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Bifurcation Analysis and Chaotic Behaviour in Discrete – Time Predator-Prey System

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ABSTRACT: *Two dimensional discrete prey-predator interactions in the closed first quadrant* R_+^2 *are investigated in this paper. The dynamics of the system such as the existence of non negative equilibria, local stability of the equilibrium states are analyzed. It is seen that the system goes through Flip and Neimark – Sacker bifurcation about axial and positive equilibrium states with prey growth rate as the bifurcation factor. It is also observed that the model is sensitive to the initial conditions and the parameter values.With the natural growth rate parameter, bifurcation diagrams are obtained. Numerical results demonstrate not only the essential outcomes of analytical observations but also give an idea about the complexity of the discrete model.*

KEYWORDS: *Prey – Predator Model, Stability, Discrete Time, Equilibria, Limit Cycle, Flip Bifurcation, Neimark – Sacker Bifurcation, Chaos.*
AMS Subject Classification: 34A08, 34K18, 39A12, 92D25.

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I. INTRODUCTION

Predator-prey interactions are one of the most important ways that species interact in ecological communities. Whenever an organism consumes another living organism, this interaction is termed as "predation". To study these interactions, mathematical models have been developed and utilized to forecast and analyse the growth and decline of populations of various species at distinctive times [1, 3, 5, 8]. The famous Lotka-Volterra model consisting of two differential equations, describing the prey-predator interactions laid the foundation to study the dynamical behavior of prey-predator population [9]. Over the years, the researchers evolved discrete-time models for non-overlapping generations as discrete-time models are more suitable compared to continuous time models. It is observed that discrete-time models present resourceful computer simulations for numerical computations which exhibit rich and varied dynamical behavior than continuous - time models [2, 6].

II. DISCRETE TIME PREY - PREDATOR MODEL

In ecology, many species have non- overlappping consecutive generations, thus their population advance into discrete-time steps [7, 10, 11] and such populations are aptly described by difference equations. We consider the discrete prey-predator interactions by the following non linear system of difference equations

$$\begin{aligned} x_{t+1} &= \mu x_t(1-x_t) - x_t y_t \\ y_{t+1} &= y_t(1-\alpha) + \beta x_t y_t \end{aligned} \tag{1}$$

In (1), x_t and y_t represent the number of prey and predator population respectively in the nth generation. The parameter μ is the intrinsic growth rate of the prey population with carrying ability one in the absence of predator. The death rate of predator is denoted by α and β denotes the growth rate of predator in the presence of the prey. The parameters α, β and μ have positive values and for mathematical and biological feasibility, we consider the dynamics of system (1) in the first octant of R_+^2.

III. EQUILIBRIA AND THEIR FEASIBILITY

We study the existence of equilibrium points for (1), the following are the equilibrium states of system (1) in the x - y plane $E_0 = (0,0)$, $E_1 = \left(1-\frac{1}{\mu}, 0\right)$ and $E_2 = \left(\frac{\alpha}{\beta}, (\mu-1) - \frac{\alpha\mu}{\beta}\right)$. The equilibrium states exist only when all its components are positive. The first equilibrium state E_0, always exists, E_1 if $\mu > 1$, and similarly $\mu > \frac{\beta}{\beta-\alpha}$ for E_2.

Stability Analysis: We now discuss the local behavior of (1) around each equilibrium state. The local stability is analyzed with the Variational matrix related to each equilibrium state. The Variation matrix of system (1) is

$$V(x,y) = \begin{bmatrix} \mu - 2\mu x - y & -x \\ \beta y & (1-\alpha)+\beta x \end{bmatrix} \tag{2}$$

For the Variation matrix (2) the characteristic equation is given by $\omega^2 - [TrV]\omega + DetV = 0$ (3)

where $TrV = 1 + \mu - \alpha + (\beta - 2\mu)x - y$ and $DetV = (\mu - 2\mu x - y)(1-\alpha+\beta x) + \beta xy$. Hence the system (1) is a dissipative dynamical system if $\left|(\mu - 2\mu x - y)(1-\alpha+\beta x) + \beta xy\right| < 1$. In order to discuss the local stability conditions, we consider the relations between eigenvalues and coefficients of the quadratic equation at the positive equilibria, for this we propose the subsequent lemma.

Lemma 3.1. Let $P(\omega) = \omega^2 + B\omega + C$ and $P(1) > 0$ with ω_1 and ω_2 as two roots of $P(\omega) = 0$. Then

- $|\omega_1| < 1$ and $|\omega_2| < 1$ if and only if $P(-1) > 0$ and $P(0) < 1$;
- $|\omega_1| < 1$ and $|\omega_2| > 1$ if and only if $P(-1) < 0$;
- $|\omega_1| > 1$ and $|\omega_2| > 1$ if and only if $P(-1) > 0$ and $P(0) > 1$;
- $\omega_1 = -1$ and $\omega_2 \neq 1$ if and only if $P(-1) = 0$ and $P(0) \neq \pm 1$;
- $|\omega_1| = 1$ and $|\omega_2| = 1$ if and only if $B^2 - 4C < 0$ and $P(0) = 1$ then ω_1 and ω_2 are complex.

Let ω_1 and ω_2 be the two roots of (3) then positive equilibrium state (x,y) is a sink if $|\omega_1| < 1$ and $|\omega_2| < 1$. A sink is always locally asymptotically stable.The state (x,y) is a source if $|\omega_1| > 1$ and $|\omega_2| > 1$. A source is unstable. The state (x,y) is a saddle if $|\omega_1| > 1$ and $|\omega_2| < 1$ and the state (x,y) is non-hyperbolic if either $|\omega_1| = 1$ or $|\omega_2| = 1$.

Theorem 3.1. If $\mu < 1$ and $0 < \alpha < 2$ then E_0 is a sink. If $\mu > 1$ and $\alpha > 2$ then E_0 is a source. If $\mu < 1$ and $\alpha > 2$ then E_0 is a saddle and if either $\mu = 1$ or $\alpha = 2$ then E_0 is non-hyperbolic.

Proof: The Variation matrix V at E_0 is

$$V(E_0) = \begin{bmatrix} \mu & 0 \\ 0 & 1-\alpha \end{bmatrix}.$$

Hence, the eigenvalues are $\omega_1 = \mu$ and $\omega_2 = 1-\alpha$. The equilibrium state E_0 is sink if $\mu < 1$ and $0 < \alpha < 2$, source if $\mu > 1$ and $\alpha > 2$, saddle if $\mu < 1$ and $\alpha > 2$ and non-hyperbolic if either $\mu = 1$ or $\alpha = 2$.

Theorem 3.2. When $\mu > 1$ there are a minimum of four dissimilar topological forms of E_1, for all permitted values of parameters

- if $1 < \mu < 3$ and $\beta\left(1-\frac{1}{\mu}\right) < \alpha < 2 + \beta\left(1-\frac{1}{\mu}\right)$ then E_1 is a sink
- if $\mu > 3$ and $\alpha > 2 + \beta\left(1-\frac{1}{\mu}\right)$ then E_1 is a source
- if $1 < \mu < 3$ and $\alpha > 2 + \beta\left(1-\frac{1}{\mu}\right)$ then E_1 is a saddle
- if $\mu = 3$ or $\alpha = 2 + \beta\left(1-\frac{1}{\mu}\right)$ then E_1 is a non-hyperbolic

Proof: The Varation matrix V at E_1 is $V(E_1)=\begin{bmatrix} 2-\mu & \frac{1}{\mu}-1 \\ 0 & (1-\alpha)+\beta\left(1-\frac{1}{\mu}\right) \end{bmatrix}$.

Hence, the eigenvalues are $\omega_1 = 2-\mu$ and $\omega_2=(1-\alpha)+\beta\left(1-\frac{1}{\mu}\right)$. The equilibrium state E_1 is sink if $1<\mu<3$ and $\beta\left(1-\frac{1}{\mu}\right)<\alpha<2+\beta\left(1-\frac{1}{\mu}\right)$, source if $\mu>3$ and $\alpha>2+\beta\left(1-\frac{1}{\mu}\right)$, saddle if $1<\mu<3$ and $\alpha>2+\beta\left(1-\frac{1}{\mu}\right)$ and non-hyperbolic if either $\mu=3$ or $\alpha=2+\beta\left(1-\frac{1}{\mu}\right)$.

Theorem 3.3. If $\mu>\frac{\beta}{\beta-\alpha}$, then the positive equilibrium state E_2

(i). is a sink if one of the below criterion holds:

(i.a) $\Delta\geq 0$ and $\frac{\beta(\alpha-4)}{\alpha(\beta-\alpha-2)}<\mu<\frac{\beta}{(\beta-\alpha-1)}$; (i.b) $\Delta<0$ and $\mu<\frac{\beta}{(\beta-\alpha-1)}$;

(ii). it is a source if one of the below criterion holds:

(ii.a) $\Delta\geq 0$ and $\mu>max\left\{\frac{\beta(\alpha-4)}{\alpha(\beta-\alpha-2)},\frac{\beta}{(\beta-\alpha-1)}\right\}$; (ii.b) $\Delta<0$ and $\mu>\frac{\beta}{(\beta-\alpha-1)}$;

(iii). it is non-hyperbolic if one of the below criterion holds:

(iii.a) $\Delta\geq 0$ and $\mu=\frac{\beta(\alpha-4)}{\alpha(\beta-\alpha-2)}$; (iii.b) $\Delta<0$ and $\mu=\frac{\beta}{(\beta-\alpha-1)}$;

(iv). for all the other parameter values of excluding those values in (i) - (iii).

Proof: If E_2 exists, the Variation matrix V at E_2 is $V(E_2)=\begin{bmatrix} 1-\frac{\alpha\mu}{\beta} & -\frac{\alpha}{\beta} \\ \beta(\mu-1)-\alpha\mu & 1 \end{bmatrix}$.

Therefore, the eigenvalues of $V(E_2)$ are $\omega_{1,2}=\left(1-\frac{\alpha\mu}{2\beta}\right)\pm\sqrt{\Delta}$, where $\Delta=\alpha(1-\mu)+\frac{\alpha^2\mu}{\beta}\left(1+\frac{\mu}{4\beta}\right)$.

It is straightforward to see that $\lambda_{1,2}$ satisfy the characteristic equation

$$P(\omega)=\omega^2-S_1\omega+S_2=0 \tag{4}$$

where $S_1=trace(V_{E_2})=2-\frac{\alpha\mu}{\beta}$ and $S_2=\det(V_{E_2})=1+\alpha(\mu-1)-\frac{\alpha\mu}{\beta}(1+\alpha)$. By means of Jury's criterion [4], the necessary and sufficient condition for local stability of the positive equilibrium state E_2 is satisfied as the above condition holds.

IV. NUMERICAL SIMULATIONS

This section presents the time plots and phase space to illustrate the results of the previous section and exhibit some interesting complex dynamics of system (1).

Example 4.1. This example considers the parameter values $\mu=0.89$, $\alpha=0.49$, $\beta=0.89$ with initial values $x_0=0.2$ and $y_0=0.3$. At equilibrium state E_0, the eigenvalues are $\omega_1=0.51$ and $\omega_2=0.89$ so that $|\omega_{1,2}|<1$. Hence E_0 is stable. The phase portraits and the time series confirm the end result, in Figure - 1. In this case prey and predator populations both will disappear over the time.

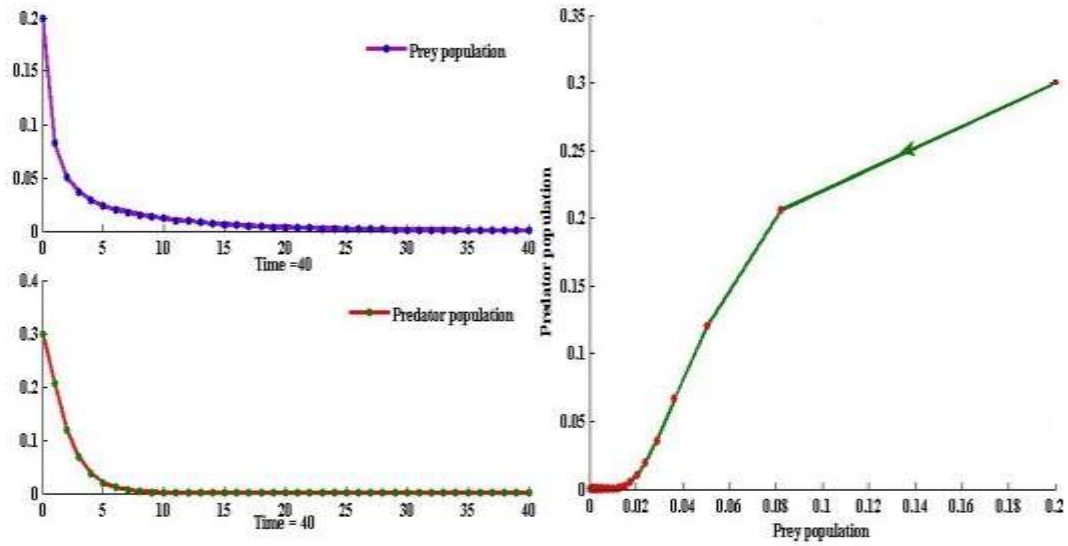

Figure 1: Time series and Phase portrait are stable at E_0.

Example 4.2. Considering the parameter values $\mu = 1.19$, $\alpha = 0.49$, $\beta = 0.89$ with initial values $x_0 = 0.2$ and $y_0 = 0.3$, we obtain $E_1 = (0.1597, 0)$ and the eigenvalues are $\omega_1 = 0.81$ and $\omega_2 = 0.6521$ so that $|\omega_1| < 1$ and $|\omega_2| < 1$. Hence the system (1) is stable for the axial equilibrium state. In Figure – 2, the time series and the phase portraits are presented. Here the prey population continues to exist whereas predator population goes to extinction.

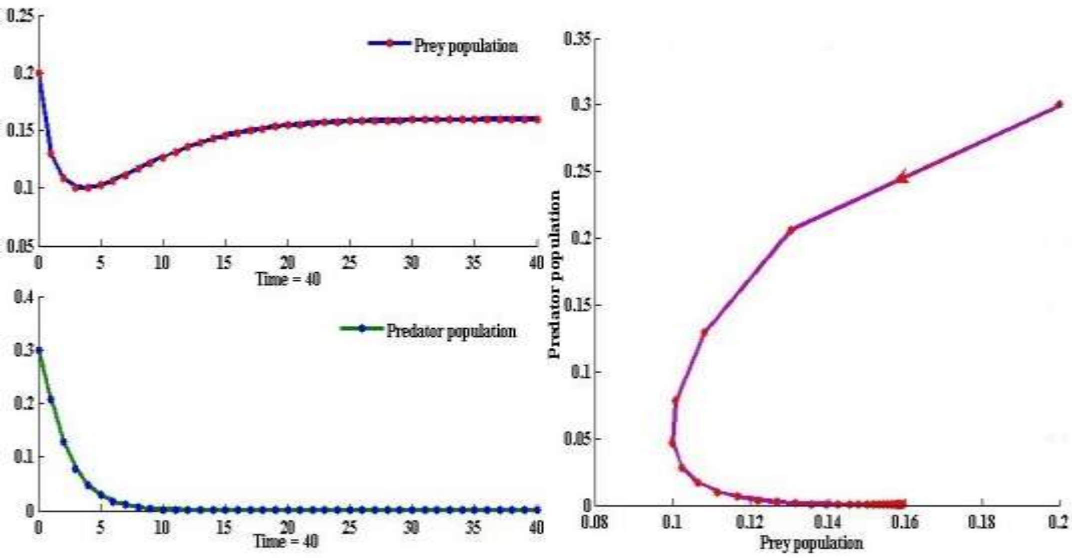

Figure 2: Time series and Phase portrait are stable at E_1.

Example 4.3. In this example we take the following set of parameter values $\mu = 2.41$, $\alpha = 0.981$, $\beta = 3.391$ with initial values $x_0 = 0.4$ and $y_0 = 0.3$. We obtain $E_2 = (0.2893, 0.7128)$ and the eigenvalues are $\omega_{1,2} = 0.6514 \pm i\,0.7601$ so that $|\omega_{1,2}| = 1.0010 > 1$. The trajectory curves inwards but fails to move towards a single point and at the end settles down as a limit cycle. Hence system (1) is unstable, see Figure - 3.

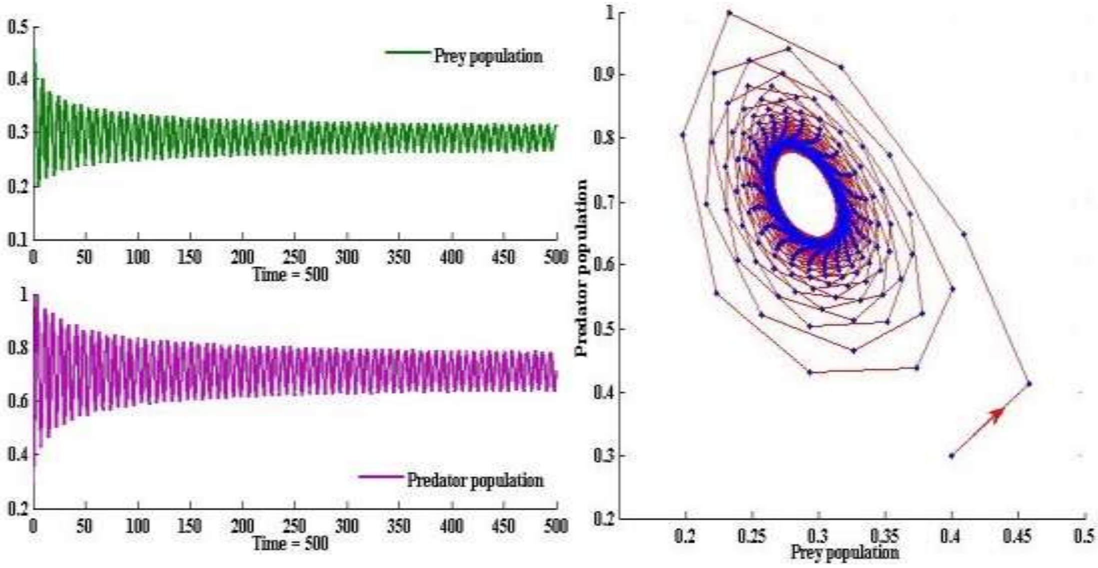

Figure 3: Time series and Phase portrait are unstable at E_2.

Whereas with $\mu = 2.51$, $\alpha = 1$, $\beta = 3.25$ with $x_0 = 0.2$ and $y_0 = 0.3$, then the positive equilibrium state $E_2 = (0.3077, 0.7377)$ and the eigenvalues are $\omega_{1,2} = 0.6138 \pm i\,0.7672$ so that $|\omega_{1,2}| = 0.9825 < 1$. Hence from Figure – 4, we conclude that system (1) is stable.

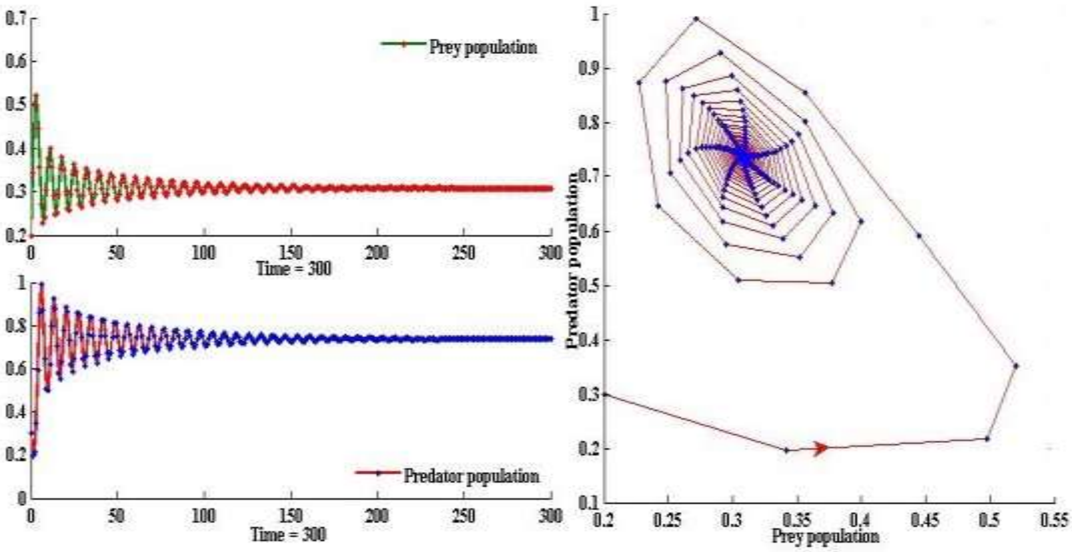

Figure 4: Time series and Phase portrait are stable at E_2.

V. BIFURCATION ANALYSIS

In this section, we discuss the bifurcation parametric conditions for the existence of flip and Neimark-Sacker bifurcation at the axial and positive equilibrium states of system (1). With intrinsic growth rate μ as the bifurcation parameter, it is observed that for axial equilibrium state E_1 the flip bifurcation will appear , if μ varies in the small neighborhood of FB_{E_1}, where $FB_{E_1} = \left\{ (\mu, \alpha, \beta) : \mu = 3, \alpha \neq 2 + \beta\left(1 - \frac{1}{\mu}\right), \mu > 1, \alpha, \beta > 0 \right\}$.

Example 5.1. First, we take $\alpha = 2.56$, $\beta = 3.82$ and $\mu \in [2.6, 4]$ with the initial values are $x_0 = 0.3$ and $y_0 = 0.5$, then for system (1), flip bifurcation emerges about the equilibrium state $E_1 = (0.6667, 0)$ at $\mu = 3$.The associated bifurcation diagram is shown in Figure -5. The characteristic polynomial evaluated at this point is given by

$\omega^2 + (0.0133)\omega - 0.9867 = 0$ (5). Furthermore, the roots of (5) are $\omega_1 = -1$ and $\omega_2 = 0.9867$ so that $\omega_1 = -1$ and $\omega_2 \neq \pm 1$. Thus we have $(\mu = 3, \alpha \neq 4.5467) \in FB_{E_1}$.

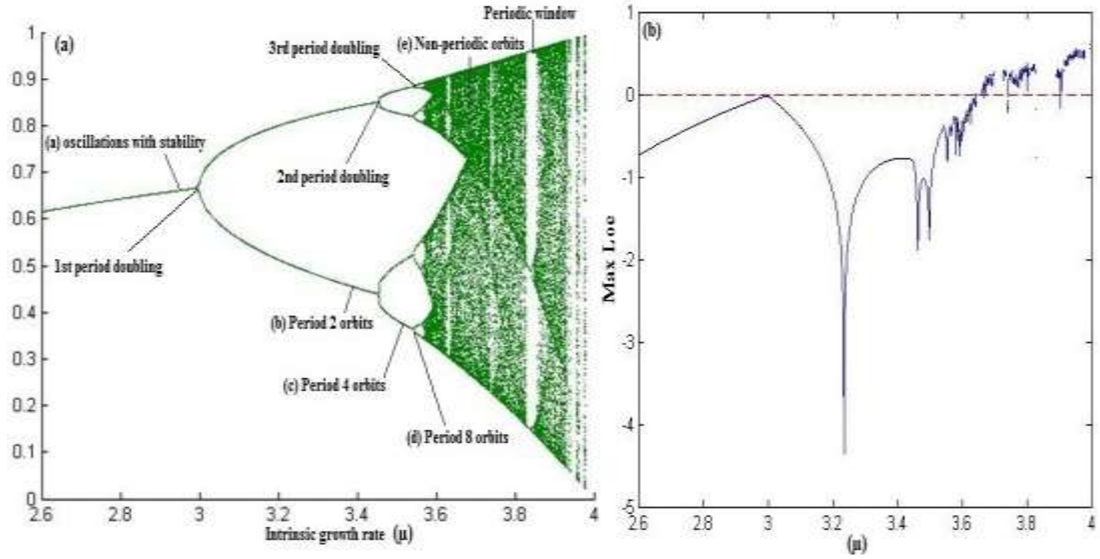

Figure 5: (a) Flip bifurcation diagram of system (1) in $(\mu - x)$ plane. (b) Maximum Lyapunov exponents associated to (a)

From Figure 5(a), it is seen that the axial equilibrium state E_1 of (1) is stable for $\mu < 3$ and is unstable at $\mu = 3$ due to flip bifurcation and for $\mu > 3$ there is a cascade of bifurcation. The maximum Lyapunov exponent related to Figure 5(a) are computed and plotted in Figure 5(b), which confirms the existence of the chaotic area and aperiodic orbits in the parametric space. Furthermore, keeping the same fixed parameter values and varying the intrinsic growth rate parameter values $\mu \in [1.8, 3]$, the time series of system (1) are presented in Figure 6. It is interesting to observe that in Figure 6(b) - 6(d) , the growth pattern becomes periodic after an initial aperiodicity. In Figure 6(b), the period is 2-value cycle at $\mu = 3.39$, 6(c) 4 value cycle at $\mu = 3.529$, 6(d) 8 value cycle at $\mu = 3.569$, and 6(e) non periodic oscillations at $\mu = 3.64$, that is usually referred to as chaos.

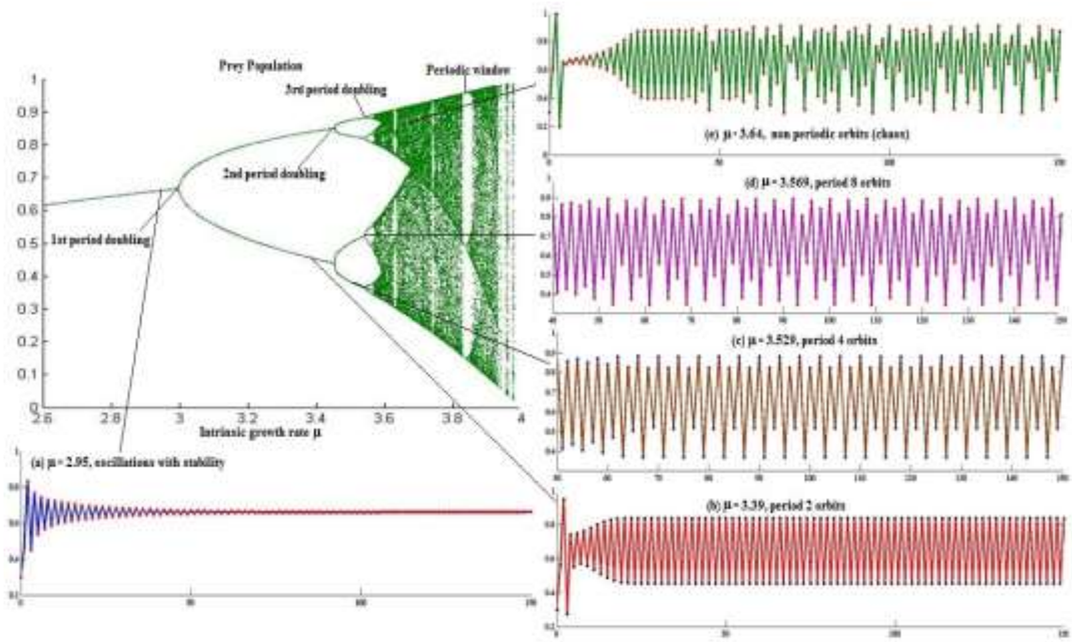

Figure 6: Time series for various parameter values of intrinsic growth rate μ corresponding to Figure 5(a).

From (4), it is easy to see that $P(\lambda) = 0$ has two complex conjugate roots with modulus one. The criterion in the terms (iii.b) of Theorem 3.1.3. can be re presented as follows $NS_{E_2} = \left\{ (\mu, \alpha, \beta) : \mu = \dfrac{\beta}{\beta - \alpha - 1}, \Delta < 0, \mu > 1, \alpha, \beta > 0 \right\}$.

then the Neimark - Sacker Bifurcation will appear in the system (1) if the intrinsic growth rate parameter μ varies in the small neighborhood of NS_{E_2} .

Example 5.2. Let $\alpha = 1.05$, $\beta = 3.25$ and $\mu \in [2.5, 4]$ with the initial values are $x_0 = 0.6$ and $y_0 = 0.5$, then the system (1) endures Neimark-Sacker bifurcation appears at the equilibrium state $E_2 = (0.3231, 0.8333)$ for $\mu = 2.7083$. The corresponding bifurcation diagram is shown in Figure - 7.

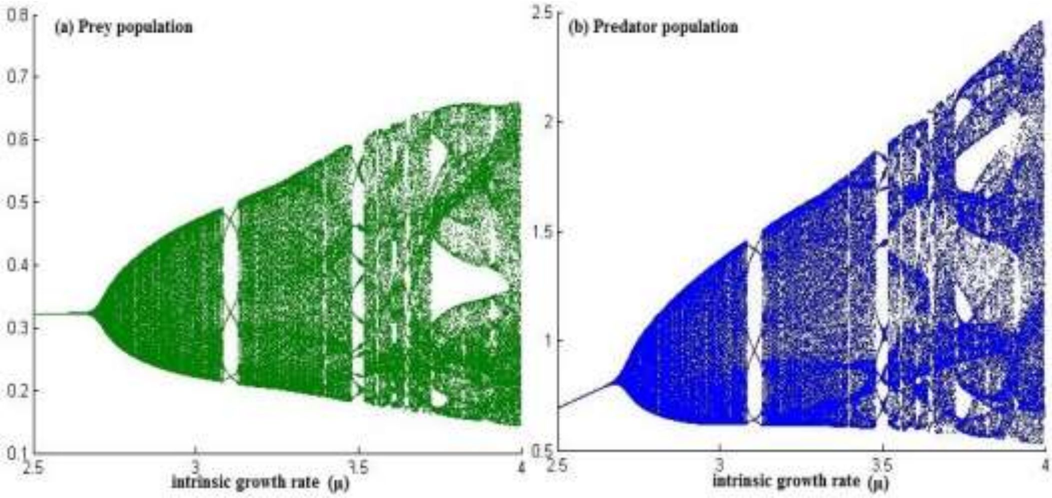

Figure 7: Neimark-Sacker Bifurcation diagram of the system (1) in (a) $(\mu - x)$ and (b) $(\mu - y)$ planes.

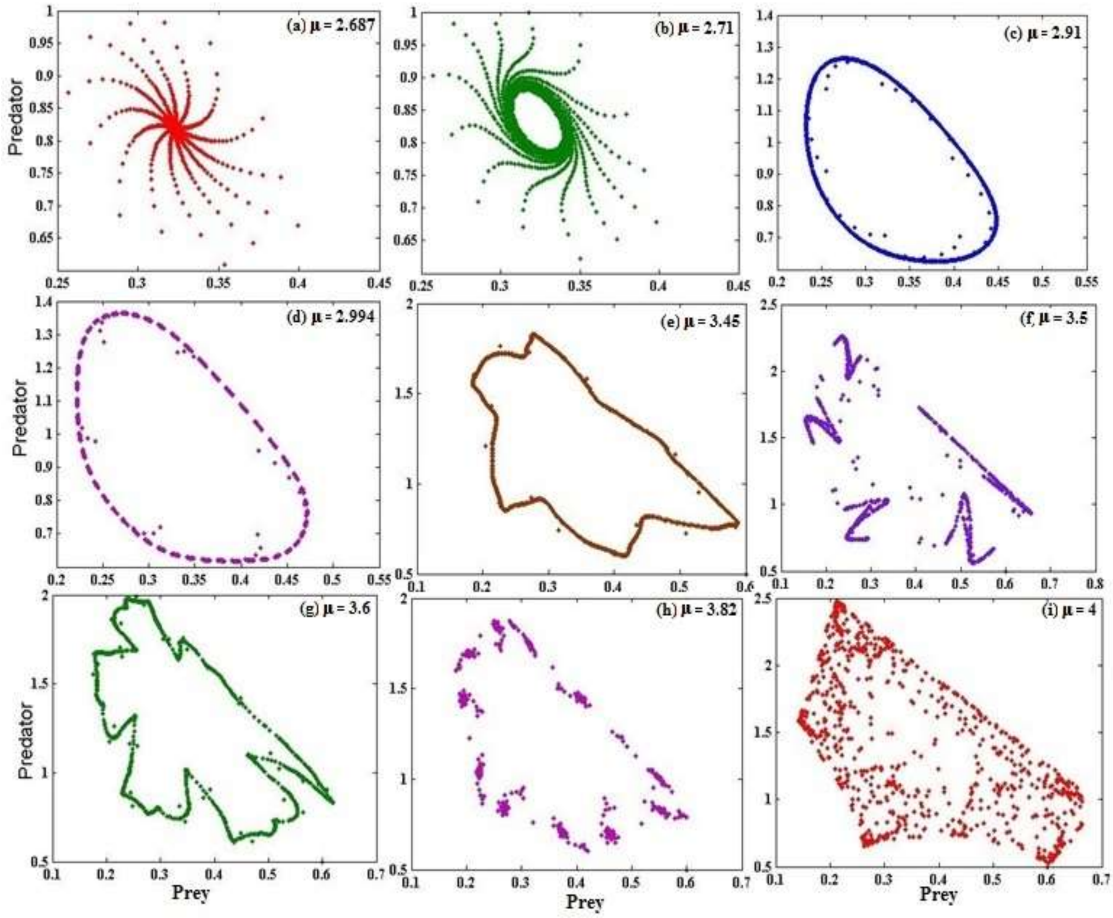

Figure 8: Phase portraits for various parameter values of intrinsic growth rate μ corresponding to Figure 7.

The characteristic equation of the Jacobian Matrix of the system (1) evaluated at this positive equilibrium state is given by $\omega^2-(1.1250)\omega+1=0$ (6). Furthermore, the roots of (6) are $\omega_1=0.5625+i0.8268$ and $\omega_2=0.5625-i0.8268$ with $|\omega_{1,2}|=1$. Thus we have $(\mu=2.7083, \Delta=-0.6836<0)\in NB_{E_2}$. Figure - 7 shows the dynamical behavior of system (1) which tends to the stable equilibrium point E_2 for $\mu<2.7083$ and loses its stability through a Neimark-Scaker bifurcation for $\mu=2.7083$. We observe that for $\mu>2.7083$, the positive equilibrium state E_2 becomes unstable and chaotic behavior is visible in the prey-predator interactions. We may conclude that for large values of the intrinsic growth rate parameter μ, there is an onset of choas in prey-predator model's behavior. For $2.5\le\mu\le4$ the system (1) exhibits complex dynamic behavior. In Figure 8 phase portrait corresponding to Figure 7(a-b) for varying values μ are presented, which shows the transition from smooth invariant circle to attracting chaotic sets. For $\mu>2.7083$ there is a circular curve and as μ increases and at $\mu=3.5$ the circle disappears and due to periodic doubling bifurcation chaotic behaviour is observed.

VI. SENSITIVE ANALYSIS

Being sensitivity to initial conditions is a distinctive quality of chaotic behaviour. To exhibit the sensitive dependence to initial values of system (1), four time series with initial points (x_0, y_0) and $(x_0+0.0001, y_0)$ are presented. The computational results are shown in Figure - 9 and Figure - 10. From these pictorial representations it is observed that at the initiation the difference is indistinguishable but over the time the difference between them is quite rapid and obvious. In addition Figure - 9 and Figure - 10 shows sensitive dependence on initial conditions, x - coordinates of the four orbits are plotted against the time with two set of parameter values $(\mu,\alpha,\beta)=(3.69, 2.56, 3.82)$ and $(\mu,\alpha,\beta)=(4, 1.05, 3.25)$ of system (1). In the initial conditions the abscissae differ by 0.0001 and the ordinates are fixed.

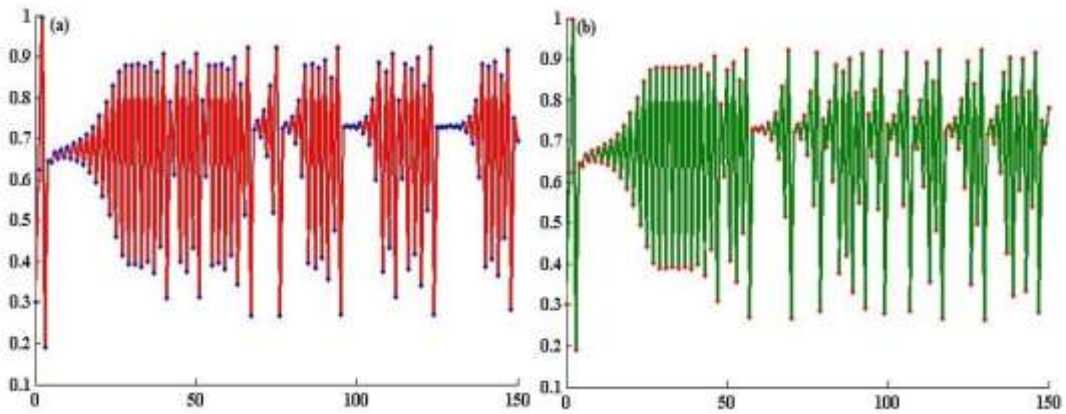

Figure 9: Time series x_t corresponding to the initial conditions (a) $(0.3, 0.5)$ and (b) $(0.3001, 0.5)$ of system (1).

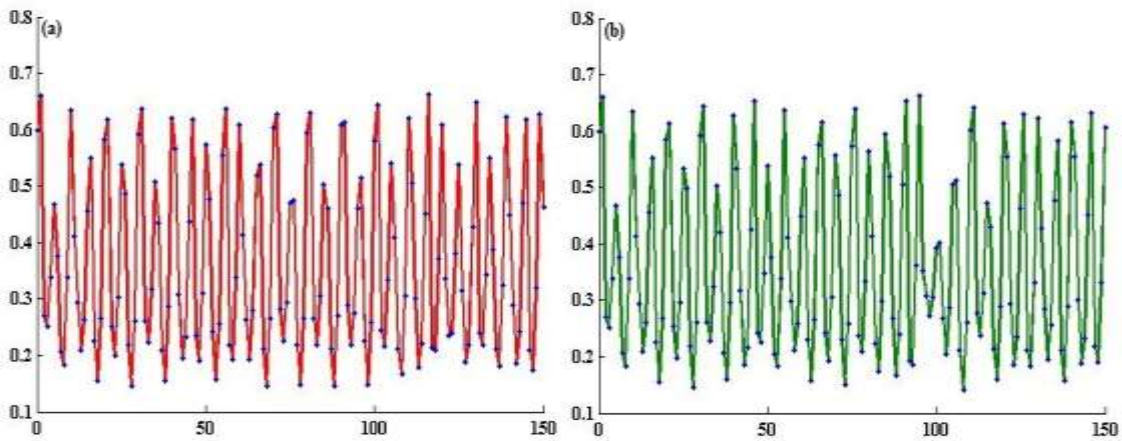

Figure 10: Time series x_t corresponding to the initial conditions (a) $(0.6, 0.5)$ and (b) $(0.6001, 0.5)$ of system (1).

VII. CONCLUSION

In this paper a two dimensional discrete prey-predator model is analyzed and it is shown that the system demonstrates rich dynamical behavior. Foremost the conditions for the existence of the equilibria of the system are obtained and also the conditions essential to study the stability of the system. It is illustrated that the system undergoes flip bifurcations and also vital conditions for flip and Neimark - Sacker Bifurcation are derived. It is interesting to note that system displays dynamical behavior such as invariant cycles, periodic 2, 3, 4 and 8 orbits, cascade of period doubling and the chaotic sets. For the co-existence equilibrium point the system is stable if the growth rate parameter lies in the range of $2.5 < \mu < 2.7083$, it is unstable for $\mu > 2.7083$ and at $\mu = 2.7083$ Neimark - Sacker Bifurcation takes place. The phase portrait corresponding to varying values of μ is presented, showing the transition from smooth invariant circle to attracting chaotic sets.

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