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Existence And Uniqueness of Non-Local Cauchy Problem For Fractional Differential Equation On Banach Space

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ABSTRACT
This article derives sufficient conditions for existence and uniqueness of solution for impulsive fractional integro-differential equations with classical and non-local conditionon Banach space using fixed point theorem. The system taken by the authors is more general nature as compared to system taken so far, as it is more relevant to physical systems encountered. An example is given to illustrate efficacy of the methodology.

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I. Introduction

There are various problems in science and engineering like seepage flow in porous media [1], anomalous diffusion [2, 3], and transport [4], the nonlinear oscillations of earthquake [5], fluid dynamics traffic model [6] etc are well modeled in fractional differential equations. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations [7]. This is because of its non-local property [8] which means, that the next state of the system depends not only upon its current state but also upon its entire historical state. The existence and uniqueness of solutions of fractional differential equations using fixed point theory has been studied by Delbosco and Rodino [5], Cheng and Guozhu [8] and El-Borai [9]. The study of non-local Cauchy problem was initiated by Byszewski [10] and followed by several authors [11]. The study of existence and uniqueness of solutions of fractional differential equations with non-local conditions using fixed point theory was initiated by N' Guerekata [12] followed by Balachandran and Park [13]. On the other hand, rapid development of impulsive differential equations played very important role in modeling of many problems of population dynamics, chemical technology and biotechnology [14]. This motivates many researchers to study existence and uniqueness solutions of the impulsive differential equations [15]. Existence and uniqueness of solutions of impulsive fractional differential equations with local conditions were studied by Benchohra and Slimani [16], Mophou [17], Ravichandran and Arjunan [18] using fixed point theory and semigroup theory.

The existence and uniqueness of solutions of impulsive differential equations with non-local condition using fixed point theory was studied by Benchohra and Slimani [16]. Balachandran et. al. [19, 20] and Gao et. al. [21] studied existence and uniqueness of solution of fractional differential equations using fixed point theory. Motivated by the work of Balachandran et. al. [19], this paper presents the existence and uniqueness of solutions of the quasi-linear impulsive fractional order Volterra-Fredholm kind of integro-differential equation of the form

$$\begin{aligned} {}^cD^\alpha x(t) &= A(t,x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 - g(x), \end{aligned}$$

over the interval $[0, T]$.

II. Preliminaries

Some basic definitions and properties of fractional calculus and fractional differential equations used in this article, are as follows:

Definition 2.1. *The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I^\alpha_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$$D^\alpha_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

This derivative has singularity at zero and also requires special initial condition which lacks physical interpretation. To overcome this difficulty, Caputo [22] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.3. *The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$$^cD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{d^n f(s)}{ds^n}ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Moreover if $0 < \alpha < 1$, then

$$^cD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{df(s)}{ds}ds.$$

The Riemann-Liouville integral I^{α} and Caputo derivative $^cD^{\alpha}_{0+}$ satisfies following properties which is mentioned in Kilbas *et. al.* [23] and Samko *et. al.* [24].

Theorem 2.1. *For $\alpha, \beta > 0$ and f has absolutely continuous derivatives up to suitable order then,*

(1)$I^{\alpha}_{0+}I^{\beta}_{0+}f(t) = I^{\alpha+\beta}_{0+}f(t)$

(2)$I^{\alpha}_{0+}I^{\beta}_{0+}f(t) = I^{\beta}_{0+}I^{\alpha}_{0+}f(t)$

(3)$I^{\alpha}_{0+}(f(t)+g(t)) = I^{\alpha}_{0+}f(t) + I^{\alpha}_{0+}g(t)$

(4)$I^{\alpha}_{0+}\,{}^cD^{\alpha}_{0+}f(t) = f(t) - f(0), \quad 0 < \alpha < 1$

(5)${}^cD^{\alpha}_{0+}I^{\alpha}_{0+}f(t) = f(t)$

(6)${}^cD^{\alpha}_{0+}f(t) = I^{1-\alpha}_{0+}f'(t), \quad 0 < \alpha < 1$

(7)${}^cD^{\alpha}_{0+}\,{}^cD^{\beta}_{0+}f(t) \neq {}^c D^{\alpha+\beta}_{0+}f(t)$

(8)${}^cD^{\alpha}_{0+}\,{}^cD^{\beta}_{0+}f(t) \neq {}^cD^{\beta}_{0+}\,{}^cD^{\alpha}_{0+}f(t)$

2.1 Notations

(N1) X = Banach space.

(N2) $\mathbb{R}_+ = [0,\infty)$

(N3) $C([0,T_0],X) = \{x : [0,T_0] \to X/x \text{ is continuous}\}$ with norm $||x|| = sup_t||x(t)||$

(N4) $PC([0,T_0],X) = \{x : [0,T_0] \to X; x \in C([t_{k-1},t_k],X),\ and\ x(t_k^-)\ and\ x(t_k^+)\ exist,\ k = 1,2,\cdots,p\ with\ x(t_k^-) = x(t_k)\}$ with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$

(N4) $PC([0,T_0],X) = \{x : [0,T_0] \to X; x \in C([t_{k-1},t_k],X),\ and\ x(t_k^-)\ and\ x(t_k^+)\ exist,\ k = 1,2,\cdots,p\ with\ x(t_k^-) = x(t_k)\}$ with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$

(N5) $B(X) = \{A : X \to X/A \text{ is bouneded and linear}\}$ with norm $||A||_{B(X)} = sup\{||A(y)||; y \in X\&||y|| \leq 1\}$

For convenience, $^cD^{\alpha}_{0+}$ is taken as $^cD^{\alpha}$ and with these definitions and properties, sufficient conditions for existence and uniqueness of solutions are derived as follows:

III. Equation with classical condition

This section presents the study of the existence and uniqueness of the solution of impulsive fractional differential equation with classical condition. Consider the Volterra-Fredholm kind of fractional quasilinear impulsive integro-differential of equation of the form

$$\begin{aligned} {}^cD^\alpha x(t) &= A(t,x)x(t) + f(t,x(t),Tx(t),Sx(t)) \quad t \neq t_k,\ k=1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t=t_k,\ k=1,2,\cdots,p \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

over the interval $[0,T]$ is bounded linear operator on X and $f: [0,T]\times X\times X\times X \to X$, $T,S: X\to X$ are defined by $Tx(t)=\int_0^t h(t,s,x(s))ds$ and $Sx(t)=\int_0^T k(t,s,x(s))ds$. Where $h: D_0\times X\to X,\ D_0=\{(t,s); 0\le s\le t\le T\}$ and $k: D_1\times X\to X,\ D_1=\{(t,s); 0\le t,s\le T\}$ are continuous. This type of nonlinear equations arise in many physical situation like mathematical problems concerned with heat flow in materials with viscoelastic problems [25].

The equation (3.1) is equivalent to the integral equation of the form

$$x(t)=\begin{cases} x_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}A(s,x(s))x(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}A(s,x(s))ds \\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds+\sum_{0<t_k<t}Ix(t_k^-). \end{cases} \tag{3.2}$$

The following conditions are assumed to show the existence and uniqueness of the solution (3.1).

(H1) $A: [0,T]\times X\to X$ is continuous bounded linear operator and there exists a positive constant M, such that $||A(t,x)-A(t,y)||_{B(X)}\le M||x-y||$, for all $x,y\in X$.

(H2) $f: [0,T]\times X\times X\times X\to X$ is continuous and there exists positive constants L_1, L_2 and L_3, such that $||f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)||\le L_1||x_1-y_1||+L_2||x_2-y_2||+L_3||x_3-y_3||$ for all x_1,x_2,x_3,y_1,y_2 and y_3 in X.

(H3) $h: D_0\times X\to X$ and $k: D_1\times X\to X$ are continuous and there exists positive constants H and K, such that $||h(t,s,x)-h(t,s,y)||\le H||x-y||$ and $||k(t,s,x)-k(t,s,y)||\le K||x-y||$ for all x and y in X.

(H4) The functions $I_k: X\to X$ are continuous and there exist positive constants I_k^* for all $k=1,2,\cdots,p$, such that $||I_kx-I_ky||\le I_k^*||x-y||$ for all x and y in X.

Set, $\gamma = \frac{T^\alpha}{\Gamma\alpha+1}$ and further assume that,

(H5) $q = \gamma\big[(p+1)[M + L_1 + THL_2 + TKL_3]\big] + \sum I_k^* \Big\} < 1$

Define $F : PC([0,T],X) \to PC([0,T],X)$ by

$$Fx(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s,x(s),Tx(s),Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,x(s),Tx(s),Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-). \end{cases} \tag{3.3}$$

Then equation (3.2) has unique solution if F defined by (3.3) has unique fixed point. This means F is well defined bounded operator on $PC([0,T],X)$ and F is contraction [26].

Lemma 3.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \cdots, p$ are continuous then F is bounded operator on $PC([0,T]X)$.*

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0,T],X)$. Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned} ||Fx_n - Fx||_{PC} \le & \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k-s)^{\alpha-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k-s)^{\alpha-1} \\ & ||f(s,x_n(s),Tx_n(s),Sx_n(s)) - f(s,x(s),Tx(s),Sx(s))||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||f(s,x_n(s),Tx_n(s),Sx_n(s)) - f(s,x(s),Tx(s),Sx(s))||ds \\ & + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)|| \end{aligned}$$

Assuming the continuity of A, f, T, S and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore F is continuous on $PC([0,T],X)$ and hence F is bounded.

Now we derive sufficient conditions for existence and uniqueness of the solution of equation (3.1).

Theorem 3.2. *If the hypotheses (H1)-(H5) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (3.1) has unique solution in $PC([0,T],X)$ for $0 < \alpha \le 1$.*

Proof. To show equation (3.1) has unique solution it is sufficient to show F defined (3.3) is contraction. Let x and y in $PC([0,T],X)$ and consider,

$$\begin{aligned}||Fx-Fy||_{PC} \le & \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\ & +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\ & +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||f(s,x(s),Tx(s),Sx(s))-f(s,y(s),Ty(s),Sy(s))||ds \\ & +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||f(s,x(s),Tx(s),Sx(s))-f(s,y(s),Ty(s),Sy(s))||ds \\ & +\sum_{0<t_k<t}||I_kx(t_k^-)-I_ky(t_k^-)||\end{aligned}$$

Applying hypotheses (H1)-(H4) we obtain,

$$\begin{aligned}||Fx-Fy|| \le & \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}M||x-y||ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}M||x-y||ds \\ & +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}\{L_1+THL_2+TKL_3\}||x-y||ds \\ & +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds\{L_1+THL_2+TKL_3\}||x-y||ds+\sum_{0<t_k<t}I_k^*||x-y|| \\ \le & \left\{\frac{T^\alpha}{\Gamma(\alpha+1)}[(p+1)[M+L_1+THL_2+TKL_3]]+\sum I_k^*\right\}||x-y|| \\ = & \left\{\gamma[(p+1)[M+L_1+THL_2+TKL_3]]+\sum I_k^*\right\}||x-y||\end{aligned}$$

Assuming hypotheses (H5) to obtain, $||Fx-Fy||_{PC} \le q||x-y||$ with $q<1$. Hence by Banach fixed point theorem the equation (3.1) has unique solution.

IV. Equation with non-local condition

In this section, classical condition is replaced by a non-local condition for existence and uniqueness of solution of the impulsive fractional differential equation.

Consider the Volterra-Fredholm kind fractional quasilinear impulsive integro-differential equation of the form

$$\begin{aligned} {}^cD^\alpha x(t) &= A(t,x)x(t)+f(t,x(t),Tx(t),Sx(t)) \quad t\neq t_k,\ k=1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t=t_k,\ k=1,2,\cdots,p \\ x(0) &= x_0-g(x) \end{aligned} \tag{4.1}$$

over the interval $[0,T]$ is bounded linear operator on X and $f:[0,T]\times X\times X\times X\to X$, $T,S:X\to X$ are defined by $Tx(t)=\int_0^t h(t,s,x(s))ds$ and $Sx(t)=\int_0^T k(t,s,x(s))ds$. Where $h:D_0\times X\to X$, $D_0=\{(t,s);0\le s\le t\le T\}$ and $k:D_1\times X\to X$, $D_1=\{(t,s);0\le t,s\le T\}$ are continuous and $g:X\to X$ is given function.

The equivalent integral equation of (4.1) is given by

$$x(t)=\begin{cases} x_0-g(x)+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}A(s,x(s))ds \\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds+\sum_{0<t_k<t}Ix(t_k^-). \end{cases} \tag{4.2}$$

The following hypotheses are assumed.

(H6) $g: X \to X$ is continuous and there exist a positive constant g^*, such that $||g(x)-g(y)|| \leq g^*||x-y||$ for each x and y in X.

(H7) $q^* = g^* + \gamma\big[(p+1)[M+L_1+THL_2+TKL_3]\big] + \sum I_k^* < 1.$

Define $G: PC([0,T],X) \to PC([0,T],X)$ by

$$Gx(t)=\begin{cases} x_0-g(x)+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s),Tx(s),Sx(s))ds+\sum_{0<t_k<t}I_kx(t_k^-). \end{cases} \tag{4.3}$$

Lemma 4.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \cdots, p$ are continuous then G is bounded operator on $PC([0,T],X)$.*

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0,T],X)$. Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned} ||Gx_n - Gx||_{PC} \leq\ & ||g(x_n(s)) - g(x(s))|| \\ & + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||f(s,x_n(s),Tx_n(s),Sx_n(s)) \\ & - f(s,x(s),Tx(s),Sx(s))||ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1} \\ & ||f(s,x_n(s),Tx_n(s),Sx_n(s)) - f(s,x(s),Tx(s),Sx(s))||ds \\ & + \sum_{0<t_k<t}||I_kx_n(t_k^-) - I_kx(t_k^-)||. \end{aligned}$$

Assuming the continuity of A, f, T, S, g and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore G is continuous on $PC([0,T], X)$ and hence G is bounded.

Now the sufficient conditions are derived as under for existence and uniqueness of the solution of equation (4.1).

Theorem 4.2. *If the hypotheses (H1)-(H4) and (H6)-(H7) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (4.1) has unique solution in $PC([0,T], X)$ for $0 < \alpha \leq 1$.*

Proof. To show equation (4.1) has unique solution it is sufficient to show F defined in (4.3) is contraction. Let x and y in $PC([0,T], X)$ **and consider,**

$$\begin{aligned}||Gx - Gy||_{PC} \leq ||g(x) - g(y)|| &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||A(s, x(s))x(s) - A(s, y(s))y(s)||ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||A(s, x(s))x(s) - A(s, y(s))y(s)||ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))||ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))||ds \\ &+ \sum_{0<t_k<t} ||I_k x(t_k^-) - I_k y(t_k^-)||\end{aligned}$$

Applying hypotheses (H1)-(H4) and (H6) the result is,

$$\begin{aligned}||Gx - Gy|| \leq g^*||x - y|| &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} M||x - y||ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} M||x - y||ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} \{L_1 + THL_2 + TKL_3\}||x - y||ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ds \{L_1 + THL_2 + TKL_3\}||x - y||ds + \sum_{0<t_k<t} I_k^*||x - y|| \\ &\leq \left\{g^* \frac{T^\alpha}{\Gamma(\alpha+1)} [(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^*\right\}||x - y|| \\ &= \left\{g^* + \gamma[(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^*\right\}||x - y||\end{aligned}$$

Assuming hypotheses (H7), it leads to $||Gx - Gy||_{PC} \leq q^*||x - y||$ with $q^* < 1$. Hence by Banach fixed point theorem the equation (4.1) has unique solution.

Example 4.2.1. *Consider the following fractional integro-*

differential equation with the impulsive condition,

$$
\begin{aligned}
{}^{c}D^{\alpha}x(t) &= \frac{1}{9}cosx(t)x(t) + \frac{1}{(t+3)^4}\frac{|x|}{1+|x|} + \frac{1}{9}\int_0^t se^{\frac{-x(s)}{4}} + \frac{1}{9}\int_0^1 (t-s)e^{-x(s)}ds \\
\Delta x(1/2) &= \frac{|x(1/2^-)|}{18+|x(1/2^-)|} \\
x(0) &= x_0 - \frac{x}{18}
\end{aligned}
\quad (4.4)
$$

where $0 < \alpha < 1$ *over the interval* $[0,1]$.

Since, $A(t,x) = \frac{1}{9}cosxI$ therefore $||A(t,x)x - A(t,y)y|| \leq \frac{1}{9}||(cosxI)x - (cosyI)y|| \leq \frac{1}{9}||x-y||$, $||Tx - Ty|| \leq \frac{1}{9}\int_0^t s||e^{\frac{-x(s)}{4}} - e^{\frac{-y(s)}{4}}|| \leq \frac{1}{36}||x-y||$, $||Sx - Sy|| \leq \frac{1}{9}\int_0^1 |(t-s)||e^{-x(s)} - e^{-y(s)}|||ds \leq \frac{1}{18}||x-y||$ $||f(t,x,Tx,Sx) - f(t,y,Ty,Sy)|| \leq \frac{7}{48}||x-y||$, $||g(x) - g(y)|| \leq \frac{1}{18}||x-y||$ and $q^* = g^* + \gamma\big[(p+1)[M + L_1 + THL_2 + TKL_3]\big] + \sum I_k^* = \frac{1}{18} + \gamma\frac{29}{72} + \frac{1}{18}$. Choose $\alpha = 1/2$ then $q^* = 0.57$ which is less than 1. Therefore by existence theorem the given system has unique solution in the interval $[0,1]$.

V. Remark

1. This method suggests not only the existence and uniqueness about the solution but it also suggests a method to find approximate solution of Volterra-Fredholm kind of impulsive fractional integro-differential equations (3.1) and (4.1).

2. This condition is not a necessary condition which means that the equations (3.1) and (4.1) may have solution if anyone of (H1) to (H7) is not satisfied.

VI. Conclusion

The system taken by Balachandran et. al. [19] is a special case of the system taken in this paper because of inclusion of the nonlinear Fredholm operator in the system which is more relevant in many physical situations including viscoelastic problems.

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