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Independent Functions of Euler Totient Cayley Graph

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ABSTRACT

*Graph Theory is the fast growing area of research in Mathematics. The concepts of Number Theory, particularly, the "Theory of Congruence" in Graph Theory, introduced by Nathanson[7], paved the way for the emergence of a new class of graphs, namely, **"Arithmetic Graphs". Cayley graphs** are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph. The Cayley graph associated with Euler Totient function is called an **Euler Totient Cayley graph** and in this paper we study the independent Functions of Euler Totient Cayley graphs. This paper is devoted to the study of independent functions of Euler Totient Cayley Graph in two cases when n is prime and when n is non-prime.*

KEYWORDS: *Euler Totient Cayley Graph, independent set, independent function.*

I. INTRODUCTION

The concept of the domination number of a graph was first introduced by Berge [3] in his book on graph theory. Ore [8] published a book on graph theory, in which the words 'dominating set' and 'domination number' were introduced. Allan and Laskar [1], Cockayne and Hedetniemi [4], Arumugam [2], Sampath kumar [9] and others have contributed significantly to the theory of dominating sets and domination numbers. An introduction and an extensive overview on domination in graphs and related topics are given by Haynes et al. [5].

II. EULER TOTIENT CAYLEY GRAPH AND ITS PROPERTIES

Definition 2.1: The **Euler totient Cayley** graph is defined as the graph whose vertex set V is given by $Z_n = \{0, 1, 2, \ldots\ldots, n-1\}$ and the edge set is E = {(x, y) / x – y ∈ S or y – x ∈ S} and is denoted by $G(Z_n, \varphi)$. where S denote the set of all positive integers less than n and relatively prime to n. That is $S = \{r / 1 \le r < n \text{ and } GCD(r, n) = 1\}$, $|S| = \varphi(n)$.

Now we present some of the properties of Euler totient Cayley graphs studied by Madhavi [6].

1. The graph $G(Z_n, \varphi)$ is $\varphi(n)$ – regular and has $\frac{n\varphi(n)}{2}$ edges.
2. The graph $G(Z_n, \varphi)$ is Hamiltonian and hence it is connected.
3. The graph $G(Z_n, \varphi)$ is Eulerian for n ≥ 3.
4. The graph $G(Z_n, \varphi)$ is bipartite if n is even.
5. The graph $G(Z_n, \varphi)$ is complete if n is a prime.

III. INDEPENDENT SETS AND INDEPENDENT FUNCTIONS

Definition 3.1: Let G(V, E) be a graph. A subset I of V is called an **independent set (IS)** of G if no two vertices of I are adjacent in G.

Definition 3.2: Let G(V, E) be a graph. A function $f : V \to [0,1]$ is called an **independent function (IF),** if for every vertex $v \in$V with $f(v) > 0$, we have $\sum_{u \in N[v]} f(u) = 1$.

RESULTS

Theorem 3.3: Let I be an IS of $G(Z_n, \varphi)$. Let a function $f : V \to [0,1]$ be defined by

$$f(v) = \begin{cases} 1, & if \ v \in I, \\ 0, & otherwise \end{cases} .$$

Then f becomes an IF of $G(Z_n, \varphi)$.

Proof: Consider $G(Z_n, \varphi)$. Let I be an IS of $G(Z_n, \varphi)$.

Let f be a function defined as in the hypothesis.

Case 1: Suppose n is a prime. Then $G(Z_n, \varphi)$ is a complete graph. So every single vertex forms an IS of $G(Z_n, \varphi)$ and every neighbourhood N[v] of $v \in V$ consists of n vertices.

Then $\sum_{u \in N[v]} f(u) = 1 + \underbrace{0 + 0 + ... + 0}_{(n-1)-times} = 1, \quad \forall v \in V.$

Hence f is an IF of $G(Z_n, \varphi)$.

Case 2: Suppose n is not a prime. Then $G(Z_n, \varphi)$ is $|S|$- regular graph. Let $|S| = r$.

Let I be an independent set of $G(Z_n, \varphi)$. Then $|I| > 1$.

If $v \in I$ then $f(v) > 0$ and since v contains no other vertex of I in its neighbourhood we have

$$\sum_{u \in N[v]} f(u) = 1 + \underbrace{0 + 0 + ... + 0}_{(r-1)-times} = 1.$$

Thus f is an IF of $G(Z_n, \varphi)$.

Therefore f is an IF of $G(Z_n, \varphi)$ for any n.

Remark 3.4: Let $f : V \to [0,1]$ be defined by

$$f(v) = \begin{cases} k, & if \ v \in I, \\ 0, & otherwise \end{cases} .$$

where 0 < k < 1.

Then f cannot be an IF of $G(Z_n, \varphi)$.

This is because for $v \in I$, $\sum_{u \in N[v]} f(u) = k < 1.$

So for $f(v) > 0$, $\sum_{u \in N[v]} f(u) = k < 1$, which implies that f cannot be an IF.

Illustration 3.5: Consider $G(Z_{11}, \varphi)$. The graph is given below.

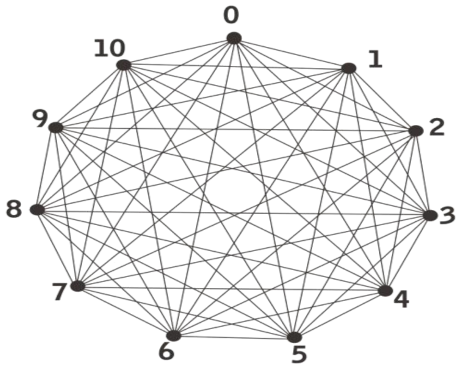

Figure 1: G(Z_{11},φ)

Let $I = \{0\}$ be the IS of $G(Z_{11}, \varphi)$.

Then $f(v) = \begin{cases} 1, & \text{if } v = 0 \\ 0, & \text{if } v = 1, 2, 3, \ldots\ldots, 10 . \end{cases}$

$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall\, v \in V$ with $f(v) > 0$.

Thus f is an IF of $G(Z_{11}, \varphi)$.

Illustration 3.6: Consider $G(Z_8, \varphi)$. The graph is given below.

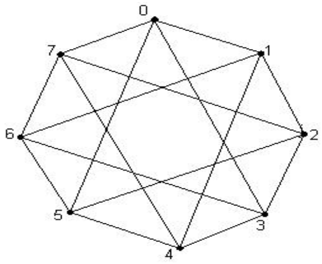

Figure 2: G(Z_8,φ)

The graph is $|S| = 4 -$ regular.

Let $I = \{0, 4\}$ be an IS of $G(Z_8, \varphi)$.

Then the summation values taken over every neighbourhood N[v] of $v \in V$ is given below.

v :	0	1	2	3	4	5	6	7
$f(v)$:	1	0	0	0	1	0	0	0
$\sum_{u \in N[v]} f(u)$:	1	2	0	2	1	2	0	2

$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall\ v \in V$ with $f(v) > 0$.

Hence f is an IF of $G(Z_8, \varphi)$.

Theorem 3.7: Let $f : V \rightarrow [0, 1]$ be a function defined by

$$f(v) = \frac{1}{r+1}, \ \forall\, v \in V .$$

where $r > 0$ denotes the degree of $v \in V$. Then f becomes an IF of $G(Z_n, \varphi)$.

Proof: Consider $G(Z_n, \varphi)$.

Let $f(v) = \frac{1}{r+1}, \ \forall\, v \in V$, where $r > 0$ denotes the degree of the vertex $v \in V$.

Case 1: Suppose n is a prime. Then every neighbourhood N[v] of $v \in V$ consists of n vertices. Then r = n − 1.

Now

$$\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{r+1} + \frac{1}{r+1} + \ldots\ldots\ldots . + \frac{1}{r+1}}_{(n-times)} = \frac{r+1}{r+1} = 1 .$$

$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \quad \forall\, v \in V$ *with* $f(v) > 0$.

Thus f is an IF of $G(Z_n, \varphi)$.

Case 2: Suppose n is not a prime. Then $G(Z_n, \varphi)$ is $|S| -$ regular graph and $|S| = r$.

Now

$$\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{r+1} + \frac{1}{r+1} + \ldots\ldots\ldots. + \frac{1}{r+1}}_{(|S|+1-times)} = \frac{r+1}{r+1} = 1.$$

$$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \quad \forall v \in V \ with \ f(v) > 0.$$

Therefore f is an IF of $G(Z_n, \varphi)$ for every n.

Illustration 3.8: Consider $G(Z_7, \varphi)$. The graph is shown below.

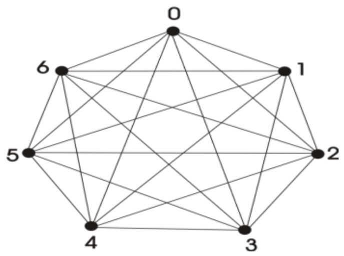

Figure 3: G(Z_7,φ)

Every neighbourhood N[v] of $v \in V$ consists of 6 vertices.

Then r + 1 = 6 + 1 = 7.
Now define a function $f : V \to [0,1]$ by

$$f(v) = \frac{1}{7}, \ \forall v \in V.$$

$$Then \sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{7} + \frac{1}{7} + \ldots\ldots\ldots \ \ldots + \frac{1}{7}}_{7-times} = \frac{7}{7} = 1.$$

$$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall v \in V \ with \ f(v) > 0.$$

Thus f is an IF of $G(Z_7, \varphi)$.

Illustration 3.9: Consider $G(Z_{15}, \varphi)$. The graph is shown below.

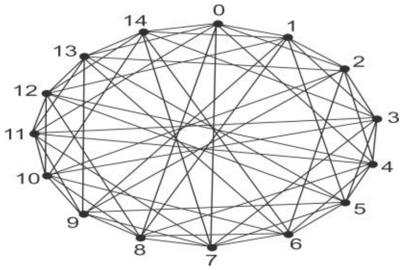

Figure 4: G(Z_{15},φ)

It is a $|S|$ = 8 - regular graph.

$$Then \sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{8} + \frac{1}{8} + \ldots\ldots\ldots \ \ldots + \frac{1}{8}}_{|S|+1-times} = \frac{8}{8} = 1.$$

$$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall v \in V \ with \ f(v) > 0.$$

Thus f is an IF of $G(Z_{15}, \varphi)$.

Theorem 3.10: Let $f : V \to [0,1]$ be a function defined by

$$f(v) = \begin{cases} r, & if \ v = v_i \in V, \\ 1 - r, & if \ v = v_j \in V, \ v_i \neq v_j, \\ 0, & otherwise \end{cases}.$$

where $0 < r < 1$.

Then f becomes an IF of $G(Z_n, \varphi)$, when n is a prime.

Proof: Consider $G(Z_n, \varphi)$, when n is a prime. Since it is a complete graph, every neighbourhood N[v] of $v \in V$ consists of n vertices.

Then

$$\sum_{u \in N[v]} f(u) = r + (1 - r) + \underbrace{0 + 0 + ... + 0}_{(n-2)-times} = r + (1 - r) = 1.$$

$$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall \ v \in V \quad with \quad f(v) > 0.$$

Thus f is an IF of $G(Z_n, \varphi)$.

Theorem 3.11: A function $f : V \to [0,1]$ is an IF of $G(Z_n, \varphi)$ if and only if $P_f \subseteq B_f$.

Proof: Consider $G(Z_n, \varphi)$.

Suppose $f : V \to [0,1]$ is an IF of $G(Z_n, \varphi)$.

The boundary set $B_f = \left\{ u \in V \Big/ \sum_{u \in N[v]} f(u) = 1 \right\}$.

Positive set $P_f = \left\{ u \in V \big/ f(u) > 0 \right\}$.

Let $v \in P_f$. Then $f(v) > 0$.

Since f is an IF, for all $f(v) > 0$, $\sum_{u \in N[v]} f(u) = 1$.

$\Longrightarrow v \in B_f$.

Therefore $P_f \subseteq B_f$.

Conversely, suppose $v \in P_f$. Then $v \in B_f$, since $P_f \subseteq B_f$.

Then $\sum_{u \in N[v]} f(u) = 1, \quad for \quad f(v) > 0$.

$\Longrightarrow f$ is an IF of $G(Z_n, \varphi)$.

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