ISSN (e): 2250 – 3005 || Volume, 07 || Issue, 12 || December – 2017 ||
International Journal of Computational Engineering Research (IJCER)

Solutions Of Heat Equation Arrived From q-Difference Operator

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ABSTRACT

Abstract: An investigation of heat equation as a model of partial difference operator is carried out in this paper. Here, we introduce solutions of the heat equation obtained by generalized q-difference equation with and without variable coefficients. The propagation of heat is studied under diverse circumstances and relevant conclusions are derived. Suitable Examples are inserted to validate our main results

Key words: Difference equation, Generalized difference operator and Heat equation.

AMS classification 2015: 39A70, 39A10, 47B39, 80A20.

Date of Submission: 15-12-2017 Date of acceptance: 23-12-2017

I. INTRODUCTION

In 1984, Jerzy Popenda [1] introduced the difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [2] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the inverse fractional difference operator $\Delta_h^{-\nu}$ [3,4]. The sum of m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$, where $\Delta_\ell u(k) = u(k+\ell) - u(k)$ [6]. In 2011, M.M.S.Manuel, et.al, [5], extended the definition of Δ_α to $\underset{\alpha(\ell)}{\Delta}$ as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$, for the real valued function $v(k)$ and $\ell > 0$. In 2014, G.Britto Antony Xavier, et.al, [7], have introduced q-difference operator defined as $\Delta_q v(k) = v(qk) - v(k)$ for the real valued function $v(k)$, $q \in (0,\infty)$ and obtained finite series solution to the corresponding generalized q-difference equation $\Delta_q v(k) = u(k)$.

For n-variable real valued function $v(k_1, k_2, ..., k_n)$, the generalized q-difference operator is defined as

$$\underset{q_1 \wedge q_2 \wedge, ..., q_n}{\Delta} v(k) = v(k_1 q_1, k_2 q_2, ..., k_n q_n) - v(k_1, k_2, ..., k_n), \qquad (1)$$

where $k = (k_1, k_2, ..., k_n) \in R^n$, $v(k) : R^n \to R$ and $q_1 \wedge q_2 \wedge, ..., q_n$.

For example $\underset{q_1 \wedge q_2}{\Delta} v(k_1, k_2) = v(k_1 q_1, k_2 q_2) - v(k_1, k_2)$.

II. PRELIMINARIES

Consider, the two side temperature distribution of a very long rod. Let $v(k_1, k_2)$ be the temperature at the real time k_2 and real position k_1 of the rod. At time k_2, if the temperature $v(\frac{k_1}{q_1}, k_2), q_1 > 0$ is higher than $v(k_1, k_2)$, heat will flow from the point $\frac{k_1}{q_1}$ to k_1. Similarly, at time k_2, if the temperature

$v(k_1q_1,k_2), q_1>0$ is higher than $v(k_1,k_2)$, heat will flow from the point k_1q_1 to k_1.

The amount of increase $v(k_1,k_2q_2)-v(k_1,k_2)$ is proportional to the differences $v(\frac{k_1}{q_1},k_2)-v(k_1,k_2)$ and $v(k_1,k_2q_2)-v(k_1,k_2)$. Let α is a positive diffusion rate constant of the rod. Then the q-heat equation is given by

$$v(k_1,k_2q_2)-v(k_1,k_2)=\alpha(v(\frac{k_1}{q_1},k_2)-v(k_1,k_2))+\alpha(v(k_1q_1,k_2)-v(k_1,k_2)),$$

$$(i.e)\ \underset{1\wedge q_2}{\Delta}v(k_1,k_2)=\alpha\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)+\alpha\underset{q_1\wedge 1}{\Delta}v(k_1,k_2). \qquad (2)$$

If $v(k_1,k_2)=k_2k_1$ is the solution of (2), then the value of α is, $\alpha=\frac{1-2q_1+q_1^2}{q_1(q_2-1)}$. Similarly, q-Heat equation for the variable coefficient is defined as,

$$\underset{1\wedge q_2}{\Delta}v(k_1,k_2)=\alpha(k_2,k_1)\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)+\alpha(k_2,k_1)\underset{q_1\wedge 1}{\Delta}v(k_1,k_2), \qquad (3)$$

where $\alpha(k_2,k_1)$ is a function of k_1 and k_2. If $v(k_1,k_2)=k_2k_1$ is a solution of (3) then the value of $\alpha(k_2,k_1)$ is $\alpha(k_2,k_1)=\frac{1-2q_1+q_1^2}{q_1(q_2-1)}$.

III. q-HEAT EQUATION WITH CONSTANT COEFFICIENT

In this section we derive a solution of equation (2) and also we obtain a function $v(k_1,k_2)$ satisfying the equation (2).

Theorem 3.1.1 *If* $\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)=u_{q_1^{-1}}(k_1,k_2)$ *and* $\underset{q_1\wedge 1}{\Delta}v(k_1,k_2)=u_{q_1}(k_1,k_2)$ *are known functions. Then the* q*-heat equation has a solution*

$$v(k_1,k_2)-v(k_1,\frac{k_2}{q_2^m})=\alpha\sum_{r=1}^{m}\left\{u_{q_1^{-1}}(k_2q_2^{-r},k_1)+u_{q_1}(k_2q_2^{-r},k_1)\right\}. \qquad (4)$$

Proof. From the linearity of $\overset{-1}{\underset{q_1,q_2}{\Delta}}$ and (2), we have

$$v(k_1,k_2)=\alpha\overset{-1}{\underset{1\wedge q_2}{\Delta}}\left(\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)+\underset{q_1\wedge 1}{\Delta}v(k_1,k_2)\right). \qquad (5)$$

Now the proof of (4) follows by taking $\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)=u_{q_1^{-1}}(k_1,k_2)$ and

$$\underset{q_1\wedge 1}{\Delta}v(k_1,k_2)=u_{q_1}(k_1,k_2).$$

Theorem 3.2.2 *If* $v(k_1,k_2)$ *is a solution of q-heat equation (2), then the following four relations are equivalent:*

$$(i)\ v(k_1,k_2)-v(k_1,\frac{k_2}{q_2^m})=\alpha\sum_{r=1}^{m}\left[v(\frac{k_1}{q_1},k_2q_2^{-r})+v(k_1q_1,k_2q_2^{-r})-2v(k_1,k_2)\right]. \qquad (6)$$

$$(ii)\ v(k_1,k_2)=\frac{1}{(1-\alpha)^m}v(k_1,k_2q_2^m)-\sum_{r=1}^{m}\frac{\alpha}{(1-\alpha)^r}\left(v(\frac{k_1}{q_1},k_2q_2^{r-1})+v(k_1q_1,k_2q_2^{r-1})\right). \qquad (7)$$

$$(iii)\ v(k_2,k_1)=\frac{1}{1-2\alpha}v(k_1,k_2q_2)-\frac{\alpha}{1-2\alpha}v(k_1q_1,k_2)-\frac{\alpha}{(1-2\alpha)^{m+1}}v(\frac{k_1}{q_1},k_2q_2^m)$$

$$+\sum_{r=1}^{m}\frac{\alpha^2}{(1-2\alpha)^{r+1}}\left\{v(\frac{k_1}{q_1^2},k_2q_2^{(r-1)})+v(k_1,k_2q_2^{(r-1)})\right\}. \qquad (8)$$

$$(iv)\ v(k_2,k_1)=\frac{1}{1-2\alpha}v(k_1,k_2q_2)-\frac{\alpha}{1-2\alpha}v(\frac{k_1}{q_1},k_2)-\frac{\alpha}{(1-2\alpha)^{m+1}}v(k_1q_1,k_2q_2^m)$$

$$+\sum_{r=1}^{m}\frac{\alpha^2}{(1-2\alpha)^{r+1}}\left\{v(k_1,k_2q_2^{(r-1)})+v(k_1q_1^2,k_2q_2^{(r-1)})\right\}. \qquad (9)$$

Proof. From the q-difference Heat equation (2), we have

$$v(k_1,k_2q_2)-v(k_1,k_2)\quad=\alpha\left[v(\frac{k_1}{q_1},k_2)-v(k_1,k_2)\right]+\alpha\left[v(k_1q_1,k_2)-v(k_1,k_2)\right]$$

$$v(k_1,k_2)=\frac{1}{1-2\alpha}v(k_1,k_2q_2)-\frac{\alpha}{1-2\alpha}v(\frac{k_1}{q_1},k_2)-\frac{\alpha}{1-2\alpha}v(k_1q_1,k_2). \qquad (10)$$

(i) Replacing k_2 by $\frac{k_2}{q_2}$ in (10), we get (6).

(ii) Replacing k_2 by k_2q_2 in (10), continuing the same process we get (7).

(iii) Replacing k_1 by $\frac{k_1}{q_1}$ in (10), we get (8).

(iv) Replacing k_1 by k_1q_1 in (10), we get (9).

Corollary 3.3.3 *Let* k_1, k_2 *and* $q_2\neq 0$ *and* $q_2\neq 1$. *Then, we have*

$$\left\{\frac{k_1+k_2}{q_2-1}+\frac{log(k_1k_2)}{logq_2}\right\}-\left\{\frac{k_1+\frac{k_2}{q_2^m}}{q_2-1}+\frac{log\,\frac{(k_1k_2)}{q_2^m}}{logq_2}\right\}=\sum_{r=1}^{m}\left\{(k_2q_2^{-r})+1\right\}. \qquad (11)$$

Proof. Taking $v(k_1,k_2)=\frac{k_1+k_2}{q_2-1}$, $u(k_1,k_2)=k_2$, $w(k_1,k_2)=1$ in (4), we get the proof of (11).

Example 3.4.4 *Taking* $k_1=0.3$, $k_2=0.2$,, $q_2=2$ *and* $m=2$ *in (11), we have* $2.15=2.15$

Corollary 3.5.5 *Let* $1-2\alpha\neq 0$, $q_1\neq 0$. *Then, we have*

$$k_1k_2 = \frac{1}{(1-2\alpha)^2}(k_1k_2q_2^2) - \frac{\alpha}{(1-2\alpha)^2}(\frac{k_1}{q_1}k_2q_2) - \frac{\alpha}{(1-2\alpha)^2}(k_1q_1k_2q_2) - \frac{\alpha}{(1-2\alpha)}(\frac{k_1}{q_1}k_2) - \frac{\alpha}{(1-2\alpha)}(k_1q_1k_2). \qquad (12)$$

Proof. The proof of (12) follows by taking $v(k_1,k_2) = k_1k_2$ and $m=2$ in (7).

Example 3.6.6 *Taking* $k_1 = 4,\ k_2 = 2,\ q_2 = 2,\ q_1 = 3$ *in (12), we get* $8=8$.

Corollary 3.7.7 *Let* $1-2\alpha \neq 0$, $q_1 \neq 0$. *Then we have*

$$k_1k_2 = \frac{1}{1-2\alpha}(k_1k_2q_2) - \frac{\alpha}{(1-2\alpha)^2}(\frac{k_1}{q_1}k_2q_2) + \frac{\alpha^2}{(1-2\alpha)^2}(\frac{k_1}{q_1}^2 k_2) + \frac{\alpha^2}{(1-2\alpha)^2}(k_1k_2) - \frac{\alpha}{1-2\alpha}(k_1q_1k_2). \qquad (13)$$

Proof. Taking $v(k_1,k_2) = k_1k_2$ and $m=1$ in (12), we get the proof of (13).

Example 3.8.8 *Taking* $k_1 = 4,\ k_2 = 2,\ q_2 = 2,\ q_1 = 3$ *in (13), we get* $8=8$.

Corollary 3.9.9 *Let* $1-2\alpha \neq 0$, $q_1 \neq 0$. *Then we have*

$$k_1k_2 = \frac{1}{1-2\alpha}(k_1k_2q_2) - \frac{\alpha}{(1-2\alpha)}(\frac{k_1}{q_1}k_2) - \frac{\alpha}{(1-2\alpha)^2}(k_1q_1k_2q_2) + \frac{\alpha^2}{(1-2\alpha)^2}(k_1k_2) + \frac{\alpha^2}{1-2\alpha^2}(k_1q_1^2k_2). \qquad (14)$$

Proof. The proof of (14) follows by taking $v(k_1,k_2) = k_1k_2$ and $m=1$ in (9).

Example 3.1010 *When* $k_1 = 4,\ k_2 = 2,\ q_1 = 3\ \ q_2 = 2,$ *in (14), we get* $8=8$.

IV. q-HEAT EQUATION WITH VARIABLE COEFFICIENT

In this section we derive a solution of q-heat equation with variable coefficient of (3) and also we obtain a function $v(k_1,k_2)$ satisfying the equation (3).

Theorem 4.1.11 *If* $v(k_1,k_2)$ *is a solution of equation (3) with variable coefficients. Then the follwoing relations are equivalent*

$$(i)\ v(k_1,k_2) - v(k_1,\frac{k_2}{q_2^m}) = \alpha(k_1,k_2)\sum_{r=1}^{m}\left[v(\frac{k_1}{q_1},k_2q_2^{-r}) + v(k_1q_1,k_2q_2^{-r}) - 2v(k_1,k_2)\right]. \qquad (15)$$

$$(ii)\ v(k_1,k_2) = \frac{1}{\prod_{r=1}^{m}[1-2\alpha(k_1,k_2q_2^{r-1})]}v(k_1,k_2q_2^m) - \sum_{r=1}^{m}\frac{\alpha(k_1,k_2q_2^{r-1})}{\prod_{s=1}^{r}[1-2\alpha(k_1,k_2q_2^{s-1})]}\left(v(\frac{k_1}{q_1},k_2q_2^{r-1}) + v(k_1q_1,k_2q_2^{r-1})\right) \qquad (16)$$

$$(iii)\ v(k_1,k_2)=\frac{1}{1-2\alpha(k_1,k_2)}v(k_1,k_2q_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(k_1q_1,k_2)$$
$$-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]\prod_{r=1}^{m}[1-2\alpha(\frac{k_1}{q_1},k_2q_2^{r-1})]}v(\frac{k_1}{q_1},k_2q_2^{m})$$

$$+\sum_{r=1}^{m}\frac{[\alpha(k_1,k_2)][\alpha(\frac{k_1}{q_1},k_2q_2^{r-1})]}{[1-2\alpha(k_1,k_2)]\prod_{s=1}^{r}[1-2\alpha(\frac{k_1}{q_1},k_2q_2^{s-1})]}\left\{v(\frac{k_1}{q_1^2},k_2q_2^{(r-1)})+v(k_1,k_2q_2^{(r-1)})\right\}. \quad (17)$$

$$(iv)\ v(k_1,k_2)=\frac{1}{1-2\alpha(k_1,k_2)}v(k_1,k_2q_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(\frac{k_1}{q_1},k_2)$$
$$-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]\prod_{r=1}^{m}[1-2\alpha(k_1q_1,k_2q_2^{r-1})]}v(k_1q_1,k_2q_2^{m})$$

$$+\sum_{r=1}^{m}\frac{\alpha(k_1,k_2)\alpha(k_1q_1,k_2q_2^{r-1})}{[1-2\alpha(k_1,k_2)]\prod_{s=1}^{r}[1-2\alpha(k_1q_1,k_2q_2^{r-1})]}\left\{v(k_1,k_2q_2^{(s-1)})+v(k_1q_1^2,k_2q_2^{(r-1)})\right\}. \quad (18)$$

Proof. (i) From the linearity of $\overset{-1}{\underset{q_2,q_1}{\Delta}}$ and (3), we have

$$v(k_1,k_2)-v(k_1,\frac{k_2}{q_2^m})=\alpha(k_1,k_2)\sum_{r=1}^{m}\left\{u(k_1,k_2q_2^{-r})+w(k_1,k_2q_2^{-r})\right\}. \quad (19)$$

From the experimental value, we take

$$\underset{q_1^{-1}\wedge 1}{\Delta}v(k_1,k_2)=u(k_1,k_2),\ \underset{q_1\wedge 1}{\Delta}v(k_1,k_2)=w(k_1,k_2). \quad (20)$$

Equation (20) is the solution of Heat Equation (3).

In (19), $u(k_1,k_2q_2^{-r_1})$ is obtained by replacing k_2 by $k_2q_2^{-r_1}$ in (20) .

Substituting (20) in (19), we get (15). (ii) From the q-Heat equation (3), we have

$$v(k_1,k_2)=\frac{1}{1-2\alpha(k_1,k_2)}v(k_1,k_2q_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(\frac{k_1}{q_1},k_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(k_1q_1,k_2)$$

(21)

Replacing k_2 by (k_2q_2) in (21), repeating the process we get (16).

(iii) Replacing k_1 by $(\frac{k_1}{q_1})$ in (21), we get (17).

(iv) Replacing k_1 by (k_1q_1) in (21), we get (18).

Corollary 4.2. 12 *Let* $q_2\neq 0$ *and* $q_2\neq 1$*. Then we have*

$$\left\{\frac{k_1+k_2}{q_2-1}+\frac{log(k_2k_1)}{logq_2}\right\}-\left\{\frac{k_1+\frac{k_2}{q_2^m}}{q_2-1}+\frac{log\frac{(k_1k_2)}{q_2^m}}{logq_2}\right\}=\sum_{r=1}^{m}\left\{(k_2q_2^{-r})+1\right\}. \quad (22)$$

Proof. The proof of (22) follows by taking $v(k_1,k_2)=\dfrac{k_1+k_2}{q_2-1}$, $u(k_1,k_2)=k_2$, $w(k_1,k_2)=1$ in (19).

Example 4.3.13 *Taking* $k_2=0.2$, $k_1=0.3$, $q_2=2$, $m=2$ *in (22), we get* $2.15=2.15$.

Corollary 4.4. 14*Let* $q_1\neq 0$, $1-2\alpha(k_1,k_2)\neq 0$ *and* $1-2\alpha(k_1,k_2q_2)\neq 0$. *Then*

$$k_1k_2=\frac{1}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1,k_2q_2)]}(k_1k_2q_2^2)-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1,k_2q_2)]}$$
$$\left\{(\frac{k_1}{q_1}k_2q_2)+(k_1q_1k_2q_2)\right\}-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]}\left\{(\frac{k_1}{q_1}k_2)+(k_1q_1k_2)\right\}. \qquad (23)$$

Proof. Taking $v(k_1,k_2)=k_1k_2$ and $m=2$ in (16), we get (23).

Example 4.5.15 *Taking* $k_1=4,\ k_2=2,\ q_1=3,$ *q_2 = 2* $in, weget$ *8 = 8*

Corollary 4.6. 16*Let* $\alpha(k_1,k_2)\neq\frac{1}{2}$ and $\alpha(\frac{k_1}{q_1},k_2)\neq\frac{1}{2}$. *Then we have*

$$k_1k_2=\frac{1}{1-2\alpha(k_1,k_2)}(k_1k_2q_2)-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(\frac{k_1}{q_1},k_2)]}(\frac{k_1}{q_1}k_2q_2)$$
$$+\frac{\alpha(k_1,k_2)\alpha(\frac{k_1}{q_1},k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(\frac{k_1}{q_1},k_2)]}\left\{(\frac{k_1^{\;2}}{q_1}k_2)+(k_1k_2)\right\}-\frac{\alpha(k_1,k_2)\alpha(\frac{k_1}{q_1},k_2)}{1-2\alpha(k_1,k_2)}(k_1q_1k_2). \qquad (24)$$

Proof. Taking $v(k_1,k_2)=k_1k_2$ and $m=1$ in (17), we get (24).

Example 4.7.17 *Taking* $k_1=4,\ k_2=2,\ q_1=3,\ q_2=2,\ in\ (24), we\ get$ *8 = 8*

Corollary 4.8.18 *Let* $1-2\alpha(k_1,k_2)\neq 0$ *and* $1-2\alpha(k_1q_1,k_2)\neq 0$. *Then we have*

$$k_2k_1=\frac{1}{1-2\alpha(k_1,k_2)}(k_1k_2q_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}(\frac{k_1}{q_1}k_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}(k_1q_1k_2q_2)$$
$$+\frac{\alpha(k_1,k_2)\alpha(k_1q_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1q_1,k_2)]}\left\{k_2k_1+k_2k_1q_1^2\right\}. \qquad (25)$$

Proof. Taking $v(k_1,k_2)=k_2k_1$ and $m=1$ in (18), we get (25).

Example 4.9.19 *Let* $k_2=2,\ k_1=4,\ q_2=2,\ q_1=3$ *in (25), we get* $8=8$.

Conclusion: In the above study, the heat equation model is studied using generalized q- difference operator. We can say that the above research helps us in reducing any wastage of heat and also enables us in making a optimal choice.

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International Journal of Computational Engineering Research (IJCER) is UGC approved Journal with Sl. No. 4627, Journal no. 47631.

G.Britto Antony Xavier "Solutions Of Heat Equation Arrived From -Difference Operator." IInternational Journal of Computational Engineering Research (IJCER), vol. 7, no. 12, 2017, pp. 61-67.