ISSN (e): 2250 – 3005 || Volume, 09 || Issue, 6 || June– 2019 ||
International Journal of Computational Engineering Research (IJCER)

Analytical expressions for arc-length of hyperbola: A hypergeometric approach

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ABSTRACT
In the present paper analytical expressions of the arc-length between two arbitrary points lying on hyperbola are obtained in terms of Gauss' hypergeometric function, Clausen's hypergeometric function and Kampé de Fériet's double hypergeometric function.
KEYWORDS: Arc-length; Hyperbola; Rectangular Hyperbola; Generalized Hypergeometric Function; Kampé de Fériet's double Hypergeometric Function.
AMS Subject Classification: *33C05, 33C20, 33C70, 33C75.*

Date of Submission: 21-06-2019 Date of acceptance:05-07-2019

I. INTRODUCTION AND PRELIMINARIES

For the sake of conciseness of the paper we have used the following notations
$\mathbb{N} := \{1,2,3,\ldots\}$; $\mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,3,\ldots\}$; $\mathbb{Z} := \{0,\pm1,\pm2,\pm3,\ldots\}$; $\mathbb{Z}^- := \{-1,-2,-3,\ldots\}$;
$\mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\} = \{0,-1,-2,-3,\ldots\}$; and $\mathbb{Z} := \mathbb{Z}_0^-\cup\mathbb{N}$;

where the symbols $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and set of integers respectively, the symbols $\mathbb{R}$ and $\mathbb{C}$ are the set of real numbers and set of complex numbers respectively.

The Pochhammer symbol $(\alpha)_p$, $(\alpha, p\in\mathbb{C})$, is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1, & (p=0;\alpha\in\mathbb{C}\backslash\{0\}) \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & (p=n\in\mathbb{N};\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-) \\ \frac{(-1)^n k!}{(k-n)!}, & (\alpha=-k;p=n;n,k\in\mathbb{N}_0;0\le n\le k) \\ 0, & (\alpha=-k;p=n;n,k\in\mathbb{N}_0;n>k) \\ \frac{(-1)^n}{(1-\alpha)_n}, & (p=-n;n\in\mathbb{N};\alpha\in\mathbb{C}\backslash\mathbb{Z}). \end{cases} \tag{1.1}$$

It being understood conventionally that $(0)_0 = 1$, and assumed tacitly that the Gamma quotient exists.

If $a, p\in\mathbb{C}$ and $r = 0,1,2,3,\ldots$, then

$$a + pr = \frac{a\left(\frac{a+p}{p}\right)_r}{\left(\frac{a}{p}\right)_r}, \text{ such that each Pochhammer symbol is well defined.} \tag{1.2}$$

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n = (\alpha)_n(\alpha+n)_m, \tag{1.3}$$

$$\Gamma(z+1) = z\Gamma(z). \tag{1.4}$$

The generalized hypergeometric function of one variable ${}_pF_q$ is defined by

$$ {}_{p}F_{q}\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_p; \\ \\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}\; z\right] = {}_{p}F_{q}\left[\begin{matrix}(\alpha_p); \\ \\ (\beta_q);\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.5}$$

where, (α_p) is a set of parameters $\alpha_1,\alpha_2,\cdots,\alpha_p$ with similar interpretation for (β_q). By convention the empty product is treated as unity and empty sum is treated as zero, $p,q\in\mathbb{N}_0$.

Convergence conditions of $_pF_q$

1. when $p\le q$ then $|z|<\infty$,
2. when $p=q+1$ then $|z|<1$,
3. when $p=q+1$ and $|z|=1$ then $\Re(\omega)>0$,
4. when $p=q+1, |z|=1$ and $z\neq 1$ then $-1<\Re(\omega)\le 0$,

where, $\omega:=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j$, and $\alpha_j\in\mathbb{C}(j=1,2,3,\dots,p)$; $\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-(j=1,2,3,\dots,q)$.

The Binomial expansion in terms of hypergeometric function can be written as

$$(1-z)^{-a} = {}_{1}F_{0}\left[\begin{matrix}a; \\ \\ -;\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!}, \tag{1.6}$$

where, $a\in\mathbb{C}$ and $|z|<1$.

Kummer's first summation theorem [2,p. 9, eq 2.3(1)]

$$ {}_{2}F_{1}\left[\begin{matrix}a\ ,\ b; \\ \\ 1+a-b;\end{matrix}\; -1\right] = \frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1+\frac{a}{2}-b\right)\Gamma(1+a)}, \tag{1.7}$$

where, $\Re(b)<1$ and $1+a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Kampé de Fériet's double hypergeometric function ([1,p. 150, eq 29], [3,p. 112], see also [4])

$$F_{\ell:m;n}^{p:q;k}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ \\ (\alpha_\ell): & (\beta_m); & (\gamma_n);\end{matrix}\; x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.8}$$

where denominator parameters $(\alpha_\ell),(\beta_m),(\gamma_n)$ are neither zero nor negative integers.

Convergence conditions of $F_{\ell:m;n}^{p:q;k}$ ([7, pp. 153-157, sections 3 & 4] see also [9,pp. 423-424, eqs. 26-27])

1. when $p+q<\ell+m+1, p+k<\ell+n+1$ then $|x|<\infty$ and $|y|<\infty$,
2. when $p+q=\ell+m+1, p+k=\ell+n+1, p>\ell$ then $|x|^{\frac{1}{p-\ell}}+|y|^{\frac{1}{p-\ell}}<1$,
3. when $p+q=\ell+m+1, p+k=\ell+n+1, p\le\ell$ then $\max\{|x|,|y|\}<1$,

The following results will be required in our present investigation.

Some basic integrals and reduction formula

$$\int \sec t\ dt = C+\ln(\sec t+\tan t) = C+\ln\left(\tan\left(\frac{t}{2}+\frac{\pi}{4}\right)\right), \tag{1.9}$$

$$\int \sec^2 t\ dt = C+\tan t, \tag{1.10}$$

$$\int \sec^3 t\ dt = C+\frac{1}{2}\ln(\sec t+\tan t)+\frac{\sec t\tan t}{2}. \tag{1.11}$$

where C is the constant of the integration.

The following reduction formula is available in all textbooks of integral calculus

$$\int \sec^n x\,dx = C+\frac{\tan x\sec^{n-2}x}{(n-1)}+\frac{(n-2)}{(n-1)}\int \sec^{n-2}x\,dx, \tag{1.12}$$

where n is positive integer greater than or equal to 2.

From reduction formula (1.12) we can write

$$\int \frac{1}{\cos^{2r-1}t}dt = C + \frac{\tan t \sec^{2r-3}t}{(2r-2)} + \frac{(2r-3)}{(2r-2)}\int \sec^{2r-3}t\, dt; \quad r \geq 2. \tag{1.13}$$

By the successive applications of reduction formula (1.12) in the right hand side of eq. (1.13) we can find same integral in finite series form containing Pochhammer symbol,

$$\int \frac{1}{\cos^{2r-1}t}dt = C + \frac{\left(\frac{1}{2}\right)_{r-1}\ln(\sec t+\tan t)}{(r-1)!} + \frac{\left(\frac{1}{2}\right)_{r-1}\sin t}{(r-1)!\cos^2 t}\left(\sum_{n=0}^{r-2}\frac{n!}{\left(\frac{3}{2}\right)_n\cos^{2n}t}\right); \quad r \geq 2. \tag{1.14}$$

The integral (1.14) can be verified with the reduction formula (1.13) by taking $r = 2,3,4 \ldots$

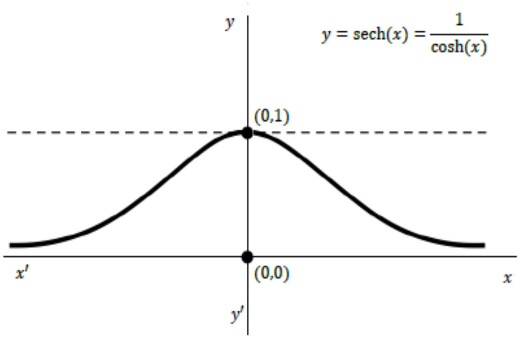

Figure 1: Graph of $\frac{1}{coshx}$

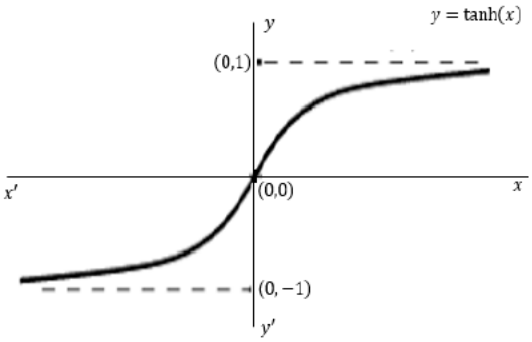

Figure 2: Graph of $tanhx$

From Figs.(1, 2) it is clear that values of $(\cosh x)^{-1}$ and $\tanh x$ lies in the interval $(0,1)$ for all values of $x > 0$.

In eqs. (1.9), (1.14) replace t by (it), use the properties of hyperbolic functions and formula: $\ln(a+ib) = \frac{1}{2}\ln(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$, (where a and b are positive real numbers), we get

$$\int \frac{dt}{\cosh t} = C + \tan^{-1}(\sinh t), \tag{1.15}$$

$$\int \frac{dt}{\cosh^{2r-1}t} = C + \frac{\left(\frac{1}{2}\right)_{r-1}}{(r-1)!}\tan^{-1}(\sinh t) + \frac{\left(\frac{1}{2}\right)_{r-1}\sinh t}{(r-1)!\cosh^2 t}\left(\sum_{n=0}^{r-2}\frac{n!}{\left(\frac{3}{2}\right)_n\cosh^{2n}t}\right); \quad r \geq 2. \tag{1.16}$$

Double series identity ([8,p. 100,eq 2.1(2)],[6,p. 57,eq 2])

$$\sum_{r=0}^{\infty}\sum_{n=0}^{r}\Phi(r,n) = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\Phi(r+n,n), \tag{1.17}$$

provided that series involved are absolutely convergent.

In this paper any values of parameters and variables leading to the results which do not make sense are tacitly excluded.

II MAIN FORMULA FOR ARC-LENGTH OF HYPERBOLA

The exact arc-length $\widehat{AB}$ between two arbitrary points $A(a\cosh t_1, b\sinh t_1)$ and $B(a\cosh t_2, b\sinh t_2)$ lying on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is obtained in terms of Clausen's hypergeometric function ${}_3F_2$ and Kampé de Fériet's double hypergeometric function $F^{2:1;2}_{2:0;1}$

$$\widehat{AB}=\left\{ae\sinh t_2-\frac{a}{2e}\tan^{-1}(\sinh t_2)-\frac{a}{16e^3}\tan^{-1}(\sinh t_2)\ {}_3F_2\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2};\\ & \frac{1}{e^2}\\ 2,\ 3;\end{matrix}\right]\right.$$

$$\left.-\frac{a\sinh t_2}{16e^3\cosh^2 t_2}F^{2:1;2}_{2:0;1}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}: & 1; & 1,1; & \\ & & & \frac{1}{e^2},\frac{1}{e^2\cosh^2 t_2}\\ 3,2: & -; & \frac{3}{2}; & \end{matrix}\right]\right\}$$

$$-\left\{ae\sinh t_1-\frac{a}{2e}\tan^{-1}(\sinh t_1)-\frac{a}{16e^3}\tan^{-1}(\sinh t_1)\ {}_3F_2\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2};\\ & \frac{1}{e^2}\\ 2,3;\end{matrix}\right]\right.$$

$$\left.-\frac{a\sinh t_1}{16e^3\cosh^2 t_1}F^{2:1;2}_{2:0;1}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}: & 1; & 1,1; & \\ & & & \frac{1}{e^2},\frac{1}{e^2\cosh^2 t_1}\\ 3,2: & -; & \frac{3}{2}; & \end{matrix}\right]\right\}$$

. (2.1)

where a is the length of semi-transverse axis and e is the eccentricity of the hyperbola. Both the series ${}_3F_2$ and $F^{2:1;2}_{2:0;1}$ are convergent since $\frac{1}{e}<1$ and $\frac{1}{e\cosh t}<1$. Therefore our expression (2.1) is convergent and is believed to be new.

Derivation of formula (2.1)

Consider the equation of hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{(Cartesian form)} \tag{2.2}$$

where, $b^2=a^2(e^2-1)$ or $\frac{a^2+b^2}{a^2}=e^2$ or $\frac{a^2}{a^2+b^2}=\frac{1}{e^2}<1$, a and b are semi-transverse and semi-conjugate axes of the hyperbola and $e(>1)$ is called the eccentricity of hyperbola.

Its parametric form is given by

$$x=\text{a}\cosh t,\quad y=b\sinh t. \tag{2.3}$$

Since given hyperbola (2.2) is symmetrical about $x-axis$ and $y-axis$ both, without any loss of generality, we shall find the arc-length between two arbitrary points lying in positive quadrant only.

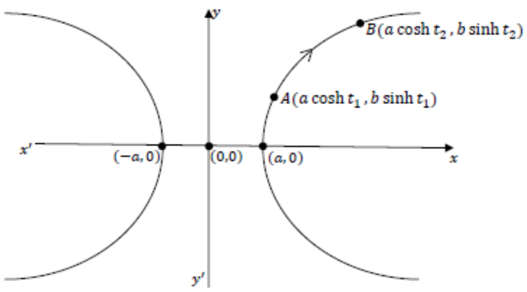

Figure 3: Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The arc-length between two arbitrary points A and B lying on any parametric curve is given by

$$\widehat{AB} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{2.4}$$

Put $\frac{dx}{dt} = a\sinh t$, $\frac{dy}{dt} = b\cosh t$, in eq. (2.4), and integrating over the interval $[t_1, t_2]$ s.t. $0 \leq t_1 < t_2 < \infty$,

$$\widehat{AB} = \int_{t_1}^{t_2} \sqrt{(a^2\sinh^2 t + b^2\cosh^2 t)}\,dt = \int_{t_1}^{t_2} \sqrt{\left((a^2+b^2)\cosh^2 t - a^2\right)}\mathrm{dt}$$

$$= \sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh t \sqrt{\left(1 - \frac{1}{e^2\cosh^2 t}\right)}\, dt,$$

$$= \sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh t \;{}_1F_0\left[\begin{matrix} -\frac{1}{2}; \\ \\ -; \end{matrix} \;\; \frac{1}{e^2\cosh^2 t}\right] dt; \quad \text{since} \quad \frac{1}{e^2\cosh^2 t} < 1, \quad \forall \quad t,$$

$$= \sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh t \left(\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r}{r!\, e^{2r}\cosh^{2r} t}\right) dt,$$

$$= \sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh t\, dt - \frac{\sqrt{(a^2+b^2)}}{2e^2} \int_{t_1}^{t_2} \frac{1}{\cosh t}\, dt + \sqrt{(a^2+b^2)} \sum_{r=2}^{\infty} \frac{\left(-\frac{1}{2}\right)_r}{r!\, e^{2r}} \int_{t_1}^{t_2} \frac{1}{\cosh^{2r-1} t}\, dt, \tag{2.5}$$

Using integrals (1.15) and (1.16) in eq. (2.5) we get,

$$\widehat{AB} = \sqrt{(a^2+b^2)} \left[\sinh t - \frac{1}{2e^2}\tan^{-1}(\sinh t) + \sum_{r=2}^{\infty} \frac{\left(-\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_{r-1}}{(r-1)!\, r!\, e^{2r}} \left\{\tan^{-1}(\sinh t) + \frac{\sinh t}{\cosh^2 t} \sum_{n=0}^{r-2} \frac{n!}{\left(\frac{3}{2}\right)_n \cosh^{2n} t}\right\}\right]_{t_1}^{t_2}$$

$$= \sqrt{(a^2+b^2)} \left[\sinh t - \frac{1}{2e^2}\tan^{-1}(\sinh t) + \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{r+2} \left(\frac{1}{2}\right)_{r+1}}{(r+1)!\,(r+2)!\, e^{2r+4}} \left\{\tan^{-1}(\sinh t) + \frac{\sinh t}{\cosh^2 t} \sum_{n=0}^{r} \frac{n!}{\left(\frac{3}{2}\right)_n \cosh^{2n} t}\right\}\right]_{t_1}^{t_2} \tag{2.6}$$

$$\widehat{AB} = \sqrt{(a^2+b^2)} \left[\sinh t - \frac{1}{2e^2}\tan^{-1}(\sinh t) + \frac{\left(-\frac{1}{2}\right)_2 \left(\frac{1}{2}\right)\tan^{-1}(\sinh t)}{2e^4} \sum_{r=0}^{\infty} \frac{\left(\frac{3}{2}\right)_r \left(\frac{3}{2}\right)_r (1)_r}{r!\,(2)_r (3)_r\; e^{2r}} + \frac{\left(-\frac{1}{2}\right)_2 \left(\frac{1}{2}\right)\sinh t}{2e^4\cosh^2 t} \sum_{r=0}^{\infty}\sum_{n=0}^{r} \frac{\left(\frac{3}{2}\right)_r \left(\frac{3}{2}\right)_r (1)_n}{(2)_r (3)_r e^{2r} \left(\frac{3}{2}\right)_n \cosh^{2n} t}\right]_{t_1}^{t_2}. \tag{2.7}$$

Now using double series identity (1.17) in eq. (2.7), we get

$$\widehat{AB} = \sqrt{(a^2+b^2)}\left[\sinh t - \frac{1}{2e^2}\tan^{-1}(\sinh t) - \frac{\tan^{-1}(\sinh t)}{16e^4}\ {}_3F_2\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2}; \\ \\ 2,3;\end{matrix}\ \frac{1}{e^2}\right] - \frac{\sinh t}{16e^4\cosh^2 t}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{r+n}\left(\frac{3}{2}\right)_{r+n}(1)_n}{(2)_{r+n}(3)_{r+n}\ e^{2r+2n}\left(\frac{3}{2}\right)_n\cosh^{2n}t}\right]_{t_1}^{t_2} \quad (2.8)$$

$$\widehat{AB} = \left\{ae\sinh t - \frac{a}{2e}\tan^{-1}(\sinh t) - \frac{a}{16e^3}\tan^{-1}(\sinh t)\ {}_3F_2\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2}; \\ \\ 2,3;\end{matrix}\ \frac{1}{e^2}\right] - \frac{a\sinh t}{16e^3\cosh^2 t}F^{2:1;2}_{2:0;1}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}: & 1; & 1,1; \\ & & \\ 3,2: & -; & \frac{3}{2};\end{matrix}\ \frac{1}{e^2},\frac{1}{e^2\cosh^2 t}\right]\right\}_{t_1}^{t_2}. \quad (2.9)$$

After simplification we get the formula (2.1).

III. MAIN FORMULAS FOR ARC-LENGTH OF RECTANGULAR HYPERBOLA

The exact arc-length $\widehat{AB}$ between two arbitrary points $A\left(ct_1,\frac{c}{t_1}\right)$ and $B\left(ct_2,\frac{c}{t_2}\right)$ lying on rectangular hyperbola $xy = c^2$ is given in terms of Gauss' hypergeometric function ${}_2F_1$

(i) When $0 < t_1 < t_2 < 1$ and $c > 0$

then $$\widehat{AB} = \frac{c}{t_1}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4}\ \ ;\end{matrix}\ -t_1^4\right] - \frac{c}{t_2}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4}\ \ ;\end{matrix}\ -t_2^4\right], \quad (3.1)$$

(ii) When $0 < t_1 < 1 < t_2 < \infty$ (or $t_1 < 1$ and $\frac{1}{t_2} < 1$) and $c > 0$

then $$\widehat{AB} = \frac{c}{t_1}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4};\end{matrix}\ -t_1^4\right] + ct_2\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4};\end{matrix}\ -\frac{1}{t_2^4}\right] - 2c\frac{\left(\Gamma\left(\frac{3}{4}\right)\right)^2}{\sqrt{\pi}}, \quad (3.2)$$

(iii) When $1 < t_1 < t_2 < \infty$ (or $\frac{1}{t_2} < \frac{1}{t_1} < 1$) and $c > 0$

then $$\widehat{AB} = c\ t_2\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4}\ \ ;\end{matrix}\ -\frac{1}{t_2^4}\right] - ct_1\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \\ \frac{3}{4}\ \ ;\end{matrix}\ -\frac{1}{t_1^4}\right]. \quad (3.3)$$

The right hand side of equations (3.1), (3.2) and (3.3) are convergent under associated conditions.

Derivations of formulas (3.1), (3.2) and (3.3)

Consider the equation of rectangular hyperbola

$$xy = c^2 \quad \text{(Cartesian form)} \quad (3.4)$$

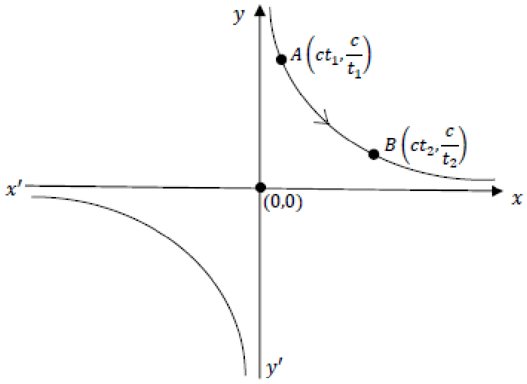

Figure 4: Rectangular Hyperbola $xy = c^2$

Its parametric form is given by

$$x = ct, \quad y = \frac{c}{t} \tag{3.5}$$

Put $\frac{dx}{dt} = c, \frac{dy}{dt} = -\frac{c}{t^2}$, in (2.4) we get

$$\widehat{AB} = \int_{t_1}^{t_2} \sqrt{\left(c^2 + \frac{c^2}{t^4}\right)}\, dt = c\int_{t_1}^{t_2} \sqrt{\left(1 + \frac{1}{t^4}\right)}\, dt. \tag{3.6}$$

Since no standard formula for indefinite integral of $\sqrt{\left(1 + \frac{1}{t^4}\right)}$ w. r. to t is available in the literature of integral calculus and other mathematical tables. Therefore we shall apply hypergeometric approach to evaluate the exact value of the arc-length. Furthermore, given rectangular hyperbola (3.4) is symmetrical about the line $y = -x$ without any loss of generality, we shall find the arc-length between two arbitrary points lying in positive quadrant only.

Three cases arise based on the position of points A and B

when $0 < t_1 < t_2 < 1$ and $c > 0$

$$\text{then} \quad \widehat{AB} = c\int_{t_1}^{t_2} \frac{1}{t^2}\sqrt{(1+t^4)}\,dt, \quad \text{since} \quad t^4 < 1 \ \text{here},$$

$$= c\int_{t_1}^{t_2} \frac{1}{t^2}\, {}_1F_0\left[\begin{matrix} -\frac{1}{2}; \\ -; \end{matrix} \; -t^4\right] dt,$$

$$= c\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{t_2} t^{4r-2}\,dt,$$

$$= c\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!\,(4r-1)} (t_2^{4r-1} - t_1^{4r-1}),$$

$$\widehat{AB} = \frac{c}{t_1}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{4}; \\ \frac{3}{4} \quad ; \end{matrix} \; -t_1^4\right] - \frac{c}{t_2}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{4}; \\ \frac{3}{4} \quad ; \end{matrix} \; -t_2^4\right] \tag{3.7}$$

Both ${}_2F_1$ series are convergent, since $t_1 < 1$ and $t_2 < 1$. Therefore, our result (3.7) is also convergent.

when $0 < t_1 < 1 < t_2 < \infty$ (or $t_1 < 1$ and $\frac{1}{t_2} < 1$) and $c > 0$

then $$\widehat{AB} = c\int_{t_1}^{t_2} \sqrt{\left(1+\frac{1}{t^4}\right)}\,dt = c\int_{t_1}^{1} \sqrt{\left(1+\frac{1}{t^4}\right)}\,dt + c\int_{1}^{t_2} \sqrt{\left(1+\frac{1}{t^4}\right)}\,dt,$$

$$= c\int_{t_1}^{1} \frac{1}{t^2}\,{}_1F_0\left[\begin{matrix}-\frac{1}{2}; \\ -;\end{matrix}\; -t^4\right]dt + c\int_{1}^{t_2} {}_1F_0\left[\begin{matrix}-\frac{1}{2}; \\ -;\end{matrix}\; -\frac{1}{t^4}\right]dt,$$

$$= c\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{1} t^{4r-2}dt + c\sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{1}^{t_2} t^{-4m}dt,$$

(3.8)

$$\widehat{AB} = c\left[\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!\,(4r-1)} (t^{4r-1})\right]_{t_1}^{1} + c\left[\sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!\,(-4m+1)} (t^{-4m+1})\right]_{1}^{t_2}$$

$$= \frac{c}{t_1}\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \frac{3}{4}\;\;;\end{matrix}\; -t_1^4\right] + ct_2\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \frac{3}{4}\;\;;\end{matrix}\; -\frac{1}{t_2^4}\right] - 2c\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \frac{3}{4}\;\;;\end{matrix}\; -1\right].$$

(3.9)

Now using Kummer's first summation theorem (1.7), we get

$$\widehat{AB} = \frac{c}{t_1}\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \frac{3}{4}\;\;;\end{matrix}\; -t_1^4\right] + ct_2\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; \\ \frac{3}{4}\;\;;\end{matrix}\; -\frac{1}{t_2^4}\right] - 2c\,\frac{\left(\Gamma\left(\frac{3}{4}\right)\right)^2}{\sqrt{\pi}},$$

(3.10)

where, $\Gamma\left(\frac{3}{4}\right) = 1.2254167024\ldots$, Both ${}_2F_1$ series are convergent, since $t_1 < 1$ and $\frac{1}{t_2} < 1$. Therefore, the result (3.10) is also convergent.

When $1 < t_1 < t_2 < \infty$ (or $\frac{1}{t_2} < \frac{1}{t_1} < 1$) and $c > 0$

then $$\widehat{AB} = c\int_{t_1}^{t_2} \sqrt{\left(1+\frac{1}{t^4}\right)}\,dt$$

$$= c\int_{t_1}^{t_2} {}_1F_0\left[\begin{matrix}-\frac{1}{2}; \\ -;\end{matrix}\; -\frac{1}{t^4}\right]dt,$$

$$= c\sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{t_1}^{t_2} t^{-4m}dt,$$

$$= c\sum_{m=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{\mathrm{m}}(-1)^{m}}{m!\,(-4m+1)}\left(t_2^{-4m+1}-t_1^{-4m+1}\right),$$

$$\widehat{AB} = ct_2\;{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; & \\ & -\frac{1}{t_2^4}\\ \frac{3}{4}\quad ; & \end{matrix}\right] - ct_1\;{}_2F_1\left[\begin{matrix}-\frac{1}{2},-\frac{1}{4}; & \\ & -\frac{1}{t_1^4}\\ \frac{3}{4}\quad ; & \end{matrix}\right] \tag{3.11}$$

Both ${}_2F_1$ series are convergent, since $\frac{1}{t_1} < 1$ and $\frac{1}{t_2} < 1$. Therefore, the result (3.11) is also convergent.

REMARK

We have also derived a formula [5] for arc-length between two arbitrary points lying on an ellipse.

We conclude our present investigation by observing that solutions of such problems can be obtained in an analogous manner.

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