF_n(2k,-k^2-3)(a^3-9ab^2+9a^2b-9b^3)$$

$$w_s = GL_n(2k,-k^2-3)(a^3-9ab^2) - 6GF_n(2k,-k^2-3)(3a^2b-3b^3)$$

The above values of x_s, y_s, w_s satisfy the following recurrence relations respectively

$$x_{s+2} - 2kx_{s+1} + (k^2+3)x_s = 0$$

$$y_{s+2} - 2ky_{s+1} + (k^2+3)y_s = 0$$

$$w_{s+2} - 2kw_{s+1} + (k^2+3)w_s = 0$$

Properties

1. $x_s(a,a,k) + y_s(a,a,k) + 16A_s t_{3,a} = 8A_s t_{4,a}$

2. $x_s(a,a+1,k) - \frac{A_s}{2}\left(6P_a^5 - 9P_{a+1}^5 + 18t_{4,a+1} - 3Ct_{6,a} - 24P_a^3 + 4t_{4,a}\right)$

$$\frac{B_s}{2i\sqrt{3}}\left(6Ct_{6,a} - 18P_a^5 + 42P_a^3 - 14\Pr_a - 7t_{4,a}\right)$$

3. $\frac{i\sqrt{3}(x_s(a,a,k) - y_s(a,a,k))}{B_s}$ is a biquadratic integer.

4. $x_s(a,a,k) - y_s(a,a,k) + \frac{B_s}{2i\sqrt{3}}\left(24(OH_a) + t_{20,a} - 9t_{4,a}\right) = 0$

5. $x_s(a,a,k) + w_s(a,a,k) + \frac{4B_s}{i\sqrt{3}}(2P_a^5 - t_{4,a}) +$

$$A_s(18(OH_a) + S_a - Ct_{4,a} + 2CT_{6,a} - 6t_{4,a}) = 0$$

6. $x_s(2a,a,k) + y_s(2a,a,k) + \frac{B_s}{2i\sqrt{3}}(10P_a^5 + 5t_{4,a}) = 5A_s(t_{4,a} - So_a - \Pr_a)$

7. $x_s(a,1,k)+y_s(a,1,k)+\frac{B_s}{2i\sqrt{3}}(4P_a^5-2Ct_{36,a}+36t_{4,a})$
$$=\frac{A_s}{2}(4P_a^5t_{4,a}-Gnomic_a-Ct_{4,a}+4t_{3,a})$$

8. $w_s(2^a,2^a,k)+24A_sJ_{3a}=\begin{cases}8\ , & \text{if a is odd}\\ -8\ , & \text{if a is even}\end{cases}$

9. $x_s(2^a,2^a,k)+y_s(2^a,2^a,k)+8A_sj_{3a}=\begin{cases}8\ , & \text{if a is odd}\\ -8\ , & \text{if a is even}\end{cases}$

10. $x_s(a,b,k)+y_s(a,b,k)=w_s(a,b,k)$

11. ${x_s}^3(a,b,k)+{y_s}^3(a,b,k)+3x_s(a,b,k)y_s(a,b,k)w_s(a,b,k)={w_s}^3(a,b,k)$

12. $(9A_s)^2\left(x_s(a,2a,k)-y_s(a,a,k)+\frac{5B_s}{2i\sqrt{3}}(2P_a^5-t_{4,a})\right)$ is a cubic integer

13.Each of the following is a nasty number

(a) $3A_s\left(x_s(a,a,k)+y_s(a,a,k)+16A_st_{3,a}\right)$

(b) $6(z(a,a))$

(c) $4A_s\{x_s(a,a,k)+w_s(a,a,k)+\frac{4B_s}{i\sqrt{3}}(2{P_a}^5-t_{4,a})+$
$A_s(18(OH_a)+S_a-3Ct_{4,a}+2Ct_{6,a})\}$

(d) $6[(x_s(a,a,k)+y_s(a,a,k))(w_s(a,a,k)]$

(e) $30A_s\left\{x_s(a,a,k)+w_s(a,a,k)+\frac{B_s}{2i\sqrt{3}}(10P_a^5+5t_{4,a})+5A_s(So_a+\Pr_a)\right\}$

14. $x_s(a,b,k)+y_s(a,b,k)=GL_n(2k,-k^2-3)u_0-6GF_n(2k,-k^2-3)v_0$

15.$x_{s+1}(a,b,k)+y_{s+1}(a,b,k)=\left[2k\left(GL_n(2k,-k^2-3)\right)-(k^2-3)\left(GL_{n-1}(2k,-k^2-3)\right)\right]u_0-$
$\left[(12k)GF_n(2k,-k^2-3)-6(k^2+3)GF_{n-1}(2k,-k^2-3)\right]v_o$

16.$x_{s+2}(a,b,k)+y_{s+2}(a,b,k)=\left[(3k^2-1)\left(GL_n(2k,-k^2-3)\right)-(2k^3+6k)\left(GL_{n-1}(2k,-k^2-3)\right)\right]u_0-$
$\left[(18k^2-1)GF_n(2k,-k^2-3)-(12k^3+36k)GF_{n-1}(2k,-k^2-3)\right]v_o$

III. CONCLUSION:

To conclude, one may search for other pattern of solutions and their corresponding properties.

REFERENCES:

[1] Dickson. L. E. "History of the Theory of Numbers'', Vol 2. Diophantine analysis, New York, Dover,2005.
[2] Mordell .L J., " Diophantine Equations Academic Press'', New York,1969.
[3] Carmichael. R.D. " The Theory of numbers and Diophantine Analysis'' ,New York, Dover, 1959.
[4] Gopalan.M.A., Manjusomnath and Vanith N., "Parametric integral solutions of $x^2+y^3=z^4$'', Acta Ciencia Indica,Vol.XXXIIIM(No.4)Pg.1261-1265(2007).
[5] Gopalan.M.A.,and Pandichelvi .V., "On Ternary biquadratic Diophantine equation
[6] $x^2+ky^3=z^4$'', Pacific- Asian Journal of Mathematics, Vol-2.No.1-2, 57-62, 2008.
[7] Gopalan.M.A., and Janaki.G., "Observation on $2(x^2-y^2)+4xy=z^4$'', Acta Ciencia Indica,Vol.XXXVM(No.2)Pg.445-448,(2009).

[8] Gopalan.M.A., Manjusomnath and Vanith N., "Integral solutions of $x^2 + xy + y^2 = (k^2 + 3)^n z^4$", Pure and Applied Mathematika Sciences,Vol.LXIX,NO.(1-2):149-152,(2009).

[9] Gopalan.M.A., and Sangeetha.G., "Integral solutions of ternary biquadratic equation $(x^2 - y^2) + 2xy = z^4$", Antartica J.Math.,7(1)Pg.95-101,(2010).

[10] Gopalan.M.A.,and Janaki.G., "Observations on $3(x^2 - y^2) + 9xy = z^4$", Antartica J.Math.,7(2)Pg.239-245,(2010).

[11] Gopalan.M.A., and Vijayasankar.A., "Integral Solutions of Ternary biquadratic equation $x^2 + 3y^2 = z^4$", Impact.J.Sci.Tech.Vol.4(No.3)Pg.47-51,2010.

[12] Gopalan.M.A,Vidhyalakshmi.S., and Devibala.S., "Ternary biquadratic diophantine equation $2^{4n+3}(x^3 - y^3) = z^4$",Impact. J.Sci.Tech.Vol(No.3)

[13] Pg.57-60,2010.

[14] 12.Gopalan.M.A,Vidhyalakshmi.S., and SumathiG., " Integral Solutions of

[15] Ternary biquadratic Non-homogeneous Equation

[16] $(\alpha + 1)(x^2 + y^2) + (2\alpha + 1)xy = z^4$",JARCEvol(6),No.2, Pg.97-98,July-Dec 2012.

[17] 13. Gopalan.M.A, SumathiG.,and Vidhyalakshmi.S., " Integral Solutions of

[18] Ternary biquadratic Non-homogeneous Equation

[19] $(k + 1)(x^2 + y^2) - (2k + 1)xy = z^4$",Archimedes J.Math,3(1),67-71,2013.