ISSN (e): 2250 – 3005 || Volume, 08 || Issue, 9|| Sepetember – 2018 ||
International Journal of Computational Engineering Research (IJCER)

Wing Signed Graph of a Signed Graph

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ABSTRACT: *In this paper we introduced a new notion wing signed graph of a signed graph and its properties are obtained. Further, we discuss structural characterization of wing signed graphs.*
KEYWORDS: *Balance, Signed graphs, Switching, Wing Signed Graph.*

Date of Submission: 01-10-2018 Date of acceptance: 13-10-2018

I. INTRODUCTION

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

The wing graph W(G) of G = (V, E) is a graph with V(W(G)) = E(G) and any two vertices e_1 and e_2 in W(G) are joined by an edge if they are non-incident edges of some induced 4-vertex path in G. This concept was introduced by Hoang [4]. Wing graphs have been introduced in connection with perfect graphs.

To model individuals preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph G = (V, E) whose edges are labeled with positive and negative signs (i.e., $\sigma: E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle. A marking of S is a function $\mu: V(G) \to \{+, -\}$. The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1. A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:

(i) Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).
(ii) There exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$ (Sampathkumar [5]).

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\mu(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S' written $S \sim S'$, whenever their exists a marking μ such that $S_\mu(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and $S'$are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be weakly isomorphic (see [6]) or cycle isomorphic (see [7]) if there exists an isomorphism $\varphi: G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\varphi(z)$ in S_2. The following result is well known (see [7]):

Theorem 1.2. (**T. Zaslavsky** [7]) Given a graph G, any two signed graphs in ψ(G), where ψ(G) denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.

II. WING SIGNED GRAPHS

We now extend the notion of wing graphs to signed graphs as follows: The wing signed graph W(S) of a signed graph $S = (G, \sigma)$ is a signed graph, whose underlying graph is W(G) and sign of any edge e_1e_2 in W(S) is $\sigma(e_1)\sigma(e_2)$. Further, a signed graph $S = (G, \sigma)$ is called wing signed graph, if $S \cong W(S^{'})$for some signed graph $S^{'}$. The following result restricts the class of wing graphs.

Theorem 2.1. For any signed graph $S = (G, \sigma)$, its wing signed graph W(S) is balanced.
Proof. Let $\sigma^{'}$ denote the signing of W (S) and let the signing of S be treated as a marking of the vertices of W (S). Then by definition of W (S) we see that $\sigma^{'}(e_1, e_2) = \sigma(e_1, e_2)$, for every edge e_1e_2 of W (S) and hence, by Theorem 1.1, W (S) is balanced.

For any positive integer k, the k^{th} iterated wing signed graph, $W^k(S)$ of S is defined as follows:

$$W^0(S) = S,\ W^k(S) = W\left(W^{k-1}(S)\right).$$

Corollary 2.2. For any signed graph $S = (G, \sigma)$ and for any integer k, $W^k(S)$ is balanced.
The following result characterizes signed graphs which are wing signed graphs.
Theorem 2.3. A signed graph $S = (G, \sigma)$ is a wing signed graph if, and only if, S is balanced signed graph and its underlying graph G is a wing graph.
Proof. Suppose that S is balanced and G is a wing graph. Then there exists a graph $G^{'}$such that $W(G^{'}) \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking μ of G such that each edge e = uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S^{'} = (G^{'}, \sigma^{'})$, where for any edge e in $G^{'}$, $\sigma^{'}(e)$ is the marking of the corresponding vertex in G. Then clearly, $W(S^{'}) \cong S$. Hence S is a wing signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a wing signed graph. Then there exists a signed graph $S^{'} = (G^{'}, \sigma^{'})$ such that $W(S^{'}) \cong S$. Hence G is the wing graph of $G^{'}$and by Theorem 2.1, S is balanced.

Theorem 2.4. For any signed graphs S_1 and S_2 with the same underlying graph, their wang signed graphs are switching equivalent.
Proof. Suppose $S_1 = (G, \sigma)$ and $S_2 = (G^{'}, \sigma^{'})$ be two signed graphs with $G \cong G^{'}$. By Theorem 2.1, $W(S_1)$ and $W(S_2)$ are balanced and hence, the result follows from Theorem 1.2.

The notion of negation η(S) of a given signed graph S defined in [3] as follows: η(S) has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator η(.) of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the W(S) is balanced. We now examine, the conditions under which negation η(S) of W(S) is balanced.

Proposition 2.5. Let S = (G, σ) be a signed graph. If W(G) is bipartite then η(W(S)) is balanced.
Proof. Since, by Theorem 2.1, W(S) is balanced, each cycle C in W(S) contains even number of negative edges. Also, since W(G) is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in W(S) is also even. Hence η(W(S)) is balanced.

In [4], the author proved that, the graph G and its wing graph W(G) are isomorphic, if $G \cong C_{2k+1}$. In view of this we have the following result:
Theorem 2.6. For any signed graph S = (G, σ), S ~ W(S) if, and only if, Sis a balanced signed graph and$G \cong C_{2k+1}$.
Proof**.** Suppose S ~ W (S). This implies,$G \cong W(G)$ and hence G is isomorphic to C_{2k+1}. Now, if S is any signed graph with underlying graph G is C_{2k+1}, Theorem 2.1 implies that W(S) is balanced and hence if S is unbalanced and its W(S) being balanced cannot be switching equivalent to S in accordance with Theorem 1.2. Therefore, S must be balanced.
Conversely, suppose that S is a balanced signed graph and G is isomorphic to C_{2k+1}. Then, since W(S) is balanced as per Theorem 2.1 and since $G \cong W(G)$, the result follows from Theorem 1.2 again.
Theorem 2.4 & 2.6 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 2.7. For any two signed graphs S_1 and S_2 with the same underlying graph, $W(S_1)$ and $W(\eta(S_2))$ are switching equivalent.
Corollary 2.8. For any two signed graphs S_1 and S_2 with the same underlying graph, $W(\eta(S_1))$ and $W(S_2)$ are switching equivalent.
Corollary 2.9. For any two signed graphs S_1 and S_2 with the same underlying graph, $W(\eta(S_1))$ and $W(\eta(S_2))$ are switching equivalent.
Corollary 2.10. For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with $G_1 \cong G_2$ and G_1, G_2 are bipartite, $\eta(W(S_1))$ and $W(S_2)$ are switching equivalent.
Corollary 2.11. For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with $G_1 \cong G_2$ and G_1, G_2 are bipartite, $W(S_1)$ and
$\eta(W(S_2))$ are switching equivalent.
Corollary 2.12. For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with $G_1 \cong G_2$ and G_1, G_2 are bipartite, $\eta(W(S_1))$ and $\eta(W(S_2))$ are switching equivalent.
Corollary 2.13. For any signed graph $S = (G, \sigma)$, $S \sim W(\eta(S))$ if, and only if, S is a balanced signed graph and $G \cong C_{2k+1}$.

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K. V. Madhusudhan "Wing Signed Graph Of A Signed Graph "International Journal of Computational Engineering Research (IJCER), vol. 08, no. 09, 2018, pp 39-41