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Bilinear and Bilateral Generating Functions of Hypergeometric Functions by means of Fractional Derivatives

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ABSTRACT

The object of the present paper is to obtain bilinear and bilateral generating functions for several classes of hypergeometric functions by employing the technique of fractional derivatives on some well-known identities.

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I. INTRODUCTION

The name generating functions was introduced by Laplace in1812. Since then the theory of generating functions has been developed in various directions and play a vital role in the investigation of various useful properties in different branches of science and technology. In the present investigation we find different types of bilinear and bilateral generating functions of different parameters by using the technique of fractional derivatives on some well-known infinite series identities.

In literature the use of fractional derivatives in the theory of hypergeometric functions have wide application in the field of modeling, physics and engineering, stochastic process, probability theory, in solving ordinary and partial differential equations and integral equations(see [4], [5], [7], [12]). etc.

In 1731, Euler's extended the derivative formula ([12] pp.285), to the general form as

$$D_z^{\mu}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} z^{\lambda-\mu} \tag{1.1}$$

where μ is an ordinary complex number.

Here we use the theorem which is mentioned below, is the application of Euler's derivative formula to some special functions.

Theorem 1: If a function $f(z)$ is analytic in the disc $|z| < \rho$, has the power series expansion,

$$f(z) = \sum_{n=0}^{\infty} (a)_n z^n, |z| < \rho \tag{1.2}$$

then ,

$$D_z^{\mu}\{z^{\lambda-1} f(z)\} = \sum_{n=0}^{\infty} (a)_n D_z^{\mu}\{z^{\lambda+\mu-1}\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{(a)_n (\lambda)_n}{(\lambda-\mu)_n} z^n \tag{1.3}$$

provided that $Re(\lambda) > 0, Re(\mu) < 0$, and $|z| < \rho$.

Some of the definition and notations used in the given manuscript are stated below:
Appell function of two variables defined by [1] are given as

$$F_1[a,b,b';\ c;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_{m+n} m!\, n!}, \tag{1.4}$$

$$F_2[a,b,b';\ c,c';\ x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m\ (c')_n m!\, n!}, \tag{1.5}$$

Generalization of Appell function of two variables by Khan M.A. and Abukhammash G.S. [2] are defined as

$$M_3(a,b,b',c,c';d,d',e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_m(d')_n(e)_m(e')_n} \frac{x^m y^n}{m!\,n!} \tag{1.6}$$

$$M_4(a,b,b',c,c';d,e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\,n!} \tag{1.7}$$

$$M_7(a,b,c,c';d,e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\,n!} \tag{1.8}$$

$$M_8(a,b,c,c';\ d,e;\ x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_{m+n}} \frac{x^m y^n}{m!\,n!} \tag{1.9}$$

Lauricella [3], generalized the Appell double hypergeometric functions $F_1, \ldots, F_4$ to functions of n variables, but we use only two $F_A^{(n)}$ and $F_D^{(n)}$ are defined by

$$F_A^{(n)}[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots\ldots,x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.10}$$
$$|x_1|,\ldots\ldots\ldots\ldots,|x_n| < 1;$$

$$F_B^{(n)}[a_1,\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1)_{m_1}\cdots(b_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.11}$$
$$\max\{|x_1|,\ldots\ldots\ldots\ldots,|x_n|\} < 1.$$

$$F_D^{(n)}[a,b_1,\ldots\ldots,b_n;c;x_1,\ldots\ldots,x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.12}$$
$$\max\{|x_1|,\ldots\ldots\ldots\ldots,|x_n|\} < 1.$$

Saran [8] initiated a systematic study of these ten hypergeometric functions of Lauricella's set. We give below the definitions of two of these functions using Saran's notations F_M and F_K and also indicating Lauricella's notations:

$$F_M[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}m!\,n!\,p!} x^m y^n z^p \tag{1.13}$$

$$F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!\,n!\,p!} x^m y^n z^p \tag{1.14}$$

A general triple hypergeometric series $F^{(3)}[x,y,z]$ defined as (see [12], pp. 69) is defined as

$$F^{(3)}[x,y,z] = F^{(3)}\left[\begin{matrix}(a) :: (b);\ (b');\ (b''): (c);\ (c');\ (c'');\\ \\ (e) :: g);\ (g');\ (g''): (h);\ (h');\ (h'');\end{matrix}\ x,y,z\right]$$
$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p) \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{1.15}$$

where, for convenience,

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}$$
$$\times \frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p} \tag{1.16}$$

where (a) abbreviates the array of A parameters $a_1,\ldots,a_A$. with similar interpretations for $(b),(b'),(b''), etcetra$.

In this manuscript we use the following fractional derivative formulas [9] and linear generating functions formulas [10] to obtain the several class of bilinear and bilateral generating functions.

$$D_x^{\alpha-\mu}\left\{x^\alpha(1-x)^{-\beta}\left(1-\frac{\omega x}{1-x}\right)^{-\gamma}\right\}$$
$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^\mu (1-x)^{-\alpha-1} F_1\left[1+\alpha,\gamma,1+\mu-\beta;1+\mu;\frac{\omega x}{1-x},\frac{-x}{1-x}\right] \tag{1.17}$$

where, $Re(\alpha) \geq 0, |x| < 1, \left|\frac{\omega x}{1-x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}\left(1-\frac{\omega_1 x}{1-x}\right)^{-\gamma}\left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^{\mu}(1-x)^{-\alpha-1}F_D^{(3)}\left[1+\alpha,\gamma,\delta,1+\mu-\beta;1+\mu;\frac{\omega_1 x}{1-x},\frac{\omega_2 x}{1-x},\frac{x}{x-1}\right] \tag{1.18}$$

where, $Re(\alpha)\geq 0, |x|<1, \left|\frac{\omega_1 x}{1-x}\right|<1, \left|\frac{\omega_2 x}{1-x}\right|<1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^{\mu}F_M[\delta,1+\alpha,1+\alpha,\beta,\gamma,\beta;\beta,1+\mu,1+\mu;\omega_2,\omega_1 x,x] \tag{1.19}$$

where, $Re(\alpha)\geq 0, |x|<1, |\omega_1 x|<1, \left|\frac{\omega_2}{1-x}\right|<1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^{\mu}F^{(3)}\begin{bmatrix}1+\alpha::--;\ \beta\,;--:\gamma\,;\ \delta\,;--; & \\ & \omega_1 x,\omega_2 x,x\\ 1+\mu::--;--;--:--;\beta;--; & \end{bmatrix} \tag{1.20}$$

where, $Re(\alpha)\geq 0, |x|<1, |\omega_1 x|<1, \left|\frac{\omega_2 x}{1-x}\right|<1.$

$$D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}\left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1x_2}\right)^{-\beta}\left(1-\frac{\omega_2}{x_1x_3}\right)^{-\gamma}\right\}$$
$$=\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)}{x_1}^{-\mu_1}{x_2}^{-\mu_2}{x_3}^{-\mu_3}M_4\left[\mu_1,\beta,\gamma,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right] \tag{1.21}$$

where, $\left|\frac{\omega_1}{x_1x_2}\right|<1, \left|\frac{\omega_2}{x_1x_3}\right|<1.$

$$D_{x_1}^{\alpha-\mu}D_{x_2}^{\alpha'-\mu'}\{{x_1}^{\alpha}{x_2}^{\alpha'}(1-\omega_1x_1x_2)^{-\beta}(1-\omega_2x_1x_2)^{-\gamma}\}$$
$$=\frac{\Gamma(1+\alpha)\Gamma(1+\alpha')}{\Gamma(1+\mu)\Gamma(1+\mu')}{x_1}^{\mu}{x_2}^{\mu}M_8[1+\alpha,1+\alpha',\beta,\gamma;1+\mu,1+\mu';\omega_1x_1x_2,\omega_2x_1x_2] \tag{1.22}$$

where, $Re(\alpha)\geq 0, Re(\alpha')\geq 0, |\omega_1x_1x_2|<1, |\omega_2x_1x_2|<1.$

$$D_x^{\mu-\alpha}\left\{x^{-\alpha}\left(1-\frac{\omega_1}{x}\right)^{-\beta}\left(1-\frac{\omega_2}{x}\right)^{-\gamma}\right\}=\frac{\Gamma(1-\alpha)}{\Gamma(1-\mu)}x^{-\mu}F_1\left[\mu\,,\beta\,,\gamma\,;\ \alpha\,;\ \frac{\omega_1}{x},\frac{\omega_2}{x}\right] \tag{1.23}$$

where, $\left|\frac{\omega_1}{x}\right|<1, \left|\frac{\omega_2}{x}\right|<1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-\omega_1 x)^{-\beta}(1-\omega_2 x)^{-\gamma}\left(1-\frac{\omega_3}{x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^{\mu}G_B\left[1+\alpha,\delta,\beta,\gamma;1+\mu;\frac{\omega_3}{x},\omega_1 x,\omega_2 x\right] \tag{1.24}$$

where, $Re(\alpha)\geq 0, |\omega_1 x|<1, |\omega_2 x|<1, \left|\frac{\omega_3}{x}\right|<1.$

$$D_x^{\alpha-\mu}\{x^{\alpha}(1-\omega_1 x)^{-\beta}(1-\omega_2 x)^{-\gamma}(1-\omega_3 x)^{-\delta}\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^{\mu}F_D^{(3)}[1+\alpha,\beta,\gamma,\delta;1+\mu;\omega_1 x,\omega_2 x,\omega_3 x] \tag{1.25}$$

where, $Re(\alpha)\geq 0, |\omega_1 x|<1, |\omega_2 x|<1, |\omega_3 x|<1.$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[\lambda+n,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right]t^n=(1-t)^{-\lambda}F_2\left[\lambda,\mu,\mu^{'};\alpha,\alpha^{'};\frac{\omega_1}{x_1(1-t)},\frac{\omega_2}{x_2(1-t)}\right] \tag{1.26}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_3\left[\lambda+n,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_3x_4}\right]t^n$$
$$=(1-t)^{-\lambda}M_3\left[\lambda,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_3x_4(1-t)}\right] \tag{1.27}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_7\left[\lambda+n,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right]t^n$$
$$=(1-t)^{-\lambda}M_7\left[\lambda,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_1x_3(1-t)}\right] \tag{1.28}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}H_A[\beta,\lambda+n,1+\alpha;\beta,1+\mu;\omega_2,\omega_1 x,x]t^n$$

$$= (1-t)^{-\lambda} H_A\left[\beta, \lambda, 1+\alpha; \beta, 1+\mu; \frac{\omega_2}{(1-t)}, \frac{\omega_1 x}{(1-t)}, x\right] \tag{1.29}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1\left[1+\alpha, \lambda+n, 1+\mu-\beta; 1+\mu; \frac{\omega x}{1-x}, \frac{x}{x-1}\right] t^n$$

$$= (1-t)^{-\lambda} F_1\left[1+\alpha, \lambda, 1+\mu-\beta; 1+\mu; \frac{\omega x}{(1-x)(1-t)}, \frac{x}{x-1}\right] \tag{1.30}$$

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} F_D^{(3)}\left[1+\alpha, \lambda_1+m, \lambda_2+n, 1+\mu-\beta; 1+\mu; \frac{\omega_1 x}{1-x}, \frac{\omega_2 x}{1-x}, \frac{x}{x-1}\right] (t_1)^m (t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_D^{(3)}\left[1+\alpha, \lambda_1, \lambda_2, 1+\mu-\beta; 1+\mu; \frac{\omega_1 x}{(1-x)(1-t_1)}, \frac{\omega_2 x}{(1-x)(1-t_2)}, \frac{x}{x-1}\right] \tag{1.31}$$

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} (t_1)^m (t_2)^n F_M[\lambda_2+n, 1+\alpha, 1+\alpha, \beta, \lambda_1+m, \beta; \beta, 1+\mu, 1+\mu; \omega_2, \omega_1 x, x]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_M\left[\lambda_2, 1+\alpha, 1+\alpha, \beta, \lambda_1, \beta; \beta, 1+\mu, 1+\mu; \frac{\omega_2}{(1-t_2)}, \frac{\omega_1 x}{(1-t_1)}, x\right] \tag{1.32}$$

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} (t_1)^m (t_2)^n F^{(3)}\left[\begin{matrix} 1+\alpha::-;\beta;-:\lambda_1+m;\lambda_2+n;-; \\ 1+\mu::--;--;--:--;\beta\,;--; \end{matrix} \omega_1 x, \omega_2 x, x\right]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F^{(3)}\left[\begin{matrix} 1+\alpha::-;\beta;-:\lambda_1;\lambda_2;-; \\ 1+\mu::-;-;-:-;\beta;-; \end{matrix} \frac{\omega_1 x}{(1-t_1)}, \frac{\omega_2 x}{(1-t_2)}, x\right] \tag{1.33}$$

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} F_1\left[\mu, \lambda_1+m, \lambda_2+n; \alpha; \frac{\omega_1}{x}, \frac{\omega_2}{x}\right] (t_1)^m (t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_1\left[\mu, \lambda_1, \lambda_2; \alpha; \frac{\omega_1}{x(1-t_1)}, \frac{\omega_2}{x(1-t_2)}\right] \tag{1.34}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!} F_D^{(3)}[1+\alpha, \lambda_1+m, \lambda_2+n, \lambda_3+p; 1+\mu; \omega_1 x, \omega_2 x, \omega_3 x](t_1)^m (t_2)^n (t_3)^p$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3} F_D^{(3)}\left[1+\alpha, \lambda_1, \lambda_2, \lambda_3; 1+\mu; \frac{\omega_1 x}{(1-t_1)}, \frac{\omega_2 x}{(1-t_2)}, \frac{\omega_3 x}{(1-t_3)}\right] \tag{1.35}$$

I. BILINEAR GENERATING FUNCTIONS

Consider the elementry identity (cf. [12], p. 297),

$$[(1-x)(1-y)-t]^{-\lambda} = (1-t)^{-\lambda}\left[\left(1-\frac{x}{1-t}\right)\left(1-\frac{y}{1-t}\right) - \frac{xyt}{(1-t)^2}\right] \tag{2.1}$$

where, $\left|\frac{t}{(1-x)(1-y)}\right| < 1$ and $\left|\frac{xyt}{(1-x-t)(1-y-t)}\right| < 1$, can be written as

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-x)^{-(\lambda+n)} (1-y)^{-(\lambda+n)} t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(1-\frac{x}{1-t}\right)^{-(\lambda+n)} \left(1-\frac{y}{1-t}\right)^{-(\lambda+n)} \left(\frac{xyt}{(1-t)^2}\right)^n. \tag{2.2}$$

In relation (2.2), replace x, y by $\frac{x\omega}{1-x}$, $\frac{yu}{1-y}$ respectively, then multiply both sides by $x^\alpha(1-x)^{-\beta} y^\gamma (1-y)^{-\delta}$. After that operate the fractional derivative operator $D_x^{\alpha-\mu} D_y^{\gamma-\eta}$ on both sides, and by using (1.17), we arrive to the bilinear generating function

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1\left[1+\alpha, \lambda+n, 1+\mu-\beta; 1+\mu; \frac{\omega x}{1-x}, \frac{-x}{1-x}\right]$$

$$\times F_1\left[1+\gamma, \lambda+n, 1+\eta-\delta; 1+\eta; \frac{uy}{1-y}, \frac{-y}{1-y}\right] t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (1+\alpha)_n (1+\gamma)_n}{(1+\mu)_n (1+\eta)_n n!} F_1\left[1+\alpha+n, \lambda+n, 1+\mu-\beta;\ 1+\mu+n;\ \frac{\omega x}{(1-x)(1-t)}, \frac{-x}{1-x}\right]$$

$$\times F_1\left[1+\gamma+n, \lambda+n, 1+\eta-\delta; 1+\eta+n; \frac{uy}{(1-y)(1-t)}, \frac{-y}{(1-y)}\right] \left(\frac{xyu\omega t}{(1-x)(1-y)(1-t)^2}\right)^n \tag{2.3}$$

In relation (2.2), replace x, y, t, λ by $\frac{\omega_1 x}{1-x}$, $\frac{u_1 y}{1-y}$, t_1, λ_1 respectively. Again, replace x, y, t, λ by $\frac{\omega_2 x}{1-x}$, $\frac{u_2 y}{1-y}$, t_2, λ_2 respectively, then multiply these two equation each other and also multiply with $x^{\alpha}(1-x)^{-\beta}y^{\gamma}(1-y)^{-\delta}$. Now on operating the fractional derivative operator $D_x^{\alpha-\mu}D_y^{\gamma-\eta}$ on both sides and with the aid of relation (1.18), we arrive to the bilinear generating function

$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_D^{(3)}\left[1+\alpha,\lambda_1+m,\lambda_2+n,1+\mu-\beta;1+\mu;\frac{\omega_1 x}{1-x},\frac{\omega_2 x}{1-x},\frac{x}{x-1}\right]$$

$$\times F_D^{(3)}\left[1+\gamma,\lambda_1+m,\lambda_2+n,1+\eta-\delta;1+\eta;\frac{u_1 y}{1-y},\frac{u_2 y}{1-y},\frac{y}{y-1}\right](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(1+\alpha)_{m+n}(1+\gamma)_{m+n}}{(1+\mu)_{m+n}(1+\eta)_{m+n}m!\,n!}$$

$$\times F_D^{(3)}\left[1+\alpha+m+n,\lambda_1+m,\lambda_2+n,1+\mu-\beta;1+\mu+m+n;\ \frac{\omega_1 x}{(1-x)(1-t_1)},\frac{\omega_2 x}{(1-x)(1-t_2)},\frac{x}{x-1}\right]$$

$$\times F_D^{(3)}\left[1+\gamma+m+n,\lambda_1+m,\lambda_2+n,1+\eta-\delta;1+\eta+m+n;\ \frac{u_1 y}{(1-y)(1-t_1)},\frac{u_2 y}{(1-y)(1-t_2)},\frac{y}{y-1}\right]$$

$$\times\left(\frac{xyu_1\omega_1 t_1}{(1-x)(1-y)(1-t_1)^2}\right)^m\left(\frac{xyu_2\omega_2 t_2}{(1-x)(1-y)(1-t_2)^2}\right)^n \tag{2.4}$$

Now, we adopt the analysis employed to obtain (2.4) and use the result (1.19) and (1.20) respectively, we then obtain the double generating functions as given below:

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_M[\lambda_2+n,1+\alpha,1+\alpha,\beta,\lambda_1+m,\beta;\beta,1+\mu,1+\mu;\omega_2,\omega_1 x,x]$$

$$\times F_M[\lambda_2+n,1+\gamma,1+\gamma,\delta,\lambda_1+m,\delta;\delta,1+\eta,1+\eta;u_2,u_1 y,y](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(1+\alpha)_m(1+\gamma)_m}{(1+\mu)_m(1+\eta)_m m!\,n!}$$

$$\times F_M\left[\lambda_2+n,1+\alpha+m,1+\alpha+m,\beta+n,\lambda_1+m,\beta+n;\beta+n,1+\mu+m,1+\mu+m;\frac{\omega_2}{(1-t_2)},\frac{\omega_1 x}{(1-t_1)},x\right]$$

$$\times F_M\left[\lambda_2+n,1+\gamma+m,1+\gamma+m,\delta+n,\ \lambda_1+m,\delta+n;\delta+n,1+\eta+m,1+\eta+m;\frac{u_2}{(1-t_2)},\frac{u_1 y}{(1-t_1)},y\right]$$

$$\times\left(\frac{xyu_1\omega_1 t_1}{(1-t_1)^2}\right)^m\left(\frac{u_2\omega_2 t_2}{(1-t_2)^2}\right)^n \tag{2.5}$$

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F^{(3)}\left[\begin{matrix}1+\alpha::--;\beta;--:\lambda_1+m;\lambda_2+n;--;\\ \\ 1+\mu::--;--;--:--;\ \beta\ ;--;\end{matrix}\ \omega_1 x,\omega_2 x,x\right]$$

$$\times F^{(3)}\left[\begin{matrix}1+\gamma::--;\delta;--:\lambda_1+m;\lambda_2+n;--;\\ \\ 1+\eta::--;--;--:--;\ \delta\ ;--;\end{matrix}\ u_1 y,u_2 y,y\right](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(1+\alpha)_{m+n}(1+\gamma)_{m+n}}{(1+\mu)_{m+n}(1+\eta)_{m+n}m!\,n!}$$

$$\times F^{(3)}\left[\begin{matrix}1+\alpha+m+n::--;\beta+n;--:\lambda_1+m;\lambda_2+n;--;\\ \\ 1+\mu+m+n::--;--;--:--;\ \beta+n\ ;--;\end{matrix}\ \frac{\omega_1 x}{(1-t_1)},\frac{\omega_2 x}{(1-t_2)},x\right]$$

$$\times F^{(3)}\left[\begin{matrix}1+\gamma+m+n :: --;\delta+n;--:\lambda_1+m;\lambda_2+n;--;\\ 1+\eta+m+n :: --;--;--:--;\ \delta+n\ ;--;\end{matrix}\ \frac{u_1 y}{(1-t_1)},\frac{u_2 y}{(1-t_2)},y\right]$$

$$\times\left(\frac{xyu_1\omega_1 t_1}{(1-t_1)^2}\right)^m\left(\frac{xyu_2\omega_2 t_2}{(1-t_2)^2}\right)^n \tag{2.6}$$

In relation (2.2), replace x, y, t, λ by $\frac{\omega_1}{x_1x_2}$, $\frac{u_1}{y_1y_2}$, t_1, λ_1 respectively. Again, replace x, y, t, λ by $\frac{\omega_2}{x_1x_3}$, $\frac{u_2}{y_1y_3}$, t_2, λ_2 respectively, then multiply these two equation each other and also multiply both side by $x^{-\alpha_1}x^{-\alpha_2}x^{-\alpha_3}y^{-\gamma_1}y^{-\gamma_2}y^{-\gamma_3}$.Now on operating the fractional derivative operator $D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}D_{y_1}^{\eta_1-\gamma_1}D_{y_2}^{\eta_2-\gamma_2}D_{y_3}^{\eta_3-\gamma_3}$ on both sides and by using the relation (1.21), we obtain the bilinear generating function .

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}M_4\left[\mu_1,\lambda_1+m,\lambda_2+n,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right]$$

$$\times M_4\left[\eta_1,\lambda_1+m,\lambda_2+n,\eta_2,\eta_3;\gamma_1,\gamma_2,\gamma_3;\frac{u_1}{y_1y_2},\frac{u_2}{y_1y_3}\right](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(\mu_1)_{m+n}(\eta_1)_{m+n}(\mu_2)_m(\eta_2)_m(\mu_3)_n(\eta_3)_n}{(\alpha_1)_{m+n}(\gamma_1)_{m+n}(\alpha_2)_m(\gamma_2)_m(\alpha_3)_n(\gamma_3)_n m!\,n!}$$

$$\times M_4\left[\mu_1+m+n,\lambda_1+m,\lambda_2+n,\mu_2,\mu_3;\alpha_1+m+n,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2(1-t_1)},\frac{\omega_2}{x_1x_3(1-t_2)}\right]$$

$$\times M_4\left[\eta_1+m+n,\lambda_1+m,\lambda_2+n,\eta_2,\eta_3;\gamma_1+m+n,\gamma_2,\gamma_3;\ \frac{u_1}{y_1y_2(1-t_1)},\frac{u_2}{y_1y_3(1-t_2)}\right]$$

$$\times\left(\frac{\omega_1u_1t_1}{x_1x_2y_1y_2(1-t_1)^2}\right)^m\left(\frac{\omega_2u_2t_2}{x_1x_3y_1y_3(1-t_2)^2}\right)^n \tag{2.7}$$

In relation (2.2), replace x, y, t, λ by $\omega_1x_1x_2$, $u_1y_1y_2$, t_1, λ_1 respectively. Again, replace x, y, t, λ by by $\omega_2x_1x_2$, $u_2y_1y_2$, t_2, λ_2 respectively, then multiply these two equation each other and also multiply both side by $x^{\alpha_1}x^{\alpha_2}y^{\gamma_1}y^{\gamma_2}$. Now on operating the fractional derivative operator $D_{x_1}^{\alpha_1-\mu_1}D_{x_2}^{\alpha_2-\mu_2}D_{y_1}^{\gamma_1-\eta_1}D_{y_2}^{\gamma_2-\eta_2}$ on both sides and by using the relation (1.22), we obtain the bilinear generating function .

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}M_8[1+\alpha_1,1+\alpha_2,\lambda_1+m,\lambda_2+n;1+\mu_1,1+\mu_2;\omega_1x_1x_2,\omega_2x_1x_2]$$

$$\times M_8[1+\gamma_1,1+\gamma_2,\lambda_1+m,\lambda_2+n;1+\eta_1,1+\eta_2;u_1y_1y_2,u_2y_1y_2](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}(1+\gamma_1)_{m+n}(1+\gamma_2)_{m+n}}{(1+\mu_1)_{m+n}(1+\mu_2)_{m+n}(1+\eta_1)_{m+n}(1+\eta_2)_{m+n}m!\,n!}$$

$$\times M_8\left[1+\alpha_1+m+n,1+\alpha_2+m+n,\lambda_1+m,\lambda_2+n;1+\mu_1+m+n,1+\mu_2+m+n;\frac{\omega_1x_1x_2}{(1-t_1)},\frac{\omega_2x_1x_2}{(1-t_2)}\right]$$

$$\times M_8\left[1+\gamma_1+m+n,1+\gamma_2+m+n,\lambda_1+m,\lambda_2+n;1+\eta_1+m+n,1+\eta_2+m+n;\frac{u_1y_1y_2}{(1-t_1)},\frac{u_2y_1y_2}{(1-t_2)}\right]$$

$$\times\left(\frac{\omega_1u_1x_1x_2y_1y_2t_1}{(1-t_1)^2}\right)^m\left(\frac{\omega_2u_2x_1x_2y_1y_2t_2}{(1-t_2)^2}\right)^n \tag{2.8}$$

In relation (2.2), replace x, y, t, λ by $\frac{\omega_1}{x}$, $\frac{u_1}{y}$, t_1, λ_1 respectively. Again, replace x, y, t, λ by by $\frac{\omega_2}{x}$, $\frac{u_2}{y}$, t_2, λ_2 respectively, then multiply these two equation each other and also multiply both side by $x^{-\alpha_1}y^{-\gamma_1}$. Now on operating the fractional derivative operator $D_x^{\mu-\alpha}D_y^{\eta-\gamma}$ on both sides and by using the relation (1.23), we obtain the bilinear generating function .

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_1\left[\mu,\lambda_1+m,\lambda_2+n;\alpha;\frac{\omega_1}{x},\frac{\omega_2}{x}\right]F_1\left[\eta,\lambda_1+m,\lambda_2+n;\gamma;\frac{u_1}{y},\frac{u_2}{y}\right](t_1)^m(t_2)^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(\mu)_{m+n}(\eta)_{m+n}}{(\alpha)_{m+n}(\gamma)_{m+n}m!\,n!}$$

$$\times F_1\left[\mu+m+n,\lambda_1+m,\lambda_2+m;\alpha+m+n;\frac{\omega_1}{x(1-t_1)},\frac{\omega_2}{x(1-t_2)}\right]$$
$$\times F_1\left[\eta+m+n,\lambda_1+m,\lambda_2+m;\gamma+m+n;\frac{u_1}{y(1-t_1)},\frac{u_2}{y(1-t_2)}\right]\left(\frac{\omega_1u_1t_1}{xy(1-t_1)^2}\right)^m\left(\frac{\omega_2u_2t_2}{xy(1-t_2)^2}\right)^n \qquad (2.9)$$

In relation (2.2), replace x, y, t, λ by ω_1x, u_1y, t_1, λ_1 respectively. Again, replace x, y, t, λ by by ω_2x, u_2y, t_2, λ_2 respectively. Further, again replace x, y, t, λ by $\frac{\omega_3}{x}$, $\frac{u_3}{y}$, t_3, λ_3, then multiply these three equation each other and also multiply both side by $x^{-\alpha}y^{-\gamma}$. Now on operating the fractional derivative operator $D_x^{\alpha-\mu}D_y^{\gamma-\eta}$ on both sides and by using the relation (1.24), one obtain the bilinear generating function .

$$\sum_{l,m,n=0}^{\infty}\frac{(\lambda_1)_l(\lambda_2)_m(\lambda_2)_n}{l!\,m!\,n!}G_B\left[1+\alpha,\lambda_3+n,\lambda_1+l,\lambda_2+m;1+\mu;\frac{\omega_3}{x},\omega_1x,\omega_2x\right]$$

$$\times G_B\left[1+\gamma,\lambda_3+n,\lambda_1+l,\lambda_2+m;1+\eta;\frac{u_3}{y},u_1y,u_2y\right](t_1)^l(t_2)^m(t_3)^n$$
$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3}\sum_{l,m,n=0}^{\infty}\frac{(\lambda_1)_l(\lambda_2)_m(\lambda_3)_n(1+\alpha)_{l+m+n}(1+\gamma)_{l+m+n}}{(1+\mu)_{l+m+n}(1+\eta)_{l+m+n}\ l!\,m!\,n!}$$

$$\times G_B\left[[1+\alpha+l+m+n,\lambda_3+n,\lambda_1+l,\lambda_2+m;1+\mu+l+m+n;\ \frac{\omega_3}{x(1-t_3)},\frac{\omega_1x}{(1-t_1)},\frac{\omega_2x}{(1-t_2)}\right]$$

$$\times G_B\left[[1+\gamma+l+m+n,\lambda_3+n,\lambda_1+l,\lambda_2+m;1+\eta+l+m+n;\ \frac{u_3}{y(1-t_3)},\frac{u_1y}{(1-t_1)},\frac{u_2y}{(1-t_2)}\right]$$

$$\times\left(\frac{xy\omega_1u_1t_1}{(1-t_1)^2}\right)^l\left(\frac{xy\omega_2u_2t_2}{(1-t_2)^2}\right)^m\left(\frac{\omega_3u_3t_3}{xy(1-t_3)^2}\right)^n \qquad (2.10)$$

where Horn's type hypergeometric functions of three variables is defined by Pandey [6] as

$$G_B[\alpha,\beta_1,\beta_2,\beta_3;\gamma;x,y,z]=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m}m!\,n!\,p!}x^my^nz^p \qquad (2.11)$$

In relation (2.2), replace x, y, t, λ by ω_1x, u_1y, t_1, λ_1 respectively. Again, replace x, y, t, λ by by ω_2x, u_2y, t_2, λ_2 respectively. Further, again replace x, y, t, λ by ω_3x, u_3y, t_3, λ_3, then multiply these three equation each other and also multiply both side by $x^{\alpha}y^{\gamma}$. Now on operating the fractional derivative operator $D_x^{\alpha-\mu}D_y^{\gamma-\eta}$ on both sides and by using the relation (1.25), one obtain the bilinear generating function .

$$\sum_{l,m,n=0}^{\infty}\frac{(\lambda_1)_l(\lambda_2)_m(\lambda_2)_n}{l!\,m!\,n!}F_D^{(3)}[1+\alpha,\lambda_1+l,\lambda_2+m,\lambda_3+n;1+\mu;\omega_1x,\omega_2x,\omega_3x]$$
$$\times F_D^{(3)}[1+\gamma,\lambda_1+l,\lambda_2+m,\lambda_3+n;1+\eta;u_1y,u_2y,u_3y](t_1)^l(t_2)^m(t_3)^n$$
$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3}\sum_{l,m,n=0}^{\infty}\frac{(\lambda_1)_l(\lambda_2)_m(\lambda_3)_n(1+\alpha)_{l+m+n}(1+\gamma)_{l+m+n}}{(1+\mu)_{l+m+n}(1+\eta)_{l+m+n}\ l!\,m!\,n!}$$
$$\times F_D^{(3)}\left[1+\alpha+l+m+n,\lambda_1+l,\lambda_2+m,\lambda_3+n;\ 1+\mu+l+m+n;\ \frac{\omega_1x}{(1-t_1)},\frac{\omega_2x}{(1-t_2)},\frac{\omega_3x}{(1-t_3)}\right]$$
$$\times F_D^{(3)}\left[1+\gamma+l+m+n,\lambda_1+l,\lambda_2+m,\lambda_3+n;\ 1+\eta+l+m+n;\ \frac{u_1y}{(1-t_1)},\frac{u_2y}{(1-t_2)},\frac{u_3y}{(1-t_3)}\right]$$
$$\times\left(\frac{xy\omega_1u_1t_1}{(1-t_1)^2}\right)^l\left(\frac{xy\omega_2u_2t_2}{(1-t_2)^2}\right)^m\left(\frac{xy\omega_3u_3t_3}{(1-t_3)^2}\right)^n \qquad (2.12)$$

II. BILATERAL GENERATING FUNCTIONS

In this section we establish various types of bilateral generating functions by using the linear generating functions discussed in the section (1), as mentioned below:

By replacing t in (1.26) by $t(1-y)$, then multiplying both sides of by y^{γ} and then operating the fractional derivative operator $D_y^{\gamma-\delta}$, one obtain

$$D_y^{\gamma-\delta}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[\lambda+n,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right]t^n y^{\gamma}(1-y)^n\right\}$$
$$=D_y^{\gamma-\delta}\left\{F_2\left[\lambda,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1[1-t(1-y)]},\frac{\omega_2}{x_2[1-t(1-y)]}\right]y^{\gamma}[1-t(1-y)]^{-\lambda}\right\} \tag{3.1}$$

Now, with some usual calculation (3.1), yields

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[\lambda+n,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right]\ {}_2F_1\left[\begin{matrix}-n,1+\gamma;\\ 1+\delta\ ;\end{matrix}y\right]t^n$$
$$=(1-t)^{-\lambda}F_A^{(3)}\left[\lambda,\mu,\mu',1+\gamma;\alpha,\alpha',1+\delta;\frac{\omega_1}{x_1(1-t)},\frac{\omega_2}{x_2(1-t)},\frac{yt}{(t-1)}\right] \tag{3.2}$$

Further, we again use the same analysis, which is used to obtain (3.2), for the generating functions (1.27), (1.28), (1.29), (1.30) to obtain four another bilateral generating functions as follows:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_3\left[\lambda+n,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_3x_4}\right]\ {}_2F_1\left[\begin{matrix}-n,1+\gamma;\\ 1+\delta\ ;\end{matrix}y\right]t^n$$
$$=(1-t)^{-\lambda}F^{1:2;2;1}_{-:2;2;1}\left[\begin{matrix}\lambda:\mu_1,\mu_2;\mu_3,\mu_4;1+\gamma;\\ -:\alpha_1,\alpha_2;\alpha_3,\alpha_4;1+\delta;\end{matrix}\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_3x_4(1-t)},\frac{yt}{(t-1)}\right] \tag{3.3}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_7\left[\lambda+n,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right]\ {}_2F_1\left[\begin{matrix}-n,1+\gamma;\\ 1+\delta\ ;\end{matrix}y\right]t^n$$
$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}\lambda::\mu_1;-;-:\mu_2\ ;\mu_3\ ;1+\gamma;\\ -::\alpha_1;-;-:\alpha_2;\alpha_3;1+\delta;\end{matrix}\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_1x_3(1-t)},\frac{yt}{(t-1)}\right] \tag{3.4}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}H_A[\beta,\lambda+n,1+\alpha;\beta,1+\mu;\omega_2,\omega_1x,x]{}_2F_1\left[\begin{matrix}-n,1+\gamma;\\ 1+\delta\ ;\end{matrix}y\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\beta)_p(1+\alpha)_p(x)^p}{(1+\mu)_p p!}$$
$$\times F_B^{(3)}\left[\lambda,\beta+p,1+\alpha+p,1+\gamma;\beta,1+\mu+p,1+\delta;\frac{\omega_2}{(1-t)},\frac{\omega_1x}{(1-t)},\frac{yt}{(t-1)}\right] \tag{3.5}$$

where hypergeometric function H_A is defined by Srivastava [10] as

$$H_A[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+n+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_m(\gamma')_{n+p}m!\,n!\,p!}x^m y^n z^p \tag{3.6}$$
$$|x|<r,|y|<s,|z|<t,r+s+t=1+st.$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1\left[1+\alpha,\lambda+n,1+\mu-\beta;1+\mu;\frac{\omega x}{1-x},\frac{-x}{1-x}\right]\ {}_2F_1\left[\begin{matrix}-n,1+\gamma;\\ 1+\delta\ ;\end{matrix}y\right]t^n$$
$$=(1-t)^{-\lambda}F_M\left[1+\gamma,1+\alpha,1+\alpha,\lambda,1+\mu-\beta,\lambda;1+\delta,1+\mu,1+\mu;\frac{yt}{(t-1)},\frac{\omega x}{(1-x)(1-t)},\frac{x}{x-1}\right] \tag{3.7}$$

Now, by replacing t_1 and t_2 in (1.31) by $t_1(1-\sigma_1 y)$ and $t_2(1-\sigma_2 y)$ respectively, such that $|\sigma_i|<1,\ i=1,2.$ Further, multiply both sides of it by y^{γ} and then on operating the fractional derivative operator $D_y^{\gamma-\delta}$, one obtain

$$D_y^{\gamma-\delta}\left\{\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_D^{(3)}\left[1+\alpha,\lambda_1+m,\lambda_2+n,1+\mu-\beta;1+\mu;\ \frac{\omega_1x}{1-x},\frac{\omega_2x}{1-x},\frac{x}{x-1}\right]\right.$$
$$\left.\times(\mathrm{t}_1)^m(t_2)^n y^{\gamma}(1-\sigma_1y)^m(1-\sigma_2y)^n\right\}$$
$$=D_y^{\gamma-\delta}\left\{F_D^{(3)}\left[1+\alpha,\lambda_1,\lambda_2,1+\mu-\beta;1+\mu;\frac{\omega_1x}{(1-x)[1-t_1(1-\sigma_1y)]},\frac{\omega_2x}{(1-x)[1-t_2(1-\sigma_2y)]},\frac{x}{x-1}\right]\right.$$
$$\left.\times y^{\gamma}[1-t_1(1-\sigma_1y)]^{-\lambda_1}[1-t_2(1-\sigma_2y)]^{-\lambda_2}\right\} \tag{3.8}$$

Now, with some usual calculation (3.8), yield

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_D^{(3)}\left[1+\alpha,\lambda_1+m,\lambda_2+n,1+\mu-\beta;\ 1+\mu;\ \frac{\omega_1x}{1-x},\frac{\omega_2x}{1-x},\frac{x}{x-1}\right]$$
$$\times F_1[1+\gamma,-m,-n;1+\delta;\sigma_1y,\sigma_2y](t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{r,s=0}^{\infty}\frac{(1+\alpha)_{r+s}(\lambda_1)_r(\lambda_2)_s}{(1+\mu)_{r+s}r!\,s!}\left(\frac{\omega_1 x}{(1-x)(1-t_1)}\right)^r\left(\frac{\omega_2 x}{(1-x)(1-t_2)}\right)^s$$
$$\times\ {}_2F_1\left[\begin{matrix}1+\alpha+r+s,1+\mu-\beta; \\ 1+\mu+r+s;\end{matrix}\frac{x}{x-1}\right]F_1\left[1+\gamma,\lambda_1+r,\lambda_2+s;1+\delta;\frac{\sigma_1 y t_1}{t_1-1},\frac{\sigma_2 y t_1}{t_2-1}\right] \tag{3.9}$$

The analysis used to obtain (3.9), is used again for the generating functions (1.32), (1.33), (1.34), (1.30) to obtain several other bilateral generating functions as follows:

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_M[\lambda_2+n,1+\alpha,1+\alpha,\beta,\lambda_1+m,\beta;\beta,1+\mu,1+\mu;\omega_2,\omega_1 x,x]$$
$$\times F_1[1+\gamma,-m,-n;1+\delta;\sigma_1 y,\sigma_2 y](t_1)^m(t_2)^n$$
$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{r,s=0}^{\infty}\frac{(1+\alpha)_r(\lambda_1)_r(\lambda_2)_s(1+\gamma)_s}{(1+\mu)_r(1+\delta)_s r!\,s!}\left(\frac{\omega_1 x}{1-t_1}\right)^r\left(\frac{\sigma_1 y t_1}{t_1-1}\right)^s$$
$$\times F_K\left[1+\alpha+r,\lambda_1,\lambda_2,\beta,1+\gamma+s,\beta;1+\mu+r,\beta,1+\delta+s;x,\frac{\omega_2}{(1-t_2)},\frac{\sigma_2 y t_2}{(t_2-1)}\right] \tag{3.10}$$

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F^{(3)}\left[\begin{matrix}1+\alpha::--;\beta;--:\lambda_1+m;\lambda_2+n;--; \\ 1+\mu::--\ ;\ --\ ;\ --:--\ ;\ \beta\ ;\ --\ ;\end{matrix}\omega_1 x,\omega_2 x,x\right]$$
$$\times F_1[1+\gamma,-m,-n;1+\delta;\sigma_1 y,\sigma_2 y](t_1)^m(t_2)^n$$
$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{r,s=0}^{\infty}\frac{(1+\alpha)_{r+s}(\lambda_1)_r(\lambda_2)_s(\beta)_s}{(1+\mu)_{r+s}r!\,s!}\left(\frac{\omega_1 x}{1-t_1}\right)^r\left(\frac{\omega_2 x}{1-t_2}\right)^s$$

$$\times_2 F_1\left[\begin{matrix}1+\alpha+r+s,\beta+s; \\ 1+\mu+r+s;\end{matrix}x\right]F_1\left[1+\gamma,\lambda_1+r,\lambda_2+s;1+\delta;\frac{\sigma_1 y t_1}{t_1-1},\frac{\sigma_2 y t_2}{t_2-1}\right] \tag{3.11}$$

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}F_1\left[\mu,\lambda_1+m,\lambda_2+n;\alpha;\frac{\omega_1}{x},\frac{\omega_2}{x}\right]F_1[1+\gamma,-m,-n;1+\delta;\sigma_1 y,\sigma_2 y](t_1)^m(t_2)^n$$
$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{r=0}^{\infty}\frac{(\mu)_r(\lambda)_r}{(\alpha)_r r!}\left(\frac{\omega_1}{x(1-t_1)}\right)^r$$
$$\times F_M\left[\mu+r,1+\gamma,1+\gamma,\lambda_2,\lambda_1+r,\lambda_2;\alpha+r,1+\delta,1+\delta;\frac{\omega_2}{x(1-t_2)},\frac{\sigma_1 y t_1}{t_1-1},\frac{\sigma_2 y t_2}{t_2-1}\right] \tag{3.12}$$

Next, on replacing t_1, t_2 and t_3 in (1.35) by $t_1(1-\sigma_1 y)$, $t_2(1-\sigma_2 y)$ and $t_3(1-\sigma_3 y)$ respectively, such that $|\sigma_i| < 1, i = 1,2,3$. Further, multiplying both sides of it by y^γ and then on operating the fractional derivative operator $D_y^{\gamma-\delta}$, one obtains

$$D_y^{\gamma-\delta}\left\{\sum_{m,n,p=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!}F_D^{(3)}[1+\alpha,\lambda_1+m,\lambda_2+n,\lambda_3+p;1+\mu;\omega_1 x,\omega_2 x,\omega_3 x]\right.$$
$$\left.\times (t_1)^m(t_2)^n(t_3)^p y^\gamma(1-\sigma_1 y)^m(1-\sigma_2 y)^n(1-\sigma_3 y)^p\right\}$$
$$= D_y^{\gamma-\delta}\left\{F_D^{(3)}\left[1+\alpha,\lambda_1,\lambda_2,\lambda_3;1+\mu;\frac{\omega_1 x}{[1-t_1(1-\sigma_1 y)]},\frac{\omega_2 x}{[1-t_2(1-\sigma_2 y)]},\frac{\omega_3 x}{[1-t_3(1-\sigma_3 y)]}\right]\right.$$
$$\left.\times y^\gamma[1-t_1(1-\sigma_1 y)]^{-\lambda_1}[1-t_2(1-\sigma_2 y)]^{-\lambda_2}[1-t_3(1-\sigma_3 y)]^{-\lambda_3}\right\} \tag{3.13}$$

Now, with some usual calculation (3.13), yields

$$\sum_{m,n,p=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!}F_D^{(3)}[1+\alpha,\lambda_1+m,\lambda_2+n,\lambda_3+p;1+\mu;\omega_1 x,\omega_2 x,\omega_3 x]$$

$$\times F_D^{(3)}[1+\gamma,-m,-n,-p;1+\delta;\sigma_1 y,\sigma_2 y,\sigma_3 y](t_1)^m(t_2)^n(t_3)^p$$
$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3}\sum_{r,s,k=0}^{\infty}\frac{(1+\alpha)_{r+s+k}(\lambda_1)_r(\lambda_2)_s(\lambda_3)_k}{(1+\mu)_{r+s+k}r!\,s!\,k!}$$

$$\times F_D^{(3)}\left[1+\gamma,\lambda_1+r,\lambda_2+s,\lambda_3+k;1+\delta;\frac{\sigma_1 y t_1}{(t_1-1)},\frac{\sigma_2 y t_2}{(t_2-1)},\frac{\sigma_3 t_3}{(t_3-1)}\right]\left(\frac{\omega_1 x}{1-t_1}\right)^r\left(\frac{\omega_2 x}{1-t_2}\right)^s\left(\frac{\omega_3 x}{1-t_3}\right)^k \tag{3.14}$$

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