Jacobian-free Newton Multigrid method for Elastohydrodynamic line contact with grease as lubricant

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ABSTRACT
In this article, a Jacobian-free Newton Multigrid (JFNMG) method is used for the analysis of isothermal, steady state and incompressible Elastohydrodynamic lubrication (EHL) line contact problem. The lubricant is grease and its rheology is described by Hershel-Bulkley constitutive equation. The problem is investigated for smooth contact surfaces. A finite difference scheme, developed after discretizing governing equations on a uniform grid, is used for the solution of the EHL problem. The JFNMG method is applied to the system of algebraic equations, which arise in the scheme, and obtain solution for a wide range of parameters of interest. The proposed scheme not only overcomes the limitations of conventional schemes but is also, stable and fast converging for the investigation of the EHL problems. The JFNMG method approximates Jacobian matrix-vector product accurately with much ease without requiring additional storage at each iteration, and avails convergence features of Newton method as well as convergence strategy (up to discretization error) of multigrid. The EHL features, namely, pressure profiles, film shapes and Petrusevich pressure spikes are studied for different parameters of interest. The minimum film thickness decreases for decrease in n, which is a power law index in Hershel-Bulkley model characterizing grease lubricant, with other parameters held constant. There is a noticeable decrease (in height as well as spread) of pressure spike for small values of n, which arrests bearings fatigue life.

Keywords: *Elastohydrodynamic lubrication, Grease lubricant, Jacobian-free, Multigrid, Full approximation scheme, Newton Inexact*

I. INTRODUCTION

Elastohydrodynamic lubrication (EHL) studies encompass the analysis of lubrication problems where solid metal surfaces deform under heavy load (viz. rolling bearings, cams, gears, etc.). Many investigators have contributed in its numerical and experimental studies. Primarily, it is used for determining minimum film thickness, which is an important aspect in the design of bearings in lubrication. Dowson and Higginson [1] presented brief history and early developments of the topic. Commonly, liquid lubrication necessitates a reservoir of liquid for its supply to lubricating surfaces. In the case of oil, it is very inconvenient and impractical. However, the use of grease avoids these difficulties and it becomes a lubricant of choice. But the rich structures of grease as well as mechanism of grease lubrication are of a very complex nature.

1.1 Grease Lubrication

Apart from mineral oil based lubricants, dry lubricants or grease are also commonly used lubricants as it is well known that grease lubricant is able to reduce friction and avoid /delay fatigue life of bearings which have been demonstrated theoretically [2] and confirmed experimentally [3]. In fact, in more than 80% of all rolling bearings, separating the contacting surfaces of the bearings, use grease. The main advantages of grease are that it is easy to use, has excellent sealing effect, protects from corrosion and experiences low friction. Thus, the grease minimizes wear of bearing components, which leads to enhanced bearing life. However, it has limited life, especially when lubricating components are in high speed. Grease acts as reservoir to supply a sufficient quantity of oil to the contact and to sustain a separating lubricant film. The grease material is normally prepared by adding thickening agents to base oil, which are metallic soaps whose fiber constitutes mesh like framework that enables it to store oil. The thixotropic property of grease, owing to its non-Newtonian rheology, is adequately described by the constitutive equation (Hershel-Bulkley model [4]):

$$\tau = \tau_s + \varphi \gamma^n \tag{1.1}$$

where τ_s is yield stress and φ is plastic viscosity, which are functions of pressure and temperature, and $n \leq 1$ is an index which characterizes the rheology of grease. With zero yield stress and index n=1, it is classical Newtonian case. This model depicts closely the experimental findings about rheology of grease.

The EHL theory of oil lubricants is fairly well established whereas that of grease lubricant continues to pose challenges due to the complexity of its rheological properties. The experimental investigations of EHL of grease reveal wide variations in performance that correspond to variations in base oil viscosity, soap type, contacts and inlet lubricant supply. The optical interferometry studies on bearing performance of grease lubricants have shown that the grease film thickness under fully flooded condition is always thicker than the base oil (Wedeven et al. [5], Palacios et al. [6]). The effect of grease composition is analyzed by Kageyama et al. [7] and it is shown that the film thickness increases with higher base oil viscosity and soap concentration in the grease. Astrom et al. [8], Kaneta et al. [9] and others used more accurate optical interferometry of fully flooded grease and also observed that the film thickness is greater than the fully flooded base oil film. Aihora and Dowson [10] measured the EHL line contact film thickness of grease film using magnetic transducer. Astrom et al. [11] performed the ball on disc experiment of grease lubrication and showed that, with increasing number of revolutions, the film thickness decreases, up to a certain value, which may cause damage to bearing surface. The degradation of grease is accompanied by the mechanism of spontaneous recovery of lubricant in the contact called replenishment. In the influential theoretical and experimental work by Kauziorich and Greenwood [12] on fully flooded fresh grease, assuming viscosity of non-Newtonian grease rheology, the empirical formula for the grease film is based on Hershel-Bulkley model and Grubin theory in EHL line contact. Their investigation also shows that, although initially, film thickness was greater than that of the base oil alone, later, it decreases gradually towards a steady state value less than that predicted from base oil alone. Later, Jonkisz and Freda [13] solved the grease EHL line contact problems using same model and reported more accurate form of film shape. In these studies, and later by Boardenet et al. [14], on point contact EHL with grease, the existence of two fluid layers in the inlet is shown. In one layer, shear stress exceeds the yield stress and grease flows as viscous fluid and in another layer, it is a plug flow, wherein the shear stress is smaller than the yield stress. More refined grease models (four parameter model, etc.) were used much earlier by Bauer [15] and, more recently, by Baart et al. [16] and others to obtain film thickness expression as function of soap concentration and temperature. Dong and Qian [17] used Bauer model and presented more refined film thickness formula for grease EHL line contact problems. Cheng [18] studied grease EHL line contact, using Hershel-Bulkley model, and obtained film thickness of fresh and shear grease. Yoo and Kim [19] attempted to show the influence of temperature and grease rheology on film shape of EHL grease line contact using the same model. In the theoretical analysis of grease EHL point contact problem, Karthikeyan et al. [20] showed the influence of thermal effects and speed on the nature of grease flow and film formation. They adapted Newton-Raphson scheme for the solution. The interest in the detailed investigation of pressure spikes is of practical importance as it affects surface fatigue and early failure of rolling element bearings. Hamrock et al. [21] used the modified scheme of Okamura and at the same time Bisset and Glander [22] adapted Newton method (with continuation algorithm) to present very useful findings in this respect covering wide range of loads and other influencing aspects of the problem. Later, Venner and Napel [23] used the advanced multilevel methods (of complexity O(nlog(n)) and partly confirm these observations. Two comprehensive review articles, one by Lugt [24] on grease lubrication in rolling bearings, and another more general review by Lugt and Morales-Espejel [25] on EHL theory, highlight many aspects of these topics in detail. Huang [26] presents the solution of grease line contact problem adopting ad hoc high and low pressure regions for limited parameter values.

In this paper, the objective is to develop a Jacobian-free Newton Multigrid (JFNMG) scheme to solve EHL line contact problem with grease as the lubricant, using Hershel-Bulkley model, and validate its performance with that of the FMG-FAS of multigrid method. The corresponding results of classical Newtonian rheology of lubricants are obtained as a particular case. Although, this method is demonstrated by its application to line contact problem, it can also be used to analyze more involved general problems in EHL. The focus is on exploring newer and more robust numerical methods for the in-depth study of emerging topics which include thermal EHL, starved EHL, roughness of the surface with non-Newtonian lubricants, with soft and hard bearing components, etc.

1.2 Jacobian-Free Newton Multigrid (JFNMG) Method

JFNMG method is a nested iterative scheme for the solution of non-linear system of algebraic equations. The Newton Multigrid method is used for solving the non-linear system of algebraic equations arising from discretization of basic governing equations of an EHL problem. Systems of non-linear equations are of the form

$$F(x) = 0 \tag{1.2}$$

where $F : R^n \to R^n$ is continuously differentiable. The multivariate Taylor expansion about a current point x^k only up to first order gives the Newton scheme, the iterations being over a sequence of linear systems:

$$J(x^k)\delta x^k = -F(x^k) \tag{1.3}$$

$$x^{k+1} = x^k + \delta x^k, \quad k = 0, 1, 2, \ldots$$

for a given x^0, $J = F'$ is Jacobian matrix, k is the non-linear index. To ensure the global convergence, Newton direction δx^k is retained, but the numerical scheme's step is restricted by τ_k, for x^{k+1} to be not going too far, i.e.

$$x^{k+1} = x^k + \tau_k \delta x^k \qquad 0 < \tau_0 < \tau_k \le 1,\ k = 0, 1, \ldots \tag{1.4}$$

which can be achieved using Armijo rule/line search method [27].

1.3 Approximation of Jacobian Matrix-Vector Product

For a scalar problem (1.2), with m+ 1 equation in m-1 unknowns, let

$$F(x) = \{F_1, F_2, F_3, \ldots, F_{m+1}\} \text{ and } x = \{x_1, x_2, x_3, \ldots, x_{m+1}\} \tag{1.5}$$

The exact expression for Jacobian J , in many situations or more so in EHL, is a tough job due to the inherent derivative computation for each of its elements. The Jacobian matrix-vector product (required at each iteration of the equation (1.3)) can be approximated by

$$Jv \cong \frac{[F(x + \in v) - F(x)]}{\in} \tag{1.6}$$

where $\in$ is small and its optimum value is $1.0\text{E}-9$ in 64-bit double precision [28]. The main advantage is that, without forming or storing the true Jacobian, it gives a vector that approximates the matrix-vector multiplication (required at each iteration) by a function evaluation without requiring any storage.

II. GOVERNING EQUATIONS AND OTHER RELATIONS

Assuming a steady, incompressible and isothermal flow, the modified Reynolds equation governing Grease EHL (GEHL) in line contact, in dimensionless form is [26],

$$\frac{d}{dX}\left(\in (P)\left(\frac{dP}{dX}\right)^{\frac{1}{n}} \right) - \frac{dH}{dX} = 0 \tag{2.1}$$

where $\in (P) = \lambda \frac{H(P)^{2+1/n}}{\eta^{1/n}}$, P(X) and H(X) are unknown pressure distribution and film thickness, respectively, η is viscosity with pressure-dependence, λ is a dimensionless speed parameter given as $\lambda = \frac{p_h^{\frac{1}{n}} b^{2+1/n}}{2U\left(2 + \frac{1}{n}\right) R^{(1+1/n)} 2^{1/n} \eta_0^{1/n}}$, b is the half width of the contact area, p_h is the maximum Hertz pressure. The boundary conditions are given as follows.

$$P(X_a) = P(X_b) = \frac{dP(X_c)}{dX} = 0 \tag{2.2}$$

The domain of the problem is (X_a, X_b) with cavitation point at X_c. Zero pressure conditions are imposed at X_a and X_b, while zero pressure gradient at X_c . The dimensionless film thickness equation is given, in integral form, as

$$H(X) = H_{00} + \frac{X^2}{2} - \frac{1}{\pi}\int_{X_a}^{X_b} \log|X - X'|P(X')dX' \tag{2.3}$$

where H_{00} is the central offset film thickness, the second term defines the un-deformed contact shape and the integral term represents the elastic deformation of the contact. The dimensionless force balance equation

$$\int_{X_a}^{X_b} P(X)dX = \frac{\pi}{2} \tag{2.4}$$

represents the balance between the applied load and the total internal pressure in the lubricant. The dimensionless form for viscosity $\eta(P)$, represented by Roelands relationship, is

$$\eta(P) = \exp\left(\frac{\alpha p_0}{z}\left[-1 + \left(1 + \frac{Pp_h}{p_0}\right)^z\right]\right) \tag{2.5}$$

where p_0 is the ambient pressure, p_h is the maximum Hertzian pressure, z is the viscosity index and α is the pressure viscosity index. The three dimensionless physical parameters that characterize the line contact problem are speed (U) load (W) and elasticity (G).

III. DISCRETIZATION OF THE EQUATIONS

The spatial domain $X \in [X_1, X_{N+1}]$ is discretized with a uniform grid of N+ 1 point $X_i \ (1 \le i \le N+1)$. The cavitation point X_c is considered to be located at an unknown internal point $X_j \ (2 \le j \le N)$. The governing equations are discretized using standard techniques. The finite difference approximation of the Reynolds equation (2.1), with mesh length $dx = \dfrac{X_b - X_a}{N}$, is

$$\frac{(\varepsilon_i + \varepsilon_{i+1})(P_{i+1} - P_i)^{1/n} - (\varepsilon_i + \varepsilon_{i-1})(P_i - P_{i-1})^{1/n}}{dx^{1+1/n}} - \frac{H_i - H_{i-1}}{dx} = 0 \tag{3.1}$$

where $\varepsilon_i = \lambda \dfrac{H(P_i)^{2+1/n}}{\eta(P_i)^{1/n}}$.

The film thickness equation (2.3) approximated at X_i on the regular grid is given by

$$H(X_i) = H_{00} + \frac{X_i^2}{2} - \frac{1}{\pi}\sum_{j=1}^{N+1} K_{ij}P(X_j) \tag{3.2}$$

where

$$K_{ij} = \left(X_i - X_j + \frac{dx}{2}\right)\left(\log|X_i - X_j + \frac{dx}{2}| - 1\right) - \left(X_i - X_j - \frac{dx}{2}\right)\left(\log|X_i - X_j - \frac{dx}{2}| - 1\right) \tag{3.3}$$

for $i = 1,2,..N+1$ and $j = 1,2,..N+1$,

and the force balance equation (2.4) in discrete from is

$$dx\sum_{j=1}^{N}\left(\frac{P_j + P_{j+1}}{2}\right) - \frac{\pi}{2} = 0 \tag{3.4}$$

The boundary conditions (2.2) are utilized in (3.1), (3.2) and (3.4), in order to solve these equations in a more general setup using Newton multigrid method. The nature of the Jacobian matrix (diagonally dominant, dense but smooth), associated with the non-linear system of algebraic equations arising from the above discretization process, poses the challenge for obtaining more effective fast converging accurate solutions.

IV. NUMERICAL IMPLEMENTATION

The discretized system of equations (3.1)-(3.2) and (3.4) is solved using Jacobian-free Newton multigrid (JFNMG) method. For each Newton iteration (outer loop), there is a linear system to be solved using multigrid (inner loop). The required Jacobian matrix-vector product is computed using relation (1.6). With this, the pressure-viscosity and pressure-density relations are used in the Jacobian-free computation with much ease. Once accurate Newton direction (solution from inner loop) is obtained, later outer loop restricts the Newton step, using line search/Armijo rule to obtain accurate global converging solution. The solution scheme comprises the following steps.

Step 1: Take initial values of P, X_c and H_{00}

Step 2: Evaluate H from film thickness equation (3.2)

Step 3: Solve Reynolds equation (3.1) for P

Step 4: Use numerical under relaxation for stability and convergence of iteration

$$P_{n+1}^{new} = P_n^{old} + C_1(P_n^{new\ predictor} - P_n^{old})$$

Step 5: Update H_{00} using force balance equation (3.4) and $H_{00} = H_{00} + R_1 c_2$, where R_1 is residual of the force balance equation

Step 6: Fix the cavitation point X_c using the condition $\frac{dP}{dX} = 0$ at $\mathrm{X} = \mathrm{X_c}$

Step 7: While not converged, go to Step 2

Hertz's pressure distribution is used as initial approximation for pressure P. For finding optimal values of the under relaxation parameters c_1 and c_2, an elegant procedure due to Durbin and Delemos [29] is used. The numerical values of these constants are in the range $0\langle c_1, c_2 \langle 1$ for different sets of physical parameters. Gauss-Seidel relaxation is used to solve the linear system and, for the termination of iteration (inner loop), the error tolerance value used is 1.0E-06. At the Step 4, for the solution of non-linear system of equations (3.1) for *P*, the inexact Newton method [27] is employed. For locating the cavitation point, the criterion given in [30] is used. The choice of the stopping criteria for linear system is guided by the behavior of the solution of the non-linear system at the previous Newton iteration. To optimize the cost of implementation and ensure accurate solution a constant tolerance of $1.0E-6$ is taken in our computations. Once accurate solution of linear system is obtained, inexact Newton scheme (requiring just 4-5 iterations) enables in finding solution for pressure P and film thickness H accurately with error norm less than $1.0E-6$, using Roelands pressure-viscosity relation. These computations are repeated for all sets of physical parameters. All the computations are done using MATLAB in double precision.

V. RESULTS AND DISCUSSIONS

Grease, with its rheological behavior described by Hershel-Bulkley model, is used as lubricant. The governing equations of isothermal incompressible steady GEHL line contact are discretized on a uniform grid. Roelands viscosity-pressure relation is used in the analysis. The detailed study is for smooth surface. The main focus is to show the potential features of the proposed numerical scheme Jacobian-free Newton multigrid method in obtaining solution of non-linear system of algebraic equations, resulting from finite difference form of the GEHL equations for wide range of parameters of interest. In fact, the proposed JFNMG method inherits convergence features of Newton methods and overcomes the limitations of classical schemes (in dealing with ill-conditioned large system of equations).Similar observations are made by Briggs et al. [31] (pg: 105-106, on Newton based methods) while illustrating various features of Newton multigrid methods by considering representative non-linear equations. But, in the absence of concrete theory, this claim cannot be made general.

As mentioned in the section 1.1, Grease forms an important component as lubricant in most of the machine elements. Its performance as a lubricant compared with Newtonian lubricant is presented in detail. Computed values of pressure profiles, film shapes and their corresponding spreads are shown in the Figures 1(a) and 1(b), respectively, for different values of index n representing constitutive property of grease. Minimum value of film thickness increases with increasing n and corresponding pressure profiles have slight changes (compared with Newtonian case). Petrusevich pressure spike decreases for smaller values of n. The pressure profiles, spread of pressure spikes and associated film thickness, its spread are given in the Figures 1(a) and 1(b), respectively, for the specific parameters of high-load and low speed (W=4.0E-05, U=6.0E-11) and for different values of n. Figures 2(a) and 2(b) present the pressure profiles, associated film thickness and their spread, respectively, for low-load and high-speed (W=2.0E-05, U=6.0E-10) parameters for different values of n. For smaller values of n, the conventional Newton-Raphson scheme does not converge, whereas the present method works up to much smaller values of n (convergence verified up to n=0.78). For decrease in the value of n, the pressure spike shrinks (almost disappears) and their locations move towards outlet. This is one of the reasons in the choice of grease as lubricant in most of the heavily loaded bearings. As n decreases, the number of mesh points required for converging solution increases. But in our computations, it is sufficient to take N=129 grid points. In addition, the required grid size is a function of load parameter (W) which increases with increase in W for a fixed n. In all the cases considered here, convergence of the scheme for P and H is achieved with N=129 grid points. Other conventional schemes, Newton-Raphson as well as Multigrid FAS, are either unstable or require much finer grids to simulate the GEHL characteristics. In [26], the author adapts Gauss-Seidel iteration in low pressure region and Jacobi iteration in high pressure region (using ad hoc procedure of selecting these regions) and reports useful findings up to n=0.8 only and has presented pressure spike and film shape just for one set of parameters whereas in our computation both high load-low speed and low load-high speed cases are considered. In the Table, details of sensitivity of convergence/non-convergence of the present scheme are given for the smooth surface with grease as lubricant for varying rheological parameter n. The method converges even for much smaller values of n requiring more number of iterations and much finer grids.

VI. CONCLUSIONS

A novel JFNMG scheme is used for the analysis of lubrication characteristics and also to analyze Petrusevich pressure spike in EHL line contact with grease as lubricant. This work does not offer comprehensive numerical investigation of all EHL systems, but it is a novel scheme whose strength is demonstrated herein by applying it to representative GEHL problem. While only a incompressible, steady isothermal EHL line contact problem is considered for study, the proposed method could equally be used for the analysis of much more involved EHL problems. The focus here is to assert stable and converging features (compared with conventional schemes) of JFNMG method in addressing more involved EHL problems. The analysis reveals decrease in minimum film thickness with decreasing n, a parameter characterizing grease lubricant. The pressure spikes almost diminish, which control fatigue life of bearings, for smaller values of n. The thermal, transient, point contact and other more involved EHL problems can be attempted using proposed numerical scheme.

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NOMENCLATURE	
b half width of the Hertzian contact, $b = 4R\sqrt{W/2\pi}$ (m)	U dimensionless speed parameter, $U = (\eta_0 u_s)/(E'R)$
E' reduced modulus of elasticity, (Pa)	u_s average rolling velocity,
G dimensionless materials parameter, $\alpha E'$	$u_s = (u_1 + u_2)/2$, (m/s)
H dimensionless film thickness, hR/b^2	u_1, u_2 velocities of lower and upper surfaces, (m/s)
h film thickness, (m)	W dimensionless load parameter, $W = w/(E'R)$
$\bar{h}$ dimensionless film thickness, $h/R(2U)^{-1/2}$	w external load per unit length, (N/m)
H_{00} dimensionless offset film thickness, (m)	x_c dimensionless location of pressure spike
K_{ij} discrete approximation of K-logarithmic kernel	X dimensionless coordinate, x/b
N number of nodes on grid	x co-ordinate
n Grease parameter	dx mesh size
P dimensionless pressure, p/P_h	z pressure viscosity parameter
p pressure, (Pa)	η Newtonian viscosity
p_h maximum Hertzian pressure,	η_0 viscosity at ambient pressure
$p_h = E'b/4R$, (Pa)	$\bar{\eta}$ dimensionless viscosity, η/η_0
R reduced radius of curvature, (m)	

FIGURES

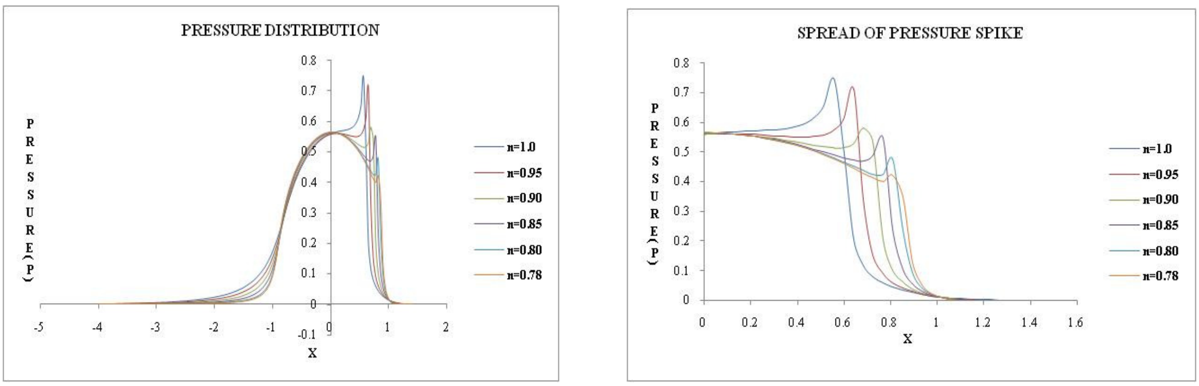

Figure 1(a) Comparison of pressure distribution and spread of pressure spike for fixed high-load (W), low-speed (U), G and varying rheological parameter n

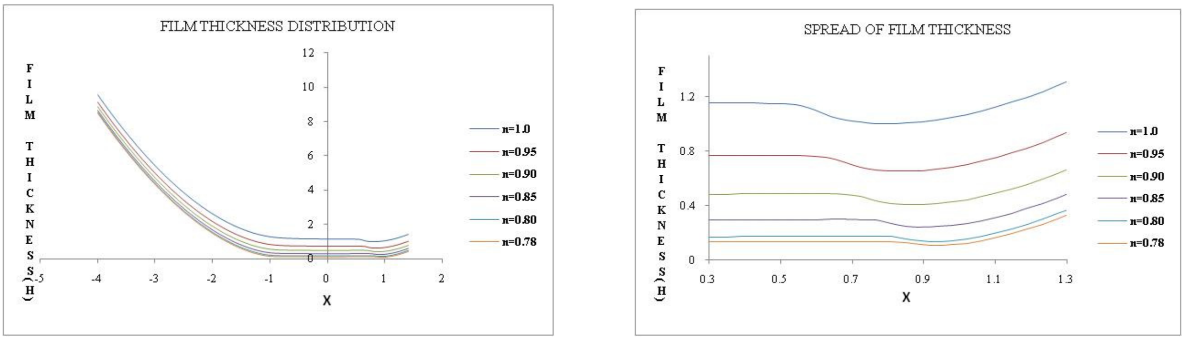

Figure 1(b) Comparison of film profile and its spread for fixed high-load (W), low-speed (U), G and varying rheological parameter n

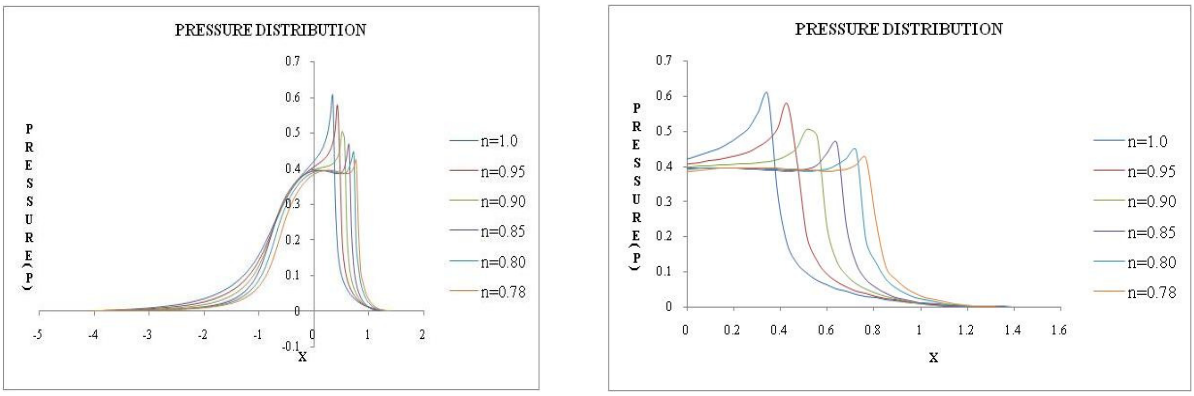

Figure 2(a) Comparison of pressure distribution and spread of pressure spike for fixed low-load (W), high-speed (U), G and varying rheological parameter n

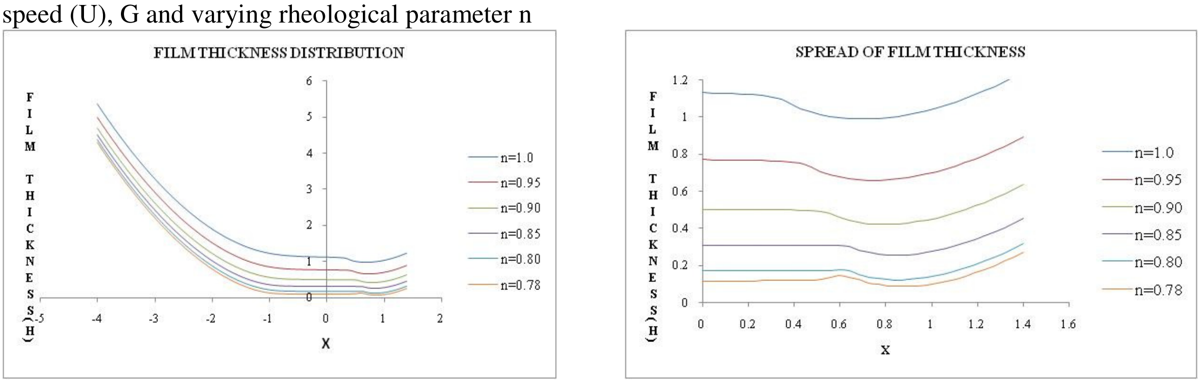

Figure 2(b) Comparison of film profile and its spread for fixed low-load (W), high-speed (U), G and varying rheological parameter n

TABLE

The sensitivity of convergence of the scheme for different physical parameters and varying rheological parameter n

Input	n	N	W	U	G	Convergence	
						Present Method	[26]
Set-I	1.0	129	4.0E-5	6.0E-11	6898	Yes	Yes
	0.95					Yes	Yes
	0.90					Yes	Yes
	0.85					Yes	Yes
	0.80					Yes	Yes
	0.78					Yes	No
Set-II	1.0	129	2.0E-5	6.0E-10	6898	Yes	-
	0.95					Yes	-
	0.90					Yes	-
	0.85					Yes	-
	0.80					Yes	-
	0.78					Yes	-