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Applications of Modified F-Expansion Method for Nonlinear Partial Differential Equations with Variable Coefficients

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ABSTRACT
The modified F-expansion method is used to obtain the new exact travelling wave solutions of Burger equation and Benjamin-Bona-Mahony (BBM) equation with variable coefficients. The obtained solutions include solitary wave solutions, trigonometric function solutions and rational solutions. In addition some figures are provided for direct viewing analysis.

KEYWORDS: *modified F-expansion method, Burgers' equation and BBM equation with variable coefficients, solitary wave solutions, trigonometric function solutions, rational solutions*

I. INTRODUCTION

Many phenomena arising in many fields of sciences and engineering have been modeled in modern times in terms of nonlinear partial differential equations (NLPDEs). As mathematical models of the phenomena, the study of exact solutions of NLPDEs will help us to understand the underlying mechanism that governs these phenomena or to provide better knowledge of its physical content and possible applications. Recently, the study of variable coefficients nonlinear partial differential equations (VC-NLPDEs) has become more and more attractive. This is because of the fact that a large number of important physical phenomena can be described by these equations. To find exact solutions of VC-NLPDEs, many powerful methods have been developed such as the tanh function method [3,10], exp-function method [3,6,13], F-expansion method [9,15], adomain decomposition method [1], sine-cosine [3] method, extended mapping transformation method [12], (G'/G) expansion method [4,5,11] and so on. In the present study, the modified F-expansion method is used to construct the exact solutions of VC-NLPDEs. As applications of the method, we will consider Burgers' equation [2] and BBM equation [14] with variable coefficients.

The plan of the paper is as follows: in section 2, we describe the modified F-expansion method. In section 3, we construct the exact solutions of Burgers' equation and BBM equation with variable coefficients. Some conclusions are given in the last section.

II. INTRODUCTION TO MODIFIED F-EXPANSION METHOD

Introduction of modified F-expansion method with constant coefficients is shown in [7,8]. In this study, we introduce modified F-expansion method with variable coefficients.

Consider a given nonlinear partial differential equation with variable coefficients as:

$$P\left(u, u_t, \alpha(t) u_x, \beta(t) u_{xx}, \gamma(t) u_{xxx}, \ldots\right) = 0 \quad (1)$$

where $u(x,t)$ is the solution of the eq.(1).

The main points of the modified F-expansion method for solving eq.(1) are as follows:

[1]. Using the transformation,

$$u(x,t) = u(\xi) \text{ where } \xi = kx + \int v(t)\,dt \quad (2)$$

where $k \neq 0$ is a constant and $v(t)$ is an integrable function of t.

Substituting eq.(2) into eq.(1) yields an ordinary differential equation for $u(\xi)$

$$P\left(u, v(t) u', k\alpha(t) u', k^2 \beta(t) u'', k^3 \gamma(t) u''', \ldots\right) = 0 \quad (3)$$

where prime denotes the derivative with respect to ξ.

[2]. Suppose that $u(\xi)$ can be expressed as

$$u(\xi)=\sum_{i=-N}^{N} a_i F^i(\xi) \text{ where } a_N \neq 0 \quad (4)$$

where $a_i\,(i=-N,\dots,-1,0,1,\dots,N)$ are constants, N is a positive integer which can be determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $u(\xi)$ in eq.(3) and $F(\xi)$ is a solution of following Riccati equation,

$$F'(\xi)=A+BF(\xi)+CF^2(\xi) \quad (5)$$

where A,B,C are constants.

[3]. Substitute eq.(4) into eq.(3) and using eq.(5) then left-hand side of eq.(3) can be converted into a finite series in $F^p(\xi)$, $(p=-N,\dots,-1,0,1,\dots,N)$. Equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations.

[4]. Solve the system of algebraic equations probably with the aid of Mathematica, we obtain the values of $a_i, v(t)$. Substituting these values into eq.(4), we can obtain the travelling wave solutions to eq.(3).

[5]. From the general form of travelling wave solutions listed in appendix, we can give a series of soliton-like solutions, trigonometric function solutions and rational solutions of eq.(1).

III. APPLICATIONS OF THE METHOD

3.1 Variable coefficient Burgers' equation

We consider the variable coefficient Burgers' equation in the form,

$$u_t+\alpha(t)uu_x-\beta(t)u_{xx}=0 \quad (6)$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions of t.

Using the transformation,

$$u(x,t)=u(\xi) \text{ where } \xi=kx+\int v(t)\,dt \quad (7)$$

where $k\neq 0$ is a constant and v is a function of t which is determined later.

Substituting eq.(7) into eq.(6), we get following ordinary differential equation

$$v(t)u'+k\alpha(t)uu'-k^2\beta(t)u''=0 \quad (8)$$

Now, balancing the orders of uu' and u'', we get integer $N=1$. So we can write solution of eq.(6) in the form,

$$u(x,t)=u(\xi) \text{ where } u(\xi)=a_0+a_{-1}F^{-1}(\xi)+a_1F(\xi) \quad (9)$$

where a_0,a_{-1},a_1 are constants and $F(\xi)$ is a solution of eq.(5).

Inserting eq.(9) together with eq.(5) into eq.(8), the left hand side of eq.(8) can be converted into a finite series in $F^p(\xi)$,($p=-3,-2,-1,0,1,2,3$). Equating each coefficient of $F^p(\xi)$ to zero, we get system of algebraic equations for $a_0,a_{-1},a_1,k,v(t)$.

$$\left.\begin{aligned}
&F^{-3}(\xi): -Aka_{-1}^2\alpha(t)-2A^2k^2a_{-1}\beta(t)=0\\
&F^{-2}(\xi): -Aa_{-1}v(t)-Aka_0a_{-1}\alpha(t)-Bka_{-1}^2\alpha(t)-3ABk^2a_{-1}\beta(t)=0\\
&F^{-1}(\xi): -Ba_{-1}v(t)-Bka_0a_{-1}\alpha(t)-Cka_{-1}^2\alpha(t)-B^2k^2a_{-1}\beta(t)\\
&\qquad -2ACk^2a_{-1}\beta(t)=0\\
&F^{0}(\xi): -Ca_{-1}v(t)+Aa_1v(t)-Cka_0a_{-1}\alpha(t)+Aka_0a_1\alpha(t)-BCk^2a_{-1}\beta(t)\\
&\qquad -ABk^2a_1\beta(t)=0\\
&F^{1}(\xi): Ba_1v(t)+Bka_0a_1\alpha(t)+Aka_1^2\alpha(t)-B^2k^2a_1\beta(t)\\
&\qquad -2ACk^2a_1\beta(t)=0\\
&F^{2}(\xi): Ca_1v(t)+Cka_0a_1\alpha(t)+Bka_1^2\alpha(t)-3BCk^2a_1\beta(t)=0\\
&F^{3}(\xi): Cka_1^2\alpha(t)-2C^2k^2a_1\beta(t)=0
\end{aligned}\right\} \quad (10)$$

Solving the above algebraic equations with the help of Mathematica, we get the following results for $a_0,a_{-1},a_1,v(t)$.

Case 1: $A = 0$, we have

$$a_0 = a_0, a_{-1} = 0, a_1 = a_1, v(t) = \frac{k^2(-2Ca_0 + Ba_1)\beta(t)}{a_1}, \alpha(t) = \frac{2Ck\beta(t)}{a_1} \qquad (11)$$

Case 2: $B = 0$, we have

$$a_0 = a_0, a_{-1} = 0, a_1 = a_1, v(t) = -\frac{2Ck^2 a_0 \beta(t)}{a_1}, \alpha(t) = \frac{2Ck\beta(t)}{a_1} \qquad (12)$$

$$a_0 = a_0, a_{-1} = -\frac{Aa_1}{C}, a_1 = a_1, v(t) = -\frac{2Ck^2 a_0 \beta(t)}{a_1}, \alpha(t) = \frac{2Ck\beta(t)}{a_1} \qquad (13)$$

Case 3: $A = B = 0$, we have

$$a_0 = a_0, a_{-1} = 0, a_1 = a_1, v = -\frac{2Ck^2 a_0 \beta(t)}{a_1}, \alpha(t) = \frac{2Ck\beta(t)}{a_1} \qquad (14)$$

Substituting these results into eq.(9) and using appendix, we obtain the following travelling wave solutions of eq.(6).

(1) Select $A = 0, B = 1, C = -1$ and $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$ from appendix and using eq.(11), we get

$$u_1(x,t) = a_0 + a_1\left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{1}{2}\left(kx + \frac{k^2(2a_0 + a_1)\int\beta(t)dt}{a_1}\right)\right]\right) \qquad (15)$$

(2) Select $A = 0, B = -1, C = 1$ and $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)$ from appendix and using eq.(11), we get

$$u_2(x,t) = a_0 + a_1\left(\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{2}\left(kx + \frac{k^2(-2a_0 - a_1)\int\beta(t)dt}{a_1}\right)\right]\right) \qquad (16)$$

(3) Select $A = \frac{1}{2}, B = 0, C = -\frac{1}{2}$ and $F(\xi) = \coth(\xi) \pm \csc h(\xi)$, $\tanh(\xi) \pm i\sec h(\xi)$ from appendix and using eq.(12) and eq.(13) respectively, we get

$$u_3(x,t) = a_0 + a_1\left(\coth\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm \csc h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right) \qquad (17)$$

$$u_4(x,t) = a_0 + a_1\left(\coth\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm \csc h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}$$
$$+ a_1\left(\coth\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm \csc h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right) \qquad (18)$$

$$u_5(x,t) = a_0 + a_1\left(\tanh\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm i\sec h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right)$$

(19)

$$u_6(x,t) = a_0 + a_1\left(\tanh\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm i\sec h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}$$
$$+ a_1\left(\tanh\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right] \pm i\sec h\left[kx + \frac{k^2 a_0\int\beta(t)dt}{a_1}\right]\right)$$

(20)

(4) Select $A = 1, B = 0, C = -1$ and $F(\xi) = \tanh(\xi), \coth(\xi)$ from appendix and using eq.(12), eq.(13) respectively, we get

$$u_7(x,t)=a_0+a_1\left(\tanh\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{21}$$

$$u_8(x,t)=a_0+a_1\left(\tanh\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}+a_1\left(\tanh\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{22}$$

$$u_9(x,t)=a_0+a_1\left(\coth\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{23}$$

$$u_{10}(x,t)=a_0+a_1\left(\coth\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}+a_1\left(\coth\left[kx+\frac{2k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{24}$$

(5) Select $A=\frac{1}{2}, B=0, C=\frac{1}{2}$ and $F(\xi)=\sec(\xi)+\tan(\xi), \csc(\xi)-\cot(\xi)$ from appendix and using eq.(12), eq.(13) respectively, we get

$$u_{11}(x,t)=a_0+a_1\left(\sec\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]+\tan\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{25}$$

$$u_{12}(x,t)=a_0-a_1\left(\sec\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]+\tan\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}+a_1\left(\sec\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]+\tan\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{26}$$

$$u_{13}(x,t)=a_0+a_1\left(\csc\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]-\cot\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{27}$$

$$u_{14}(x,t)=a_0-a_1\left(\csc\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]-\cot\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right)^{-1}+a_1\left(\csc\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]-\cot\left[kx-\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{28}$$

(6) Select $A=-\frac{1}{2}, B=0, C=-\frac{1}{2}$ and $F(\xi)=\sec(\xi)-\tan(\xi), \csc(\xi)+\cot(\xi)$ from appendix and using eq.(12), eq.(13) respectively, we get

$$u_{15}(x,t)=a_0+a_1\left(\sec\left[kx+\frac{k^2a_0\int\beta(t)dt}{a_1}\right]-\tan\left[kx+\frac{k^2a_0\int\beta(t)dt}{a_1}\right]\right) \tag{29}$$

$$u_{16}(x,t) = a_0 - a_1\left(\sec\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right] - \tan\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right)^{-1} + a_1\left(\sec\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right] - \tan\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right) \quad (30)$$

$$u_{17}(x,t) = a_0 + a_1\left(\csc\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right] + \cot\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right) \quad (31)$$

$$u_{18}(x,t) = a_0 - a_1\left(\csc\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right] + \cot\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right)^{-1} + a_1\left(\csc\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right] + \cot\left[kx + \frac{k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right) \quad (32)$$

(7) Select $A = 1, B = 0, C = 1$ and $F(\xi) = \tan(\xi)$ from appendix and using eq.(12), eq.(13) respectively, we get

$$u_{19}(x,t) = a_0 + a_1\tan\left[kx - \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right] \quad (33)$$

$$u_{20}(x,t) = a_0 - a_1\left(\tan\left[kx - \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right)^{-1} + a_1\tan\left[kx - \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right] \quad (34)$$

(8) Select $A = -1, B = 0, C = -1$ and $F(\xi) = \cot(\xi)$ from appendix and using eq.(12), eq.(13) respectively, we get

$$u_{21}(x,t) = a_0 + a_1\cot\left[kx + \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right] \quad (35)$$

$$u_{22}(x,t) = a_0 - a_1\left(\cot\left[kx + \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right]\right)^{-1} + a_1\cot\left[kx + \frac{2k^2 a_0\int\beta(t)\,dt}{a_1}\right] \quad (36)$$

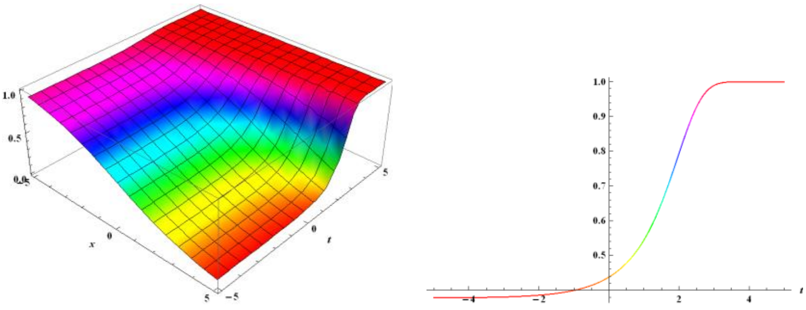
(9) Select $A = 0, B = 0, C \neq 0$ and $F(\xi) = -\frac{1}{C\xi + m}$ from appendix and using eq.(14), we get

$$u_{23}(x,t) = a_0 - \frac{a_1^2}{Cka_1x - 2C^2k^2a_0\int\beta(t)\,dt + a_1 m} \quad (37)$$

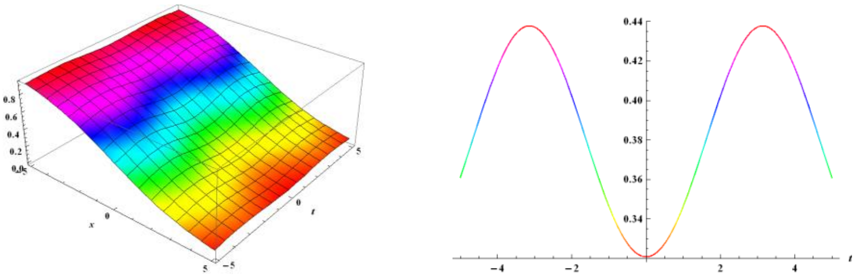
where m is an arbitrary constant.

Graphical presentation:

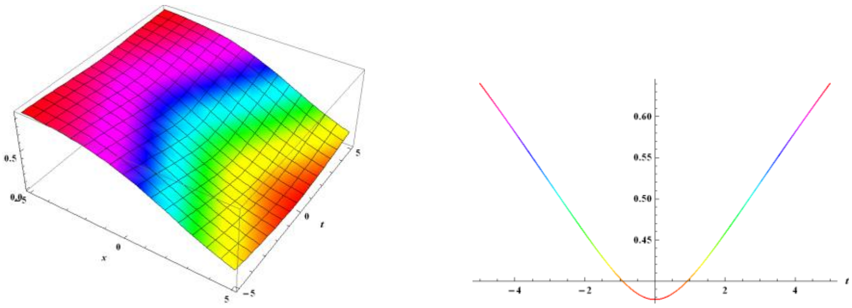
For the graphical presentation, we taking solution $u_1(x,t)$ given by eq.(15) as an example. For simplicity we set $a_0 = 1, a_1 = -1$ and $k = \frac{1}{2}$. Here we plot solution $u_1(x,t)$ for different values of $\beta(t) = (a)\,e^t, (b)\sin(t), (c)\tanh(t)$.



(a) $\beta(t) = e^{t}$



(b) $\beta(t) = \sin(t)$



(c) $\beta(t) = \tanh(t)$

Figure 1. Solution of Burgers' equation with variable coefficients in different form

3.2 Variable coefficient BBM equation

We consider following variable coefficient BBM equation

$$u_t + a(t)u_x + b(t)u^p u_x - u_{xxt} = 0 \quad (38)$$

where $p > 0$ and $a(t)$, $b(t)$ are arbitrary functions of t.

For simplicity here we consider $p = 1$

$$u_t + a(t)u_x + b(t)uu_x - u_{xxt} = 0 \quad (39)$$

Taking the transformation,

$$u(x,t) = u(\xi) \text{ where } \xi = kx + \int v(t)\,dt \quad (40)$$

where $k \neq 0$ and v is a function of t which is determined later.

Substituting eq.(40) into eq.(39), we get ordinary differential equation

$$v(t)u' + k\,a(t)u' + k\,b(t)uu' - k^2 v(t)u''' = 0 \quad (41)$$

Balancing the orders of uu' and u''', we get integer $N = 2$. So we can write solution of eq.(39) in the form,

$$u(\xi) = a_0 + a_{-2}F^{-2}(\xi) + a_{-1}F^{-1}(\xi) + a_1 F(\xi) + a_2 F^2(\xi) \quad (42)$$

Substituting eq.(42) together with eq.(5) into eq.(41), the left hand side of eq.(41) can be converted into a finite series in $F^p(\xi)$,($p = -5,-4,-3,-2,-1,0,1,2,3,4,5$). Equating each coefficient of $F^p(\xi)$ to zero, we get system of algebraic equations for $a_0, a_{-2}, a_{-1}, a_1, a_2, k, v(t)$.

$$\left.\begin{aligned}
&F^{-5}(\xi): -2Ab(t)k\,a_{-2}^{2} + 24A^{3}k^{2}a_{-2}v = 0\\
&F^{-4}(\xi): -2b(t)Bka_{-2}^{2} - 3Ab(t)ka_{-2}a_{-1} + 54A^{2}Bk^{2}a_{-2}v(t) + 6A^{3}k^{2}a_{-1}v(t) = 0\\
&F^{-3}(\xi): -2a(t)Aka_{-2} - 2Ab(t)ka_{0}a_{-2} - 2b(t)Cka_{-2}^{2} - 3b(t)Bka_{-2}a_{-1}\\
&\qquad - Ab(t)ka_{-1}^{2} - 2Aa_{-2}v(t) + 38AB^{2}k^{2}a_{-2}v(t) + 40A^{2}Ck^{2}a_{-2}v(t)\\
&\qquad + 12A^{2}Bk^{2}a_{-1}v(t) = 0\\
&F^{-2}(\xi): -2a(t)Bka_{-2} - 2b(t)Bka_{0}a_{-2} - a(t)Aka_{-1} - Ab(t)ka_{0}a_{-1} - 3b(t)Cka_{-2}a_{-1}\\
&\qquad - b(t)Bka_{-1}^{2} - Ab(t)ka_{-2}a_{1} - 2Ba_{-2}v(t) + 8B^{3}k^{2}a_{-2}v(t) + 52ABCk^{2}a_{-2}v(t)\\
&\qquad - Aa_{-1}v(t) + 7AB^{2}k^{2}a_{-1}v(t) + 8A^{2}Ck^{2}a_{-1}v(t) = 0\\
&F^{-1}(\xi): -2a(t)Cka_{-2} - 2b(t)Cka_{0}a_{-2} - a(t)Bka_{-1} - Bb(t)ka_{0}a_{-1} - b(t)Cka_{-1}^{2}\\
&\qquad - b(t)Bka_{-2}a_{1} - 2Ca_{-2}v(t) + 14B^{2}Ck^{2}a_{-2}v(t) + 16AC^{2}k^{2}a_{-2}v(t) - Ba_{-1}v(t)\\
&\qquad + B^{3}k^{2}a_{-1}v(t) + 8ABCk^{2}a_{-1}v(t) = 0\\
&F^{0}(\xi): -a(t)Cka_{-1} - b(t)Cka_{0}a_{-1} + a(t)Aka_{1} + Ab(t)ka_{0}a_{1} - b(t)Cka_{-2}a_{1} + 6BC^{2}k^{2}a_{-2}v(t)\\
&\qquad - Ca_{-1}v(t) + B^{2}Ck^{2}a_{-1}v(t) + 2AC^{2}k^{2}a_{-1}v(t) + Aa_{1}v(t) - AB^{2}k^{2}a_{1}v(t) - 2A^{2}Ck^{2}a_{1}v(t)\\
&\qquad + Ab(t)ka_{-1}a_{2} - 6A^{2}Bk^{2}v(t)a_{2} = 0\\
&F^{1}(\xi): a(t)Bka_{1} + Bb(t)ka_{0}a_{1} + Ab(t)ka_{1}^{2} + Ba_{1}v(t) - B^{3}k^{2}a_{1}v(t) - 8ABCk^{2}a_{1}v(t) + 2a(t)Aka_{2}\\
&\qquad + 2Ab(t)ka_{0}a_{2} + b(t)Bka_{-1}a_{2} + 2Av(t)a_{2} - 14AB^{2}k^{2}a_{2}v(t) - 16A^{2}Ck^{2}a_{2}v(t) = 0\\
&F^{2}(\xi): a(t)Cka_{1} + Cb(t)ka_{0}a_{1} + Bb(t)ka_{1}^{2} + Ca_{1}v(t) - 7B^{2}Ck^{2}a_{1}v(t) - 8AC^{2}k^{2}a_{1}v(t)\\
&\qquad + 2a(t)Bka_{2} + 2b(t)Bka_{0}a_{2} + b(t)Cka_{-1}a_{2} + 3Ab(t)ka_{1}a_{2} + 2Bv(t)a_{2} - 8B^{3}k^{2}v(t)a_{2}\\
&\qquad - 52ABCk^{2}v(t)a_{2} = 0\\
&F^{3}(\xi): b(t)Cka_{1}^{2} - 12BC^{2}k^{2}a_{1}v(t) + 2a(t)Cka_{2} + 2b(t)Cka_{0}a_{2} + 3b(t)Bka_{1}a_{2} + 2Cv(t)a_{2}\\
&\qquad - 38B^{2}Ck^{2}v(t)a_{2} - 40AC^{2}k^{2}v(t)a_{2} + 2Ab(t)ka_{2}^{2} = 0\\
&F^{4}(\xi): -6C^{3}k^{2}a_{1}v(t) + 3b(t)Cka_{1}a_{2} - 54BC^{2}k^{2}a_{2}v(t) + 2b(t)Bka_{2}^{2} = 0\\
&F^{5}(\xi): -24C^{3}k^{2}a_{2}v(t) + 2b(t)Cka_{2}^{2} = 0
\end{aligned}\right\}$$

(43)

Solving above algebraic equations with the help of Mathematica, we obtain following results:

Case 1: $A = 0$, we have

$$a_0 = a_0, a_{-2} = 0, a_{-1} = 0, a_1 = a_1, a_2 = \frac{Ca_1}{B}, v(t) = \frac{b(t)a_1}{12BCk},$$
$$a(t) = \frac{-12b(t)BCk^2a_0 - b(t)a_1 + b(t)B^2k^2a_1}{12BCk^2} \qquad (44)$$

Case 2: $B = 0$, we have

$$a_0 = a_0, a_{-2} = 0, a_{-1} = 0, a_1 = 0, a_2 = a_2, v(t) = \frac{b(t)a_2}{12C^2k},$$
$$a(t) = \frac{-12b(t)C^2k^2a_0 - b(t)a_2 + 8b(t)ACk^2a_2}{12C^2k^2} \qquad (45)$$

$$a_0 = a_0, a_{-2} = \frac{A^2a_2}{C^2}, a_{-1} = 0, a_1 = 0, a_2 = a_2, v(t) = \frac{b(t)a_2}{12C^2k},$$
$$a(t) = \frac{-12b(t)C^2k^2a_0 - b(t)a_2 + 8b(t)ACk^2a_2}{12C^2k^2} \qquad (46)$$

Case 3: $A = B = 0$, we have

$$a_0 = a_0, a_{-2} = 0, a_{-1} = 0, a_1 = 0, a_2 = a_2, v(t) = \frac{b(t)a_2}{12C^2k},$$
$$a(t) = \frac{-12b(t)C^2k^2a_0 - b(t)a_2}{12C^2k^2} \qquad (47)$$

Substituting these results into eq.(42) and using appendix, we obtain the travelling wave solutions of eq.(39) as follows:

(1) Select $A = 0, B = 1, C = -1$ and $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$ from appendix and using eq.(44), we get

$$u_1(x,t) = a_0 + \frac{a_1}{4} sech\left[\frac{kx}{2} - \frac{a_1\int b(t)dt}{24k}\right]^2 \qquad (48)$$

(2) Select $A = 0, B = -1, C = 1$ and $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)$ from appendix and using eq.(44), we get

$$u_2(x,t) = a_0 - \frac{a_1}{4}\operatorname{csc}h\left[\frac{kx}{2} - \frac{a_1\int b(t)dt}{24k}\right]^2 \qquad (49)$$

(3) Select $A = \frac{1}{2}, B = 0, C = -\frac{1}{2}$ and $F(\xi) = \coth(\xi) \pm \operatorname{csc}h(\xi)$, $\tanh(\xi) \pm i\sec h(\xi)$ from appendix and using eq.(45) and eq.(46) respectively, we get

$$u_3(x,t) = a_0 + a_2\left(\operatorname{csc}h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm \coth\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^2 \qquad (50)$$

$$u_4(x,t) = a_0 + a_2\left(\operatorname{csc}h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm \coth\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^{-2}$$
$$+ a_2\left(\operatorname{csc}h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm \coth\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^2 \qquad (51)$$

$$u_5(x,t) = a_0 + a_2\left(\tanh h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm i\,\text{sech}\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^2 \qquad (52)$$

$$u_6(x,t) = a_0 + a_2\left(\tanh h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm i\,\text{sech}\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^{-2}$$
$$+ a_2\left(\tanh h\left[kx + \int\frac{b(t)a_2}{3k}dt\right] \pm i\,\text{sech}\left[kx + \int\frac{b(t)a_2}{3k}dt\right]\right)^2 \qquad (53)$$

(4) Select $A = 1, B = 0, C = -1$ and $F(\xi) = \tanh(\xi), \coth(\xi)$ from appendix and using eq.(45) and eq.(46) respectively, we get

$$u_7(x,t) = a_0 + a_2\left(\tanh\left[kx + \int\frac{b(t)a_2}{12k}dt\right]\right)^2 \qquad (54)$$

$$u_8(x,t) = a_0 + a_2\left(\tanh\left[kx + \int\frac{b(t)a_2}{12k}dt\right]\right)^{-2}$$
$$+ a_2\left(\tanh\left[kx + \int\frac{b(t)a_2}{12k}dt\right]\right)^2$$

(55) $$u_9(x,t) = a_0 + a_2\left(\coth\left[kx + \int\frac{b(t)a_2}{12k}dt\right]\right)^2 \qquad (56)$$

$$u_{10}(x,t) = a_0 + a_2\left(\coth\left[kx + \int \frac{b(t)a_2}{12k}dt\right]\right)^{-2} + a_2\left(\coth\left[kx + \int \frac{b(t)a_2}{12k}dt\right]\right)^{2} \quad (57)$$

(5) Select $A = \frac{1}{2}, B = 0, C = \frac{1}{2}$ and $F(\xi) = \sec(\xi) + \tan(\xi), \csc(\xi) - \cot(\xi)$ from appendix and using eq.(45) and eq.(46) respectively, we get

$$u_{11}(x,t) = a_0 + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (58)$$

$$u_{12}(x,t) = a_0 + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{-2} + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (59)$$

$$u_{13}(x,t) = a_0 + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (60)$$

$$u_{14}(x,t) = a_0 + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{-2} + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (61)$$

(6) Select $A = -\frac{1}{2}, B = 0, C = -\frac{1}{2}$ and $F(\xi) = \sec(\xi) - \tan(\xi), \csc(\xi) + \cot(\xi)$ from appendix and using eq.(45) and eq.(46) respectively, we get

$$u_{15}(x,t) = a_0 + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (62)$$

$$u_{16}(x,t) = a_0 + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{-2} + a_2\left(\sec\left[kx + \int \frac{b(t)a_2}{3k}dt\right] - \tan\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (63)$$

$$u_{17}(x,t) = a_0 + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (64)$$

$$u_{18}(x,t) = a_0 + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{-2} + a_2\left(\csc\left[kx + \int \frac{b(t)a_2}{3k}dt\right] + \cot\left[kx + \int \frac{b(t)a_2}{3k}dt\right]\right)^{2} \quad (65)$$

(6) Select $A = 1(-1), B = 0, C = 1(-1)$ and $F(\xi) = \tan(\xi), \cot(\xi)$ from appendix and using eq.(45) and eq.(46) respectively, we get

$$u_{19}(x,t) = a_0 + a_2\left(\tan\left[kx + \frac{b(t)a_2}{12k}\right]\right)^{2} \quad (66)$$

$$u_{20}(x,t) = a_0 + a_2\left(\tan\left[kx + \frac{b(t)a_2}{12k}\right]\right)^{-2} + a_2\left(\tan\left[kx + \frac{b(t)a_2}{12k}\right]\right)^{2} \quad (67)$$

$$u_{21}(x,t) = a_0 + a_2\left(\cot\left[kx + \frac{b(t)a_2}{12k}\right]\right)^2 \quad (68)$$

$$u_{22}(x,t) = a_0 + a_2\left(\cot\left[kx + \frac{b(t)a_2}{12k}\right]\right)^{-2} + a_2\left(\cot\left[kx + \frac{b(t)a_2}{12k}\right]\right)^2 \quad (69)$$

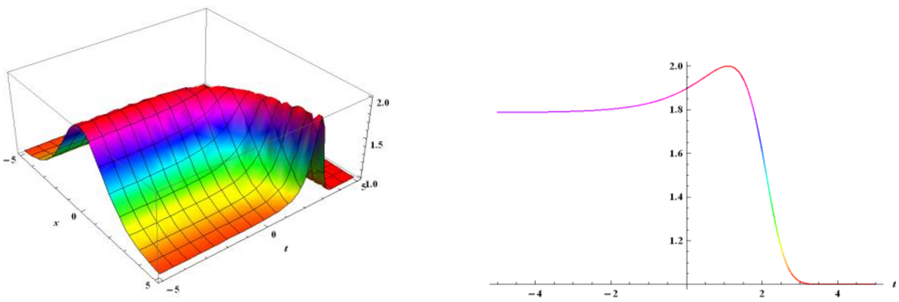
(7) Select $A = B = 0, C \neq 0$ and $F(\xi) = \left(-\frac{1}{C\xi + m}\right)$ from appendix and using eq.(47), we get

$$u_{23}(x,t) = a_0 + a_2\left(-\frac{1}{C\left(kx + \int \frac{b(t)a_2}{12C^2k}dt\right) + m}\right)^2 \quad (70)$$

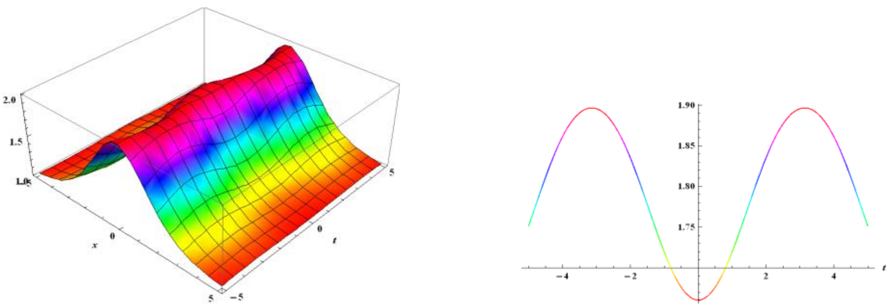
where m is an arbitrary constant.

Graphical presentation:

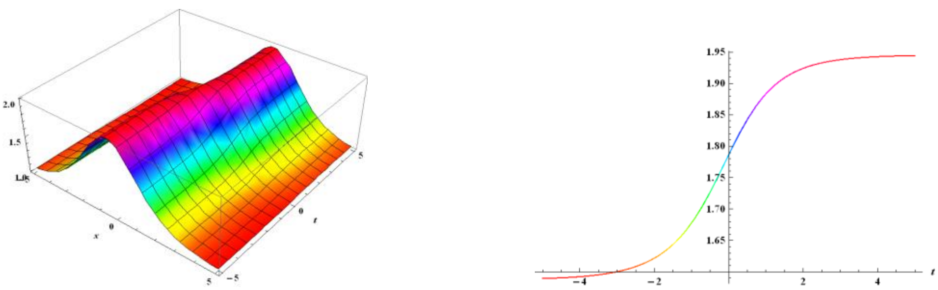
Here we taking solution $u_1(x,t)$ given by eq.(48) for the graphical presentation. Fixing $a_0 = k = 1, a_2 = 4$ in eq.(48). When $b(t) = e^t, \sin(t)$ and $\sec h(t)$, the structures of eq.(48) are illustrated in figures 2(a), 2(b) and 2(c).



(a) $b(t) = e^t$



(b) $b(t) = \sin(t)$



(c) $b(t) = \sec h(t)$

Figure 2. Solution of BBM equation with variable coefficients in different form

IV. CONCLUSION

In this paper, exact travelling wave solutions of the Burgers' equation and BBM equation with variable coefficients have been obtained using the modified F-expansion method. These solutions are expressed in terms of hyperbolic, trigonometric and rational functions with arbitrary parameters. The obtained solutions may be useful to further understand the variable coefficients Burger equation and BBM equation and mechanism of the physical phenomena. The modified F-expansion method is promising and powerful method for handling other nonlinear partial differential equations with variable coefficients arising in mathematical physics.

Appendix: Relations between values of (A, B, C) and corresponding $F(\xi)$ in Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi)$$

A	B	C	$F(\xi)$
0	1	-1	$\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$
0	-1	1	$\frac{1}{2}-\frac{1}{2}\coth\left(\frac{\xi}{2}\right)$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth(\xi) \pm \csc h(\xi), \tanh(\xi) \pm i \sec h(\xi)$
1	0	-1	$\tanh(\xi), \coth(\xi)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec(\xi)+\tan(\xi), \csc(\xi)-\cot(\xi)$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec(\xi)-\tan(\xi), \csc(\xi)+\cot(\xi)$
$1(-1)$	0	$1(-1)$	$\tan(\xi), \cot(\xi)$
0	0	$\neq 0$	$-\frac{1}{C\xi+m}$ (m is an arbitrary constant)

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