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On the Hankel Integral Transform with Wright's Generalized Hypergeometric Function and Applications

Yashwant Singh
Government College, Kaladera, Jaipur, Rajasthan, India

Abstract: *The aim of this paper is to evaluate Hankel transform of Wright's generalized hypergeometric function defined by Dotsenko [1, 2]. The author has given two applications of Hankel transform of Wright's generalized hypergeometric function by connecting this, first with the Weyl integral and second is with Riemann-Liouville type of fractional derivative.The results obtained are basic in nature and are likely to find useful applications.*
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Key words: *Hankel transform, Wright's Generalized Hypergeometric Function, Weyl Integral, Riemann-Liouville type fractional derivative.*

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I. INTRODUCTION

Generalized Wright's function ${}_2R_1(a,b;c,w;\mu;z)$ defined by Dotsenko [1, 2] hs been denoted as

$$ {}_2R_1(a,b;c,w;\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma\left(b+k\frac{w}{\mu}\right)}{\Gamma\left(c+k\frac{w}{\mu}\right)} \frac{z^k}{k!} $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2\psi_1\left[z \Big|_{\left(c,\frac{w}{\mu}\right)}^{(a,1),\left(b,\frac{w}{\mu}\right)} \right] \qquad (1.1) $$

Provided $\text{Re}(c) > 0, \text{Re}(b) > 0, \text{Re}\left(\frac{w}{k}\right) > 0$.

Virchenko et. al. [6] defined the Wright type hypergeometric function by taking $\frac{w}{k} = \tau > 0$ in (1.1) as

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c,w;\mu;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma\left(b+k\frac{w}{\mu}\right)}{\Gamma\left(c+k\frac{w}{\mu}\right)} \frac{z^k}{k!}, \tau > 0, |z| < 1 \qquad (1.2) $$

If $\tau = 1$, then (1.2) reduces to a Gauss's hypergeometric function.
The Hankel transform of a function $f(x)$, denoted by $g(p,v)$ or in short by simply $g(p)$ is defined as

$$ g(p;v) = \int_0^{\infty} (px)^{\frac{1}{2}} J_v(px) f(x) dx; p > 0 \qquad (1.3) $$

Where p as a complex parameter.

II. HANKEL TRANSFORM OF WRIGHT'S GENERALIZED WRIGHT'S FUNCTION

If $a,b,c,p,A,v\in C;\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(v)>0,\mathrm{Re}(A)>0$ and

$\frac{w}{k}\in N$, then

$$\int_0^\infty x^{\rho-1}J_v(Ax)\,{}_2R_1(a,b;c,w;\mu;px^\delta)dx$$

$$=\frac{\Gamma(c)2^{\rho-1}}{\Gamma(a)\Gamma(b)}\,{}_3\psi_2\left[\left(\frac{2^\delta p}{A^\delta}\right)\middle|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),\left(\frac{\rho+v}{2},\frac{-\delta}{2}\right)\\ \left(c,\frac{w}{\mu}\right),\left(\frac{\rho-v}{2},\frac{-\delta}{2}\right)\end{matrix}\right] \quad (2.1)$$

Proof: $\int_0^\infty x^{\rho-1}J_v(Ax)\,{}_2R_1(a,b;c,w;\mu;px^\delta)dx$

$$=\int_0^\infty x^{\rho-1}J_v(Ax)\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma\left(b+\frac{w}{\mu}k\right)}{\Gamma\left(c+\frac{w}{\mu}k\right)}\frac{p^kx^{\delta k}}{k!}dx$$

$$=\int_0^\infty x^{\rho+\delta k-1}J_v(Ax)\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma\left(b+\frac{w}{\mu}k\right)}{\Gamma\left(c+\frac{w}{\mu}k\right)}\frac{p^k}{k!}dx$$

Changing the order of integration and summation therein (which is permissible under the conditions mentioned with (2.1), we find that

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma\left(b+\frac{w}{\mu}k\right)}{\Gamma\left(c+\frac{w}{\mu}k\right)}\frac{p^k}{k!}\int_0^\infty x^{\rho+\delta k-1}J_v(Ax)dx$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma\left(b+\frac{w}{\mu}k\right)\Gamma\left(\frac{\rho+\delta k+v}{2}\right)}{\Gamma\left(c+\frac{w}{\mu}k\right)\Gamma\left(1-\frac{-v+\rho+\delta k}{2}\right)}\frac{2^{2+\delta k-1}A^{-\rho-\delta k}p^k}{k!}$$

$$=\frac{\Gamma(c)2^{\rho-1}}{\Gamma(a)\Gamma(b)A^\rho}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma\left(b+\frac{w}{\mu}k\right)\Gamma\left(\frac{\rho+\delta k+v}{2}\right)}{\Gamma\left(c+\frac{w}{\mu}k\right)\Gamma\left(1-\frac{-v+\rho+\delta k}{2}\right)}\frac{1}{k!}\left(\frac{2^\delta p}{A^\delta}\right)^k$$

$$=\frac{\Gamma(c)2^{\rho-1}}{\Gamma(a)\Gamma(b)}\,{}_3\psi_2\left[\left(\frac{2^\delta p}{A^\delta}\right)\middle|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),\left(\frac{\rho+v}{2},\frac{-\delta}{2}\right)\\ \left(c,\frac{w}{\mu}\right),\left(1-\frac{\rho-v}{2},\frac{-\delta}{2}\right)\end{matrix}\right]$$

III. APPLICATIONS

The Weyl integral ([3], p.91) of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$$\left({}_xW_\infty^\alpha f\right)(x)=\left({}_xI_\infty^\alpha f\right)(x)=\left(I_-^\alpha f\right)(x)$$

$$=\frac{1}{\Gamma(\alpha)}\int_x^\infty (x-t)^{\alpha-1} f(t)dt,-\infty,x<\infty \qquad (3.1)$$

Where $\alpha\in C,\mathrm{Re}(\alpha)>0$.

The Weyl Integral of Hankel Transform in Association with Wright's Generalized Hypergeometric Function

The main integral (2.1) can be rewritten as the following Weyl integral formula:

$$\left({}_0W_\infty^\rho J_v(At)\,{}_2R_1(a,b;c,w;\mu;pt^\delta)\right)(t)$$

$$=\frac{1}{\Gamma(\rho)}\int_0^\infty t^{\rho-1}J_v(At)\,{}_2R_1(a,b;c,w;\mu;pt^\delta)dt$$

$$=\frac{\Gamma(c)2^{\rho-1}}{\Gamma(\rho)\Gamma(a)\Gamma(b)A^\rho}\,{}_3\psi_2\left[\left(\frac{2^\delta p}{A^\delta}\right)\middle|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),\left(\frac{\rho+v}{2},\frac{-\delta}{2}\right)\\ \left(c,\frac{w}{\mu}\right),\left(1-\frac{\rho-v}{2},\frac{-\delta}{2}\right)\end{matrix}\right] \qquad (3.2)$$

Provided

$a,b,c,p,A,v\in C;\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(v)>0,\mathrm{Re}(A)>0$

and

$\frac{w}{k}\in N$,

Fractional Derivatives

Following Miller ([4],p.82), let $g\in A$ (Where A is a class of good functions). Then

$$ {}_zD_\infty^q g(z)=\frac{(-1)^q}{\Gamma(-q)}\int_z^\infty (u-z)^{-q-1}g(u)du,\ for q<0 \qquad (3.3)$$

For $q\geq 0$

$$ {}_zD_\infty^q g(z)=\frac{d^r}{dz^r}\left({}_zD_\infty^{q-r}g(z)\right) \qquad (3.4)$$

r being a positive integer such that $r>q$.

Fractional Derivatives of Hankel Transform in Association with Wright's Generalized Hypergeometric Function

The main integral (2.1) can be rewritten as the following Riemann-Liouville fractional derivative formula:

$$ {}_0D_\infty^{-\rho}\left(J_v(Ax)\,{}_2R_1(a,b;c,w;\mu;px^\delta)\right)$$

$$=\frac{(-1)^\rho}{\Gamma(\rho)}\int_0^\infty x^{\rho-1}J_v(Ax)\,{}_2R_1(a,b;c,w;\mu;px^\delta)dx$$

$$=\frac{(-1)^\rho\Gamma(c)2^{\rho-1}}{\Gamma(\rho)\Gamma(a)\Gamma(b)A^\rho}\,{}_3\psi_2\left[\left(\frac{2^\delta p}{A^\delta}\right)\middle|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),\left(\frac{\rho+v}{2},\frac{-\delta}{2}\right)\\ \left(c,\frac{w}{\mu}\right),\left(1-\frac{\rho-v}{2},\frac{-\delta}{2}\right)\end{matrix}\right];\ for\ \rho>0. \qquad (3.5)$$

It is being assumed that the conditions given in (2.1) and (3.3) are satisfied.

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