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On An Optimal control Problem for Parabolic Equations

M. H. FARAG[1,2], T. A. NOFAL[1,2], A. I. EL-NASHAR[3,4] and N. M. AL-BAQMI[5]
[1]Mathematics Department, Faculty of Science, Taif University, Hawia (888), Taif, KSA
[2]Mathematics Department, Faculty of Science, Minia University, Mina, Egypt
[3]Department of Inf. Tech., Coll. of Computers and Inf. Tech., Hawia (888), Taif, KSA
[4]Department computer sciences, Faculty of Science, Minia University, Mina, Egypt
[5]College of Applied Medical Sciences, Taif University, Torba, KSA

ABSTRACT:
Consideration was given to the problem of optimal control of parabolic equations. The existence solution of the considering optimal control parabolic problem is proved. The gradient of the cost functional by the adjoint problem approach is obtained. Lipschitz continuity of the gradient is derived.

Keywords: *Optimal control problem, parabolic Equations, Existence solution, Fréchet gradient, Adjoint problem, Lipschitz continuity.*

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

The optimal control problems governed by partial differential equations have developed very fast in the last 30 years, and it has brought a promising and vital researching domain to the subject of mathematics. The optimal control problems governed by partial differential equations concern many applications in physics, chemistry, biology, etc., such as materials design, crystal growth, temperature control, petroleum exploitation, and so on. The relative details can be seen in [1-4], and so on. The partial differential equations involved in these problems include elliptic equations, parabolic equations and hyperbolic equations [5-7]. The reference [1] considered the problem of optimization of the optimal control parabolic problem with control in boundary and right hand side of the equation. In the present paper, the control action in the initial and boundary conditions of the considering optimal control parabolic problem.

Let the considered process be described in $\Omega_T = \{(x,t): x \in (0,l)\ ,\ 0 < t < T\}$ by the following problem:

$$\begin{cases} u_t = (\lambda(x)\, u_x)_x + v_o(x,t), & (x,t) \in \Omega_T\,, \\ u(x,0) = v_1(x)\,, & x \in (0,l), \\ u_x(0,t) = 0\ ,\ -\lambda(l)\, u_x(l,t) = k\,[u(l,t) - v_2(t)] & t \in (0,T]. \end{cases} \qquad (1)$$

where $u(x,t)$ is the solution of the system (1), the constant $k > 0$ is called the convection coefficient or heat transfer coefficient , $\lambda(x) > 0$, $\lambda(x) \in L_\infty[0,l]$ and $V = V_\circ \times V_1 \times V_2$ is the set of admissible controls where $V = \{v = (v_o(x,t), v_1(x), v_2(t)) : v_o(x,t) \in L_2(\Omega_T), v_1(x) \in L_2(0,l), v_2(t) \in L_2(0,T)\}$ is a closed and convex subset in $V \subset L_2(\Omega_T) \times L_2[0,l] \times L_2[0,T]$.

The problem of optimal control lies in determining admissible controls $v \in V$ minimizing together with the corresponding generalized solution of problem (1) the functional

$$f_\alpha(v) = \beta \int_0^l [u(x,T) - z(x)]^2\, dx + \alpha \int_0^T [v_2(t) - w(t)]^2\, dt \qquad (2)$$

In (2), $z(x)$ and $w(t)$ are given functions, respectively, from $L_2[0,l]$ and $L_2[0,T]$, T is a fixed time and β, α are given positive numbers.

II. EXISTENCE SOLUTION OF THE OPTIMAL CONTROL PROBLEM

In this section, we give the definition of the weak solution of the problem (1) and the existence solution of the optimal control problem (1)-(2).

The weak solution of the problem (1) will be defined as the function $u \in L_2(\Omega_T)$, which satisfies the following integral identity:

$$\int_0^l u(x,T)\,\zeta(x,T)\,dx - \int_0^l v_1(x)\,\zeta(x,0)\,dx - \iint_{\Omega_T}[u(x,t)\,\zeta_t(x,t) - \lambda(x)\,\zeta_x(x,t)\,u_x(x,t)]\,dxdt$$

$$-k\int_0^T [u(l,t) - v_2(t)]\zeta(l,t)dt = \iint_{\Omega_T} v_0(x,t)\,\zeta(x,t)\,dxdt \quad , \ \forall\ \zeta(x,t) \in L_2(\Omega_T). \tag{3}$$

Evidently, under the above conditions with respect to the given data, the weak solution $u \in L_2(\Omega_T)$ of the direct problem (1) exists and unique [8,9].

We define a solution of the optimal control problem (1)–(2), according to [10], as a solution of the minimization problem for the cost functional $f_\alpha(v)$, given by (2):

$$f_\alpha(v_*) = \inf_{v\in V} f_\alpha(v) \tag{4}$$

Evidently, if $f_\alpha(v_*) = 0$, then the solution $v_* \in V$ is also a strict solution of the optimal control problem (1)–(2), since $v_* \in V$ satisfies the functional equation $u(x,t;v)\big|_{t=T} = z(x)$, $x \in (0,l)$. Further, in the view of the weak solution theory for parabolic problems, one can prove that if the sequence $\{v^{(n)}\} \subset V$ weakly converges to the function $v \in V$, then the sequence of traces $\{u(x,T;v^{(n)})\}$ of corresponding solutions of problem (1) converges in $L_2(\Omega_T)-norm$ to the solution $\{u(x, T; v)\}$, which means $f_\alpha(v^{(n)}) \to f_\alpha(v)$, as $n\to\infty$. This means the functional $f_\alpha(v)$ is weakly continuous on V, hence due to the Weierstrass existence theorem [11] the set of solutions $V_* = \{v \in V : f_\alpha(v_*) = (f_\alpha)_* = \inf f_\alpha(v)\}$ of the minimization problem (2) is not an empty set.

III. FRÉCHET DIFFERENTIABILITY OF THE COST FUNCTIONAL AND ITS GRADIENT

The principal result in this section is Theorem 3.1. Its proof will be prepared by two lemmas.
Let us consider the first variation

$$\Delta f_\alpha(v) = f_\alpha(v+\Delta v) - f_\alpha(v) = 2\beta\int_0^l [u(x,T;v) - z(x)]\,\Delta u(x,T;v)\,dx + \beta\int_0^l [\Delta u(x,T;v)]^2\,dx$$

$$+\alpha\int_0^T [\Delta v_2(t)]^2\,dt + 2\alpha\int_0^T [v_2(t) - w(t)]\,\Delta v_2(t)\,dt \tag{5}$$

of the cost functional (2), where

$$v + \Delta v = \{v_0(x,t) + \Delta v_0(x,t), v_1(x) + \Delta v_1(x), v_2(t) + \Delta v_2(t)\} \in V$$

$$\Delta u(x,t;v) = u(x,t,v+\Delta v) - u(x,t,v) \in L_2(\Omega_T).$$

Evidently the function $\Delta u = \Delta u(x,t;v)$ is the solution of the following parabolic problem

$$\begin{cases} \Delta u_t = (\lambda(x)\,\Delta u_x)_x + \Delta v_0(x,t) & , \quad (x,t)\in\Omega_t \\ \Delta u(x,0) = \Delta v_1(x) & , \quad x\in(0,l) \\ \Delta u_x(0,t) = 0 \quad , \; -\lambda(l)\Delta u_x(l,t) = k[\Delta u(l,t) - \Delta v_2(t)] & , \quad t\in(0,T] \end{cases} \tag{6}$$

Lemma 3.1.

Let $v = \{v_0(x,t), v_1(x), v_2(t)\}$, $v+\Delta v = \{v_0(x,t)+\Delta v_0(x,t), v_1(x)+\Delta v_1(x), v_2(t)+\Delta v_2(t)\}\in V$ be given elements. If $u = u(x,t;v)\in L_2(\Omega_T)$ is the corresponding solution of the direct problem (1) and $\psi(x,t;v)\in L_2(\Omega_T)$ is the solution of the adjoint parabolic problem

$$\begin{cases} \psi_t = -(\lambda(x)\,\psi_x)_x & , \quad (x,t)\in\Omega_t \\ \psi(x,T) = 2\beta\,[u(x,T;v) - z(x)] & , \quad x\in(0,l) \\ \psi_x(0,t) = 0 \quad , \; -\lambda(l)\,\psi_x(l,t) = k\,\psi(l,t) & , \quad t\in(0,T] \end{cases} \tag{7}$$

then for all v ∈ V the following integral identity holds:

$$2\beta\int_0^l [u(x,T;v) - z(x)]\,\Delta u(x,T;v)\,dx = \int_0^l \psi(x,0;v)\,\Delta v_1(x)\,dx$$

$$+\iint_{\Omega_T}\psi(x,t;v)\,\Delta v_0(x,t)\,dx\,dt + k\int_0^T \psi(l,t;v)\,\Delta v_2(t)\,dt \quad , \quad \forall v\in V \tag{8}$$

Proof: Let us use the final condition at $t = T$ in (7) to transform the left-hand side of (8) as follows:

$$2\beta\int_0^l [u(x,T;v) - z(x)]\,\Delta u(x,T;v)\,dx$$

$$= \int_0^l \psi(x,T;v)\,\Delta u(x,T;v)\,dx$$

$$= \iint_{\Omega_T}[\psi(x,t;v)\,\Delta u(x,t;v)]_t\,dx\,dt$$

$$= \iint_{\Omega_T}[\psi_t(x,t;v)\,\Delta u(x,t;v) + \psi(x,t;v)\,\Delta u_t(x,t;v)]\,dx\,dt$$

$$= \iint_{\Omega_T}[-(\lambda(x)\,\psi_x(x,t;v))_x\,\Delta u(x,t;v) + \psi(x,t;v)\,(\lambda(x)\,\Delta u_x(x,t;v))_x]\,dx\,dt$$

$$+\iint_{\Omega_T} v_0(x,t)\,\psi(x,t;v)\,dx\,dt$$

$$= \int_0^T [-\lambda(x)\,\psi_x(x,t;v)\,\Delta u(x,t;v) + \psi(x,t;v)\,\lambda(x)\Delta u_x(x,t;v)]_{x=0}^{x=l}\,dt$$

$$+\iint_{\Omega_T} v_0(x,t)\psi(x,t;v)\,dx\,dt\;.$$

Taking into account the boundary conditions in (7) and (8) for the functions $\psi(x,t;v)$ and $\Delta u(x,t;v)$; we obtain (8).

We will define the parabolic problem (7) as an adjoint problem, corresponding to the inverse problem (1)–(2). The parabolic equation (7) is a backward one, and due to the "final condition" at t = T it is a well-posed initial boundary-value problem under a reversal of time.

Now we use the integral identity (8) on the right-hand side of formula (5) for the first variation of the cost functional $f_\alpha(v)$. Then we have

$$\Delta f_\alpha(v) = \iint_{\Omega_T} \Delta v_0(x,t)\psi(x,t;v)\,dx\,dt + k\int_0^T \psi(l,t;v)\Delta v_2(t)\,dt + \int_0^l \psi(x,0;v)\,\Delta v_1(x)\,dx$$

$$+\,2\alpha\int_0^T [v_2(t) - w(t)]\Delta v_2(t)\,dt + \alpha\int_0^T [\Delta v_2(t)]^2\,dt + \beta\int_0^l [\Delta u(x,T;v)]^2\,dx \qquad (9)$$

Taking into account the above definition of the scalar product in V and the definition of the Fréchet-differential we need to transform the right-hand side of (9) into the following form:

$$\Delta f_\alpha(v) = \left\langle f_\alpha^{/}(v), \Delta v\right\rangle_V + \alpha\int_0^T [\Delta v_2(t)]^2\,dt + \beta\int_0^l [\Delta u(x,T;v)]^2\,dx \qquad (10)$$

This formula provides further insight into the gradient of the functional $f_\alpha(v)$ via the solution of the adjoint parabolic problem (7). Due to the definition of the Fréchet-differential, we need to show that the last two terms on the right-hand side of (10) are of order $O\left(\|v\|_V^p\right)$, with $p \ge 1$.

The following result precisely shows an estimate for the last two terms in (1) in order $O\left(\|v\|_V^2\right)$.

Lemma 3.2.

Let $\Delta u = \Delta u(x,t;v) \in L_2(\Omega_T)$ be the solution of the parabolic problem (6) corresponding to a given $v \in V$. Then the following estimate holds:

$$\int_0^l [\Delta u(x,T;v)]^2\,dx \le \frac{c_0}{\varepsilon}\|\Delta v\|_V^2 \quad , \quad \forall\ \Delta v \in V \qquad (11)$$

where $\|\Delta v\|_V = \left[\iint_{\Omega_T} |\Delta v_0(x,t)|^2\,dx\,dt + \int_0^l |\Delta v_1(x)|^2\,dx + \int_0^T |\Delta v_2(t)|^2\,dt\right]^{1/2}$

is the norm $L_2(\Omega_T) - norm$ of the function $\Delta v \in V$, and the constants c_0, $\varepsilon > 0$ are defined as follows:

$$c_0 = \max\{1, k\} \quad , \quad \lambda_* = \min_{0\le x\le l}\lambda(x) > 0 \quad , \quad \varepsilon = \min\left\{\frac{l^2}{\lambda_*}, \frac{2k}{k+2l}\right\} \qquad (12)$$

Proof.

Multiplying both sides of the parabolic equation (10) by Δu, integrating on Ω_T,

$\Delta u\,\Delta u_t = \frac{1}{2}\left[\Delta u^2\right]_t$, $(\lambda(x)\Delta u_x)_x\Delta u = (\lambda(x)\Delta u_x\,\Delta u)_x - \lambda(x)(\Delta u_x)^2$ and using the initial and boundary conditions we obtain:

$$0 = \iint_{\Omega_T} \Delta u\left[\Delta u_t - (\lambda(x)\,\Delta u_x)_x - \Delta v_0(x,t)\right]dx\,dt$$

$$= \frac{1}{2}\iint_{\Omega_T} \left[\Delta u^2\right]_t \, dt \; dx + \iint_{\Omega_T} \left(\lambda(x)\Delta u_x \,\Delta u\right)_x dx \; dt + \iint_{\Omega_T} \lambda(x)(\Delta u_x)^2 dx \; dt + \iint_{\Omega_T} v_0(x,t)\Delta u(x,t) \, dx \; dt$$

$$= \frac{1}{2}\int_0^l \left[\Delta u(x,T;v)\right]^2 dx + k\int_0^T \left[\Delta u(l,t;v)\right]^2 dt - k\int_0^T \Delta u(l,t;v)\Delta v_2(t) dt$$

$$+ \iint_{\Omega_T} \lambda(x)(\Delta u_x(x,t;v))^2 dx \; dt - \iint_{\Omega_T} v_0(x,t)\Delta u(x,t;v) dx \; dt - \frac{1}{2}\int_0^l \left[\Delta v_1(x)\right]^2 dx$$

This implies the following energy identity

$$\frac{1}{2}\int_0^l \left[\Delta u(x,T;v)\right]^2 dx + k\int_0^T \left[\Delta u(l,t;v)\right]^2 dt + \iint_{\Omega_T} \lambda(x)(\Delta u_x)^2 dx \; dt$$
$$= \iint_{\Omega_T} v_0(x,t)\Delta u(x,t;v) dx \; dt + k\int_0^T \Delta u(l,t;v)\Delta v_2(t) dt + \frac{1}{2}\int_0^l \left[\Delta v_1(x)\right]^2 dx \qquad (13)$$

for the solution $\Delta u = \Delta u(x,t;v)$ of the parabolic problem (13). We use the ε-inequality $\alpha\beta \le \frac{\varepsilon\alpha^2}{2} + \frac{\beta^2}{2\varepsilon}, \forall \alpha,\beta \in R, \; \forall \varepsilon > 0$, on the right-hand side integrals of this identity. Then we have

$$\iint_{\Omega_T} v_0(x,t)\Delta u(x,t;v) dx \; dt + k\int_0^T \Delta u(l,t;v)\Delta v_2(t) dt + \frac{1}{2}\int_0^l \left[\Delta v_1(x)\right]^2 dx$$

$$\le \frac{\varepsilon}{2}\iint_{\Omega_T} \left[\Delta u(x,t;v)\right]^2 dxdt + \frac{1}{2\varepsilon}\iint_{\Omega_T} \left[\Delta v_0(x,t)\right]^2 dxdt + \frac{k\varepsilon}{2}\int_0^T \left[\Delta u(l,t;v)\right]^2 dt$$

$$+ \frac{k}{2\varepsilon}\int_0^T \left[\Delta v_2(t)\right]^2 dt + \frac{1}{2}\int_0^l \left[\Delta v_1(x)\right]^2 dx \quad , \quad \forall \varepsilon > 0 \qquad (14)$$

Further, we estimate the term $\left[\Delta u(x,t)\right]^2$ by applying the Cauchy inequality,

$$\left[\Delta u(x,t)\right]^2 = \left[\int_x^l \Delta u_\xi(\xi,t;v) d\xi - \Delta u(l,t;v)\right]^2$$

$$\le 2\left(\int_x^l \Delta u_\xi(\xi,t;v) d\xi\right)^2 + 2\left(\Delta u(l,t;v)\right)^2$$

$$\le 2l\int_0^l \left[\Delta u_x(x,t;v)\right]^2 dx + 2\left(\Delta u(l,t;v)\right)^2$$

Now integrate the both sides of this inequality on Ω_T

$$\iint_{\Omega_T} \left[\Delta u(x,t;v)\right]^2 dxdt \le 2l^2 \iint_{\Omega_T} \left[\Delta u_x(x,t;v)\right]^2 dxdt + 2l\int_0^T \left[\Delta u(l,t;v)\right]^2 dt \qquad (15)$$

and use this estimate on the right-hand side of (14):

$$\iint_{\Omega_T} v_0(x,t)\,\Delta u(x,t;v)\,dx\,dt + k\int_0^T \Delta u(l,t;v)\,\Delta v_2(t)\,dt + \frac{1}{2}\int_0^l [\Delta v_1(x)]^2\,dx$$

$$\le \varepsilon\, l^2 \iint_{\Omega_T} [\Delta u_x(x,t;v)]^2\,dxdt + (\varepsilon\, l + \frac{k\varepsilon}{2})\int_0^T [\Delta u(l,t;v)]^2\,dt + \frac{1}{2\varepsilon}\iint_{\Omega_T} [\Delta v_0(x,t)]^2\,dxdt$$

$$+ \frac{k}{2\varepsilon}\int_0^T [\Delta v_2(t)]^2\,dt + \frac{1}{2}\int_0^l [\Delta v_1(x)]^2\,dx$$

This inequality with (14) implies

$$(\lambda_* - \varepsilon\, l^2)\iint_{\Omega_T} [\Delta u_x(x,t;v)]^2\,dxdt + (k - \varepsilon\, l - \frac{k\varepsilon}{2})\int_0^T [\Delta u(l,t;v)]^2\,dt + \frac{1}{2}\int_0^l [\Delta u(x,T;v)]^2\,dx$$

$$\le \frac{1}{2\varepsilon}\iint_{\Omega_T} [\Delta v_0(x,t)]^2\,dxdt + \frac{k}{2\varepsilon}\int_0^T [\Delta v_2(t)]^2\,dt + \frac{1}{2}\int_0^l [\Delta v_1(x)]^2\,dx \qquad (16)$$

Requiring the positivity of the terms $k - \varepsilon l - \frac{k\varepsilon}{2}$ and $\lambda_* - \varepsilon l^2$ we get bound (12) for the parameter $\varepsilon > 0$. With this parameter $\varepsilon > 0$, from estimate (16) finally we obtain:

$$\frac{1}{2}\int_0^l [\Delta u(x,T;v)]^2\,dx \le \frac{1}{2\varepsilon}\iint_{\Omega_T} [\Delta v_0(x,t)]^2\,dxdt + \frac{k}{2\varepsilon}\int_0^T [\Delta v_2(t)]^2\,dt + \frac{1}{2}\int_0^l [\Delta v_1(x)]^2\,dx$$

The required estimate (13) follows from this inequality by choosing the constant $c_0 > 0$ as in (16), which completes the proof.

The lemmas 3.1, 3.2 imply that the last integral in (10) is bounded by the term $O\left(\|\Delta v\|_V^2\right)$. Thus by the definition of Fréchet-differential at $v \in V$

$$\Delta f_\alpha(v) = \left\langle f_\alpha^{/}(v), \Delta v\right\rangle_V + \alpha\int_0^T [\Delta v_2(t)]^2\,dt + \beta\int_0^l [\Delta u(x,T;v)]^2\,dx \quad ,$$

We obtain the following theorem:-

Theorem 3.1.

Let conditions in the considered problem hold. Then the cost functional is Fréchet-differentiable, $f_\alpha(v) \in C^1(V)$. Moreover, Fréchet derivative at $v \in V$ of the cost functional $f_\alpha(v)$ can be defined by the solution $\psi \in W_2^{1,0}(\Omega_T)$ of the adjoint problem (7) as follows:

$$f_\alpha^{/}(v) = \{\, \psi(x,t;v),\ \psi(x,0;v);\ k\ \psi(l,t;v)\} \qquad (17)$$

IV. LIPSCHITZ CONTINUITY OF THE GRADIENT

Any gradient method for the minimization problem (4) requires an estimation of the iteration parameter $\alpha_k > 0$ in the iteration process

$$v^{(n+1)} = v^{(n)} - \alpha_n f'_\alpha\left(v^{(n)}\right) \qquad , \qquad \text{n=0,1,2….} \tag{18}$$

where $v^{(0)} \in V$ is a given initial iteration. Choice of the parameter α_k defines various gradient methods [12], although in many situation estimations of this parameter is a difficult problem. However, in the case of Lipschitz continuity of the gradient $f'_\alpha(v)$ the parameter α_n can be estimated via the Lipschitz constant as follows:

$$0 < \delta_0 \le \alpha_n \le 2/(L + 2\delta_1) \tag{19}$$

where δ0, δ1 > 0 are arbitrary parameters.

Lemma 4.1.

Let conditions of Theorem 3.1 hold. Then the functional $f_\alpha(v)$ is of Hölder class $c^{1,1}(V)$ and

$$\left\| f'_\alpha(v + \Delta v) - f'_\alpha(v) \right\|_V \le L \left\| \Delta v \right\|_V , \tag{20}$$

where

$$\left\| f'_\alpha(v + \Delta v) - f'_\alpha(v) \right\|_V^2 = \iint_{\Omega_T} (\Delta \psi(x,t;v))^2 dxdt + k^2 \int_0^T (\Delta \psi(l,t;v))^2 dt$$

$$+ \left(\frac{l^2}{\lambda_*} + \frac{l}{k} + \frac{k}{2}\right) \int_0^l (\Delta \psi(x,0;v))^2 dx \tag{21}$$

and the Lipschitz constant $L > 0$ is defined via the parameters c_0, ε >0 in (16) as follows:

$$L = 4 \sqrt{\frac{\beta c_0}{\varepsilon} \left(\frac{l^2}{\lambda_*} + \frac{l}{k} + \frac{k}{2} \right)} > 0 \tag{22}$$

Proof.

The function $\Delta \psi(x,t;v) = \psi(x,t;v + \Delta v) - \psi(x,t;v) \in W^{1,0}(\Omega_T)$ is the solution of the following backward parabolic problem:

$$\begin{cases} \Delta \psi_t = (-\lambda(x) \Delta \psi_x)_x & , \quad (x,t) \in \Omega_t \\ \Delta \psi(x,T) = 2\beta \, \Delta u(x,T;v) & , \quad x \in (0,l) \\ \Delta \psi_x(0,t) = 0 \quad , \quad -\lambda(l) \Delta \psi_x(l,t) = k \, \Delta \psi(l,t) & , \quad t \in (0,T] \end{cases} \tag{23}$$

Multiplying both sides of Eq. (23) by $\Delta \psi(x,t;v)$, integrating on Ω_T and using the initial and boundary conditions, as in the proof of Lemma 3.2, we can obtain the following energy identity:

$$\iint_{\Omega_T} \lambda(x) [\Delta \psi_x(x,t;v)]^2 dxdt + k \int_0^T [\Delta \psi(l,t;v)]^2 dt + \frac{1}{2} \int_0^l [\Delta \psi(x,0;v)]^2 dx$$

$$= 2\beta \int_0^l [\Delta u(x,T;v)]^2 dx \tag{24}$$

This identity implies the following two inequalities:

$$\begin{cases} \lambda_* \iint_{\Omega_T} [\Delta \psi_x(x,t;v)]^2 dx\, dt + \frac{1}{2} \int_0^l [\Delta \psi(x,0;v)]^2 dx \le 2\beta \int_0^l [\Delta u(x,T;v)]^2 dx \\ k \int_0^T [\Delta \psi(l,t;v)]^2 dt + \frac{1}{2} \int_0^l [\Delta \psi(x,0;v)]^2 dx \le 2\beta \int_0^l [\Delta u(x,T;v)]^2 dx \end{cases}$$

Multiplying the first and the second inequality by $\frac{2\, l^2}{\lambda_*}$ and $\frac{2\, l}{k}$, correspondingly, summing up them, and then using the inequality (15) we obtain:

$$\iint_{\Omega_T} \left[\Delta\psi(x,t;v)\right]^2 dxdt \; + \; \left(\frac{l^2}{\lambda_*}+\frac{l}{k}\right)\int_0^l \left[\Delta\psi(x,0;v)\right]^2 dx \;\; \le 4\beta\left(\frac{l^2}{\lambda_*}+\frac{l}{k}\right)\int_0^l \left[\Delta u(x,T;v)\right]^2 dx \; .$$

Let us estimate now the second integral on the right-hand side of (21) by the same term. From the energy identity (24) we can also conclude

$$k^2\int_0^T \left[\Delta\psi(l,t;v)\right]^2 dt \; + \frac{k}{2}\int_0^l \left[\Delta\psi(x,0;v)\right]^2 dx \;\; \le \; 2\,\beta\, k\int_0^l \left[\Delta u(x,T;v)\right]^2 dx$$

This, with the above estimate, implies

$$\iint_{\Omega_T} \left[\Delta\psi(x,t;v)\right]^2 dxdt \; + k^2\int_0^T \left[\Delta\psi(l,t;v)\right]^2 dt \; + \left(\frac{l^2}{\lambda_*}+\frac{l}{k}+\frac{k}{2}\right)\int_0^l \left[\Delta\psi(x,0;v)\right]^2 dx$$

$$\le 4\beta\left(\frac{l^2}{\lambda_*}+\frac{l}{k}+\frac{k}{2}\right)\int_0^l \left[\Delta u(x,T;v)\right]^2 dx$$

Using this in (21) and taking into account Lemma 3.2 we obtain (20) with the Lipschitz constant (22).

V. CONCLUSION

Consideration was given to the problem of optimal control of parabolic equations. The existence solution of the considering optimal control parabolic problem is proved. The gradient of the cost functional by the adjoint problem approach is obtained. Lipschitz continuity of the gradient is derived.

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