ISSN (e): 2250 – 3005 || Vol, 04 || Issue, 5 || May – 2014 ||
International Journal of Computational Engineering Research (IJCER)

# Extensions of Enestrom-Kakeya Theorem

M. H. Gulzar
*Department of Mathematics*
*University of Kashmir, Srinagar 190006*

**ABSTRACT:**
In this paper we give an extension of the famous Enestrom-Kakeya Theorem, which generalizes many generalizations of the said theorem as well.

**Mathematics Subject Classification:** 30 C 10, 30 C 15

**Keywords and Phrases:** Coefficient, Polynomial, Zero

## I. INTRODUCTION AND STATEMENT OF RESULTS

A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [8]:

**Theorem A:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \le 1$ .

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [6] gave the following generalization of Theorem A:

**Theorem B:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \le \dfrac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz and Zargar [1] generalized Theorem B by proving the following result:

**Theorem C:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

Gulz ar [4,5] generalized Theorem C to polynomials with complex coefficients and proved the following results:

**Theorem D:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ ,

$\mathrm{Im}(a_j)=\beta_j,\ j=0,1,......,\ n$ such that for some $k \ge 1, 0 < \tau \le 1$ ,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left| z + (k-1)\frac{\alpha_n}{a_n} \right| \le \frac{k\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

**Theorem E:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1, 0 < \tau \le 1$,

$$k\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \tau\beta_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left| z + (k-1)\frac{\beta_n}{a_n} \right| \le \frac{k\beta_n + 2|\beta_0| - \tau(\beta_0 + |\beta_0|) + 2\sum_{j=0}^{n} |\alpha_j|}{|a_n|}.$$

**Theorem F: :** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real $\alpha, \beta; \left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}, j = 0,1,2,......, n$, and for some $k \ge 1, 0 < \tau \le 1$,

$$k|a_n| \ge |a_{n-1}| \ge ...... \ge |a_1| \ge \tau|a_0| .$$

Then all the zeros of P(z) lie in the closed disk

$$|z| \le \frac{k|a_n|(1 + \cos + \sin\alpha) - |a_n| + 2|a_0| - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|}{|a_n|}.$$

Some questions which have been raised by some researchers in connection with the Enestrom-Kakeya Theorem are[2]:

What happens , if ( i) instead of the leading coefficient $a_n$, there is some $a_j$ with $a_{j+1} \ge a_j < a_{j-1}$ such that for some $k \ge 1$, $a_n \ge a_{n-1} \ge ...... \ge a_{j+1} \ge ka_j \ge a_{j-1} ...... \ge \alpha_1 \ge \alpha_0$, j=1,2,…..,n and (ii) for some $k_1 \ge 1, k_2 \ge 1; k_1 a_n \ge k_2 a_{n-1} \ge ...... \ge a_1 \ge a_0$.

In this direction, Liman and Shah [7, Cor.1] have proved the following result:

**Theorem G:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$, ,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge ka_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0| + (k-1)\left\{\sum_{j=\lambda}^{n}(a_j + |a_j|) - |a_n|\right\}}{|a_n|} .$$

Unfortunately, the conclusion of the theorem is not correct and their claim that it follows from Theorem 1 in [7] is false. The correct form of the result is as follows:

**Theorem H:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1$, ,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge ka_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0| + 2(k-1)|a_\lambda|}{|a_n|}.$$

In this paper , we are going to prove the following more general result:

**Theorem 1:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1, 0 < \tau \le 1$,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge k a_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge \tau a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + 2(k-1)|a_\lambda| - \tau(a_0 + |a_0|) + 2|a_0|}{|a_n|}.$$

**Remark 1:** For $\tau = 1$ ,Theorem 1 reduces to Theorem H.

Taking in particular $k = \frac{a_{\lambda-1}}{a_\lambda} \ge 1$ in Theorem 1, we get the following

**Corollary 1:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ ,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge a_\lambda \le a_{\lambda-1}. \ge ..... \ge a_1 \ge \tau a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + 2(\frac{a_{\lambda-1} - a_\lambda}{a_\lambda})|a_\lambda| - \tau(a_0 + |a_0|) + 2|a_0|}{|a_n|}.$$

For $\tau = 1$ , Cor. 1 reduces to the following

**Corollary 2:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ ,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge a_\lambda \le a_{j-1}. \ge ..... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + (\frac{a_{\lambda-1} - a_\lambda}{a_\lambda})|a_\lambda| - a_0 + |a_0|)}{|a_n|}.$$

Theorem 1 is a special case of the following more general result:

**Theorem 2:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ ,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge k\alpha_\lambda \ge \alpha_{\lambda-1} ...... \ge \alpha_1 \ge \tau\alpha_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

**Remark 2:** If $\beta_j = 0, \forall j = 0,1,2,......, n$ i.e. $a_j$ is real, then Theorem 2 reduces to Theorem 1.

Applying Theorem 2 to the polynomial –iP(z), we get the following result:

**Theorem 3:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ , $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,......,n$ such that for some $k\geq 1, 0<\tau\leq 1$ ,

$$\beta_n \geq \beta_{n-1}\geq ......\geq \beta_{j+1}\geq k\beta_j \geq \beta_{j-1}......\geq \beta_1 \geq \tau\beta_0 .$$

Then P(z) has all its zeros in

$$|z|\leq \frac{a_n + 2(k-1)|\beta_\lambda| - \tau(\beta_0+|\beta_0|) + 2|\beta_0| + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|} .$$

For polynomials with complex coefficients, we have the following form of Theorem 1:

**Theorem 4:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k\geq 1, 0<\tau\leq 1$ ,

$$|a_n|\geq |a_{n-1}|\geq ......\geq |a_{\lambda+1}|\geq k|a_\lambda|\geq |a_{\lambda-1}|......\geq |a_1|\geq \tau|a_0| .$$

Then P(z) has all its zeros in

$$|z|\leq \frac{1}{|a_n|}[|a_n|(\cos\alpha+\sin\alpha) - k|a_\lambda|(\cos\alpha-\sin\alpha-1) + 2|a_\lambda|(k+k\sin\alpha-1)$$

$$-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|]$$

**Remark 3:** For k=1, Theorem 4 reduces to Theorem F with k=1.
Next, we prove the following result:

**Theorem 5:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ , $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,......,n$ such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$ ,

$$k_1\alpha_n \geq k_2\alpha_{n-1}\geq \alpha_{n-2}......\alpha_1\geq \tau\alpha_0 .$$

Then P(z) has all its zeros in

$$|z|\leq \frac{(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

**Remark 4:** If $\beta_j=0, \forall j=0,1,2,......,n$ i.e. $a_j$ is real, we get the following result:

**Corollary 3:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$ ,

$$k_1 a_n \geq k_2 a_{n-1}\geq a_{n-2}......a_1\geq \tau a_0 .$$

Then P(z) has all its zeros in

$$|z|\leq \frac{(k_1|a_n|+k_2|a_{n-1}|)+(k_1a_n-k_2a_{n-1})+a_{n-1}-(|a_n|+|a_{n-1}|)+2|a_0|}{|a_n|} .$$

Applying Theorem 2 to the polynomial –iP(z), we get the following result from Theorem 4:

**Theorem 6:** Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ , $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,......,n$ such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$ ,

$$k_1\beta_n \geq k_2\beta_{n-1} \geq \beta_{n-2} ...... \geq \beta_1 \geq \tau\beta_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{(k_1|\beta_n| + k_2|\beta_{n-1}|) + (k_1\beta_n - k_2\beta_{n-1}) + \beta_{n-1} - (|\beta_n| + |\beta_{n-1}|) + 2|\beta_0| + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|} .$$

**Theorem 7:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some

real $\alpha, \beta; |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0,1,2,......, n$, and for some $k_1 \geq 1, k_2 \geq 1, 0 < \tau \leq 1$,

$$k_1|a_n| \geq k_2|a_{n-1}| \geq |a_{n-2}| ...... \geq |a_1| \geq \tau|a_0| .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\begin{bmatrix} k_1|a_n|(1+\cos\alpha+\sin\alpha) + k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha) - |a_n| - |a_{n-1}|(1-\cos\alpha) + 2|a_0| \\ -\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j| \end{bmatrix} .$$

**Remark 4:** For $k_1 = k, k_2 = 1$, Theorem 6 reduces to Theorem F.

Taking $\tau = 1$ in Theorem 7, we get the following

**Corollary 4:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some

real $\alpha, \beta; |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0,1,2,......, n$, and for some $k_1 \geq 1, k_2 \geq 1,$ ,

$$k_1|a_n| \geq k_2|a_{n-1}| \geq |a_{n-2}| ...... \geq |a_1| \geq |a_0| .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha) + k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha) - |a_n|$$

$$-|a_{n-1}|(1-\cos\alpha) - \tau|a_0|(1+\cos\alpha-\sin\alpha) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|] .$$

## II. LEMMA

For the proof of Theorem 6, we need the following lemma:

**Lemma:** Let $a_1$ and $a_2$ be any two complex numbers such that $|a_1| \geq |a_2|$ and for some real numbers $\alpha$ and $\beta$ , $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$ ,then

$$|a_1 - a_2| \leq (|a_1| - |a_2|)\cos\alpha + (|a_1| + |a_2|)\sin\alpha .$$

The above lemma is due to Govil and Rahman [3] .

## 3. Proofs of Theorems

**Proof of Theorem 2:** Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1}$$

$$+ (a_\lambda - a_{\lambda-1})z^\lambda + ...... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1}) z^n + (\alpha_{n-1} - \alpha_{n-2}) z^{n-1} + \ldots\ldots$$
$$+ \{(\alpha_{\lambda+1} - k\alpha_\lambda) + (k\alpha_\lambda - \alpha_\lambda)\} z^{\lambda+1} + \{(k\alpha_\lambda - \alpha_{\lambda-1}) - (k\alpha_\lambda - \alpha_\lambda)\} z^\lambda + \ldots\ldots$$
$$+ \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\} z + \alpha_0 + i\{(\beta_n - \beta_{n-1}) z^n + \ldots\ldots(\beta_1 - \beta_0) z + \beta_0\}$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,\ldots\ldots, n$ so that, by using the hypothesis,

$$|F(z)| \geq |a_n||z|^{n+1} - [|\alpha_n - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-1}||z|^{n-1} + \ldots\ldots + |\alpha_{\lambda+1} - k\alpha_\lambda||z|^{\lambda+1}$$
$$+ (k-1)|\alpha_\lambda||z|^{\lambda+1} + |k\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda + (k-1)|\alpha_\lambda||z|^\lambda + \ldots\ldots + |\alpha_1 - \tau a_0||z|$$
$$+ (1-\tau)|\alpha_0||z| + |\alpha_0| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}||z|^j + |\beta_0|$$
$$= |z|^n [|a_n||z| - \{[|\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-1}|\frac{1}{|z|} + \ldots\ldots + |\alpha_{\lambda+1} - k\alpha_\lambda|\frac{1}{|z|^{n-\lambda-1}}$$
$$+ (k-1)|\alpha_\lambda|\frac{1}{|z|^{n-\lambda-`1}} + |k\alpha_\lambda - \alpha_{\lambda-1}|\frac{1}{|z|^{n-\lambda}} + (k-1)|\alpha_\lambda|\frac{1}{|z|^{n-\lambda}} + \ldots\ldots + |\alpha_1 - \tau a_0|\frac{1}{|z|^{n-1}}$$
$$+ (1-\tau)|\alpha_0|\frac{1}{|z|^{n-1}} + |\alpha_0|\frac{1}{|z|^n} + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}|\frac{1}{|z|^{n-j}} + |\beta_0|\frac{1}{|z|^n}\}]$$
$$> |z|^n [|a_n||z| - [\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \ldots\ldots + \alpha_{\lambda+1} - k\alpha_\lambda + (k-1)|\alpha_\lambda|$$
$$+ k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + \ldots\ldots + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\alpha_0|$$
$$+ \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|) + |\beta_0|]$$
$$= |z|^n [|a_n||z| - [\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|]$$
$$> 0$$

if

$$|z| > \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \leq \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \leq \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

**Proof of Theorem 4: :** Consider the polynomial

$$F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_{n-1} z^{n-1} + ...... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...... + \{(a_{\lambda+1} - ka_\lambda) + (ka_\lambda - a_\lambda)\} z^{\lambda+1}$$
$$+ \{(ka_\lambda - a_{\lambda-1}) - (ka_\lambda - a_\lambda)\} z^\lambda + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + ......$$
$$\{(a_1 - \tau a_0) + (\tau a_0 - a_0)\} z + a_0 .$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,......., \ n$ , so that, by using the hypothesis and the Lemma,

$$|F(z)| \geq |a_n||z|^{n+1} - [|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-1}||z|^{n-1} + ...... + |a_{\lambda+1} - ka_\lambda||z|^{\lambda+1}$$
$$+ (k-1)|a_\lambda||z|^{\lambda+1} + |ka_\lambda - a_{\lambda-1}||z|^\lambda + (k-1)|a_\lambda||z|^\lambda + ...... + |a_1 - \tau a_0||z|$$
$$+ (1-\tau)|a_0||z| + |a_0|$$
$$= |z|^n [|a_n||z| - \{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \frac{1}{|z|} + ...... + |a_{\lambda+1} - ka_\lambda| \frac{1}{|z|^{n-\lambda-1}}$$
$$+ (k-1)|a_\lambda| \frac{1}{|z|^{n-\lambda-1}} + |ka_\lambda - a_{\lambda-1}| \frac{1}{|z|^{n-\lambda}} + (k-1)|a_\lambda| \frac{1}{|z|^{n-\lambda}} + ...... + |a_1 - \tau a_0| \frac{1}{|z|^{n-1}}$$
$$+ (1-\tau)|a_0| \frac{1}{|z|^{n-1}} + |a_0| \frac{1}{|z|^n}\}]$$
$$> |z|^n [|a_n||z| - \{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ...... + |a_{\lambda+1} - ka_\lambda|$$
$$+ (k-1)|a_\lambda| + |ka_\lambda - a_{\lambda-1}| + (k-1)|a_\lambda| + ...... + |a_1 - \tau a_0|$$
$$+ (1-\tau)|a_0| + |a_0|\}]$$
$$\geq |z|^n [|a_n||z| - \{(|a_n| - |a_{n-1}|)\cos\alpha + (|a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha$$
$$+ (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ...... + (|a_{\lambda+1}| - k|a_\lambda|)\cos\alpha + (|a_{\lambda+1}| + k|a_\lambda|)\sin\alpha$$
$$+ (k-1)|a_\lambda| + (k|a_\lambda| - |a_{\lambda+1}|)\cos\alpha + (k|a_\lambda| + |a_{\lambda+1}|)\sin\alpha + (k-1)|a_\lambda|$$
$$+ (|a_{\lambda+1}| - |a_{\lambda-2}|)\cos\alpha + (|a_{\lambda+1}| + |a_{\lambda-2}|)\sin\alpha + ......(|a_1| - \tau|a_0|)\cos\alpha$$
$$+ (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0| + |a_0|\}]$$
$$= |z|^n [|a_n||z| - \{|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|\}]$$
$$> 0$$

if

$$|z| > \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \leq \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \leq \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

**Proof of Theorem 5:** Consider the polynomial

$$F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + \{(k_1\alpha_n - k_2\alpha_{n-1}) - (k_1\alpha_n - \alpha_n) + (k_2\alpha_{n-1} - \alpha_{n-1})\} z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1}$$

$$+ ...... + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\} z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + ......$$

$$+ (\beta_1 - \beta_0)z + \beta_0\}$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,......., n$, so that, by using the hypothesis,

$$|F(z)| \geq |a_n||z|^{n+1} - [|k_1\alpha_n - k_2\alpha_{n-1}||z|^n + |k_1\alpha_n - \alpha_n||z|^n + |k_2\alpha_{n-1} - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-2}||z|^{n-1}$$

$$+ ...... + |\alpha_1 - \tau a_0||z| + (1 - \tau)|\alpha_0||z| + |\alpha_0| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}||z|^j + |\beta_0|]$$

$$= |z|^n[|a_n||z| - [|k_1\alpha_n - k_2\alpha_{n-1}| + |k_1\alpha_n - \alpha_n| + |k_2\alpha_{n-1} - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}|\frac{1}{|z|}......$$

$$+ |\alpha_1 - \tau\alpha_0|\frac{1}{|z|^{n-1}} + (1 - \tau)|\alpha_0|\frac{1}{|z|^{n-1}} + |\alpha_0|\frac{1}{|z|^n} + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}|\frac{1}{|z|^{n-j}} + |\beta_0|\frac{1}{|z|^n}\}]$$

$$> |z|^n[|a_n||z| - \{k_1\alpha_n - k_2\alpha_{n-1} + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + \alpha_{n-1} - \alpha_{n-2} + ......$$

$$+ \alpha_1 - \tau\alpha_0 + (1 - \tau)|\alpha_0| + |\alpha_0| + \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|) + |\beta_0|\}]$$

$$= |z|^n[|a_n||z| - \{k_1\alpha_n - k_2\alpha_{n-1} + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + \alpha_{n-1} + ......$$

$$- \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|\}]$$

$$= |z|^n[|a_n||z| - \{(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|)$$

$$+ 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|\}]$$

$$> 0$$

if

$$|z| > \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \le \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|] .$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \le \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

Since the zeros of P(z) are also the zeros of F(z), Theorem 4 follows.

**Proof of Theorem 6.** Consider the polynomial

$$\begin{aligned}F(z) &= (1 - z)P(z) = (1 - z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0)\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...... + (a_1 - a_0)z + a_0\\ &= -a_n z^{n+1} + \{(k_1 a_n - k_2 a_{n-1}) - (k_1 a_n - a_n) + (k_2 a_{n-1} - a_{n-1})\} z^n + (a_{n-1} - a_{n-2})z^{n-1}\\ &\quad + ...... + \{(a_1 - \tau a_0) + (\tau a_0 - a_0)\} z + a_0\end{aligned}$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,......., n$, so that, by using the hypothesis and the Lemma,

$$\begin{aligned}|F(z)| &\ge |a_n||z|^{n+1} - [|k_1 a_n - k_2 a_{n-1}||z|^n + |k_1 a_n - \alpha_n||z|^n + |k_2 a_{n-1} - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1}\\ &\quad + ...... + |a_1 - \tau a_0||z| + (1 - \tau)|a_0||z| + |a_0|]\end{aligned}$$

$$\begin{aligned}&= |z|^n[|a_n||z| - [|k_1 a_n - k_2 a_{n-1}| + |k_1 a_n - a_n| + |k_2 a_{n-1} - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + ......\\ &\quad + |a_1 - \tau a_0|\frac{1}{|z|^{n-1}} + (1 - \tau)|a_0|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}]\\ &> |z|^n[|a_n||z| - \{|k_1 a_n - k_2 a_{n-1}| + |k_1 a_n - a_n| + |k_2 a_{n-1} - a_{n-1}| + |a_{n-1} - a_{n-2}| + ......\\ &\quad + |a_1 - \tau a_0| + (1 - \tau)|a_0| + |a_0|\}]\\ &\ge |z|^n[|a_n||z| - \{(k_1|a_n| - k_2|a_{n-1}|)\cos\alpha + (k_1|a_n| + k_2|a_{n-1}|)\sin\alpha + (k_1 - 1)|a_n|\\ &\quad + (k_2 - 1)|a_{n-1}| + (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ......\\ &\quad + (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1 - \tau)|a_0|\}]\end{aligned}$$

$$\begin{aligned}&= |z|^n[|a_n||z| - \{k_1|a_n|(1 + \cos\alpha + \sin\alpha) + k_2|a_{n-1}|(1 - \cos\alpha + \sin\alpha) - |a_n|\\ &\quad - |a_{n-1}|(1 - \cos\alpha) - \tau|a_0|(1 + \cos\alpha - \sin\alpha) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|\}]\end{aligned}$$

$$>0$$

if

$$|z| > \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|] .$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \le \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|] .$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \le \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|] .$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

## REFERENCES


[1] A.Aziz and B.A.Zargar, Some Extensions of Enestrom-Kakeya Theorem, Glasnik Math. 31(1996), 239-244.
[2] N. K. Govvil, International Congress of ASIAM, Jammu University, India March 31-April 3, 2007.
[3] N. K. Govil and Q.I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math.J.20 (1968), 126-136.
[4] M. H. Gulzar, Some Refinements of Enestrom-Kakeya Theorem, Int. Journal of Mathematical Archive -2(9, 2011),1512-1529.
[5] M. H. Gulzar, Ph.D thesis, Department of Mathematics, University of Kashmir Srinagar, 2012.
[6] A. Joyal, G. Labelle and Q. I. Rahman, On the Location of Zeros of Polynomials, Canad. Math. Bull., 10(1967), 53-66.
[7] A. Liman and W. M. Shah, Extensions of Enestrom-Kakeya Theorem, Int. Journal of Modern Mathematical Sciences, 2013, 8(2), 82-89.
[8] M. Marden, Geometry of Polynomials, Math. Surveys, No.3; Amer. Math. Soc. Providence R.I. 1966.