Synchronization, Anti-Synchronization and Hybrid-Synchronization Of A Double Pendulum Under The Effect Of External Forces

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Abstract

In the present manuscript, an investigation on synchronization, anti-synchronization and hybrid-synchronization behavior of a double pendulum under the effect of external forces using active control technique based on the Lyapunov stability theory and the Routh-Hurwitz criteria, have been made. The designed controller with a new choice of co-efficient matrix of the error-dynamics are found to be effective in the stabilization of error states at the origin, thereby achieving synchronization between the states variables of two dynamical systems under consideration. Numerical simulations have been presented to illustrate the effectiveness of the proposed control techniques using mathematica.

Keywords: Double Pendulum under the effect of external forces, Lyapunov stability theory and Routh- Hurwitz Criteria, Synchronization, Anti-synchronization and Hybrid- synchronization.

1 Introduction

Classically, synchronization means adjustment of rhythm of self-sustained periodic oscillations due to their weak interaction and this adjustment can be described in terms of phase-locking and frequency entrainment. In the modern context we call such type of objects as rotators and chaotic systems. The history of synchronization actually goes back to the 17th century. In 1673, when the famous Dutch scientist Huygens [1] observed weak synchronization of two double pendulum clocks, which is about two model shape of vibration. He had invented shortly before: “It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motion of each pendulum in opposite swings were so much in agreement and sound of each was always heard simultaneously”. Furthermore he described that if one of the pendulum was disturbed by interference, it would return back to its normal state. This was first discovery of synchronization. After careful observation, finally he found that the cause of this is due to motion of the beam, even though this is hardly perceptible [2].Synchronization of periodic self-sustained oscillators are based on the existence of a special variable, called phase. If the coupled two pendulums have small oscillations with the same initial conditions or the zero initial phase difference, the two pendulums will be synchronized. If the initial phase difference is 180◦, the anti-synchronization of two pendulums can be observed. For a general case, the motion of the two pendulums will be combined by the synchronization and anti-synchronization modes of vibration.The recent progress on the Huygens synchronization was presented in [3].

Chaotic synchronization did not attract much attention until Pecora and Carroll [4] introduced a method to synchronize two identical chaotic systems with different initial conditions in 1990. From then on, enormous studies have been done by researchers on the synchronization of dynamical systems. In 1994, Kapitaniak [5] used continuous control to achieve a synchronization of two chaotic systems. In 1996, Peng et al.[6] presented chaotic synchronization of n-dimensional system. In the past few decades, the concept of synchronization from the traditional point of view has also been extended. In 2002, Boccaletti et al. [7] gave a review on the synchronization of chaotic systems and clarified definitions and concepts of dynamical system synchronization. In 2004, Compos and Urias [8] mathematically described multimodel synchronization with chaos, and introduced a multi-valued synchronized function. In 2005, Chen [9] investigated the synchronization of two different chaotic systems. Such synchronization is based on the error dynamics of the slave and master systems. The active control functions were used to remove non-linear terms, and the Lyapunov function was used to determine the stability of the synchronization. Lu and Cao [10] used the similar technique of Chen [9] to discuss the adaptive complete synchronization of two identical or different chaotic systems with unknown parameters. Thus in the continuation, a wide variety of methods have successfully been applied to achieve synchronization of chaotic systems. These methods including adaptive control [11, 12], backstepping design [13, 14, 15], active control [16, 17, 18] nonlinear control [19, 20, 21, 22] and observer based control method [23, 24]. Using these methods, numerous synchronization problem of well-known chaotic systems such as Lorenz, Chen, L¨u and R¨ossler system have been worked on by many researchers. Recently, Ge et al. [25, 26, 27, 28, 29, 30] also studied chaotic synchronization of many practical physical systems and obtained interesting results. Among these methods, chaos synchronization using active control has recently been widely accepted because it can be used to synchronize identical as well as non-identical systems. In order to achieve stable synchronization this method has been applied to many practical systems such as the electronic circuits, in which model there is third order "Jerk" equation [31], Lorenz, Chen and Lu system [32], geophysical systems [33], nonlinear equations waves (Lorenz Stenflo system)[34], Van-der Pol-duffing oscillator [35], forced damped pendulum [36], RCL-shunted Josephson function [37], modified projective synchronization [38]. In this paper, we have applied the active control techniques based on Lyapunov stability theory and Routh-Hurwitz criteria to study the synchronization, anti-synchronization and hybrid synchronization behavior of a double pendulum under the effect of external forces. It is well known that double pendulum is a chaotic system, its long term behavior can not be predicted. Slight changes in the initial conditions can result in drastic long term differences. If one starts the system at slightly different angles, perhaps by fraction of a degree, the resulting motion will not look same in the long run. In synchronization, two systems (master and slave) are synchronized and start with different initial conditions. The problem may be treated as the design of control laws for full chaotic slave system using known information of the master system so as to ensure that the controlled receiver synchronizes with the master system. Hence, the slave chaotic system completely traces the dynamics of the master system in the course of time. The aim of this study is to investigate the synchronization, anti-synchronization and hybrid synchronization of a double pendulum under the effect of external forces.

2 Equations of Motion Of Double Pendulum Under The Effect Of External Forces

In the figure given below, a double pendulum consists of two point masses $m1$ and $m2$ connected by massless rods to the pivot point. Let $\ell1$ and $\ell2$ are the lengths of rods respectively and $\theta1$ and $\theta2$ be the angles that two rods make with the vertical. Let F be the external force exerted on the pivot point by pendulum and ϕ be the angle that F makes with rod $\ell1$.

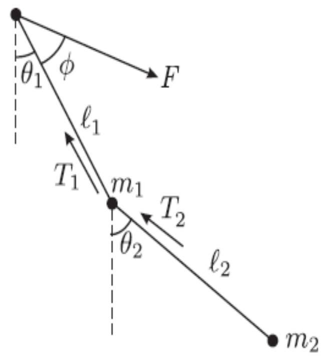

Fig. 1

The equations of motion are [39]:

$$\ddot{\theta}_1 = -\frac{T_1 \sin(\theta_1 - \theta_2) + g m_1 \sin\theta_2 - \ell_1 m_1 \sin(\theta_1 - \theta_2)\cdot \theta_1^2}{\cos(\theta_1 - \theta_2)\ell_1 m_1} \tag{1}$$

$$\ddot{\theta}_2 = -\frac{g m_2 - T_2\cos\theta_2 + \ell_1 m_2 \cos\theta_1 \cdot \dot{\theta}_1^2 + \ell_1 m_2 \sin\theta_1 \cdot \ddot{\theta}_1 + \ell_2 m_2 \cos\theta_2 \cdot \dot{\theta}_2^2}{\sin\theta_2 \cdot \ell_2 m_2}, \tag{2}$$

where g is the acceleration due to gravity, T_1 and T_2 be the tensions on each rod and are respectively given by

$$T_1 = F\cos(\phi - \theta) - \frac{(m_1\ddot{x}_1 + m_2\ddot{x}_2)}{\sin\theta_1}$$

and

$$T_2 = -\frac{m_2\ddot{x}_2}{\sin\theta_2} + F\cos(\phi - \theta_1)\cos(\theta_1 + \theta_2).$$

3 Synchronization Via Active Control

The systems defined by (1) and (2) can be written as a system of four first order differential equations, the four variables are introduced as below:

$$\begin{cases} x_1 = \theta_1 \\ x_2 = \dot{\theta}_1 \\ x_3 = \theta_2 \\ x_4 = \dot{\theta}_2 \end{cases} \tag{3}$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{T_1 \sin(x_1 - x_3) + g m_1 \sin x_3 - l_1 m_1 \sin(x_1 - x_3) \cdot x_2^2}{\cos(x_1 - x_3) l_1 m_1}, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -\frac{g m_2 - T_2 \cos x_3 + l_1 m_2 \cos x_1 \cdot x_2^2 + l_1 m_2 \sin x_1 \cdot \dot{x}_2 + l_2 m_2 \cos x_3 \cdot x_4^2}{\sin x_3 \cdot m_2 l_2}. \end{aligned}$$

After expanding trigonometrical terms and neglecting higher order terms only (for reducing non-linearity) in the above equations, we get:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{T_1(x_1 - x_3) + g m_1 x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \dfrac{(x_1 - x_3)^2}{2} \ell_1 m_1} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{g m_2 - T_2 \left(1 + \dfrac{x_3^2}{2}\right) + l_1 m_2 x_2^2 + l_1 m_2 x_1 \dot{x}_2 + l_2 m_2 \left(x_4^2 + \dfrac{x_4^2 x_3^2}{2}\right)}{l_2 m_2 x_3} \end{cases} \tag{4}$$

Corresponding to the master system (4), the identical slave system is defined as:

$$\begin{cases} \dot{y}_1 = y_2 + u_1(t) \\ \dot{y}_2 = -\dfrac{T_1(y_1 - y_3) + g m_1 y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \dfrac{(y_1 - y_3)^2}{2} \ell_1 m_1} + u_2(t) \\ \dot{y}_3 = y_4 + u_3(t) \\ \dot{y}_4 = -\dfrac{g m_2 - T_2 \left(1 + \dfrac{y_3^2}{2}\right) + l_1 m_2 y_2^2 + l_1 m_2 y_1 \dot{y}_2 + l_2 m_2 \left(y_4^2 + \dfrac{y_4^2 y_3^2}{2}\right)}{l_2 m_2 y_3} \\ \qquad + u_4(t) \end{cases} \tag{5}$$

where $u_i(t)$, $i = 1, 2, 3, 4$ are control functions to be determined. Now defining error functions such that in synchronization state $\lim_{t \to \infty} e_i(t) \to 0$, $i = 1, 2, 3, 4$.

$$\begin{aligned} e1 &= y1 - x1 \\ e2 &= y2 - x2 \\ e3 &= y3 - x3 \\ e4 &= y4 - x4 \end{aligned}$$

and the error dynamics are expressed as:

$$\begin{cases} \dot{e}_1 = \dot{y}_1 - \dot{x}_1 \\ \dot{e}_2 = \dot{y}_2 - \dot{x}_2 \\ \dot{e}_3 = \dot{y}_3 - \dot{x}_3 \\ \dot{e}_4 = \dot{y}_4 - \dot{x}_4 \end{cases} \quad (6)$$

from equations (4), (5) and (6), we get:

$$\begin{cases} \dot{e}_1(t) = e_2(t) + u_1(t) \\ \dot{e}_2(t) = -\dfrac{T_1(y_1 - y_3) + gm_1 y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \frac{(y_1 - y_3)^2}{2}\ell_1 m_1} \\ \qquad + \dfrac{T_1(x_1 - x_3) + gm_1 x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \frac{(x_1 - x_3)^2}{2}\ell_1 m_1} + u_2(t) \\ \dot{e}_3(t) = e_4(t) + u_3(t) \\ \dot{e}_4(t) = -\dfrac{gm_2 - T_2\left(1 + \frac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot{y}_2 + \ell_2 m_2\left(y_4^2 + \frac{y_4^2 y_3^2}{2}\right)}{\ell_2 m_2 y_3} \\ \qquad + \dfrac{gm_2 - T_2(1 + \frac{x_3^2}{2}) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot{x}_2 + \ell_2 m_2\left(x_4^2 + \frac{x_4^2 x_3^2}{2}\right)}{\ell_2 m_2 x_3} \\ \qquad + u_4(t). \end{cases} \quad (7)$$

The error dynamical system (7) to be controlled must be a linear system with control inputs. Therefore we redefine the control functions such as to eliminate non-linear terms in $e1(t)$, $e2(t)$, $e3(t)$ and $e4(t)$ of equation (7) as follows:

$$\begin{cases} u_1(t) = v_1(t) \\ u_2(t) = \dfrac{T_1(y_1 - y_3) + gm_1 y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \frac{(y_1 - y_3)^2}{2}\ell_1 m_1} \\ \quad - \dfrac{T_1(x_1 - x_3) + gm_1 x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \frac{(x_1 - x_3)^2}{2}\ell_1 m_1} + v_2(t) \\ u_3(t) = v_3(t) \\ u_4(t) = \dfrac{gm_2 - T_2\left(1 + \frac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot{y}_2 + \ell_2 m_2\left(y_4^2 + \frac{y_4^2 y_3^2}{2}\right)}{\ell_2 m_2 y_3} \\ \quad - \dfrac{gm_2 - T_2\left(1 + \frac{x_3^2}{2}\right) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot{x}_2 + \ell_2 m_2\left(x_4^2 + \frac{x_4^2 x_3^2}{2}\right)}{\ell_2 m_2 x_3} \\ \quad + v_4(t) \end{cases} \quad (8)$$

using (7) and (8), we have

$$\begin{cases} \dot{e}_1(t) = e_2(t) + v_1(t) \\ \dot{e}_2(t) = v_2(t) \\ \dot{e}_3(t) = e_4(t) + v_3(t) \\ \dot{e}_4(t) = v_4(t) \end{cases} \quad (9)$$

Equation (9) is the error dynamics, which can be interpreted as a control problem where the system to be controlled is a linear system with control inputs $v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$. We choose $v_1(t)$, $v_2(t)$, $v_3(t)$, $v_4(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{pmatrix} = A \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix},$$

where A is a 4×4 constant feedback matrix to be determined. The error dynamical system (9) can be written as:

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{pmatrix} = B \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix}, \quad (10)$$

where B is 4×4 co-efficient matrix. According to the Lyapunov stability theory and Routh-Hurwitz criteria, eigen-values of the co-efficient matrix of error system must be real or complex with negative real parts. We can choose elements of matrix arbitrarily; there are several ways to choose in order to satisfy Lyapunov and Routh-Herwitz criteria. Consequently, for

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \quad (11)$$

becomes a matrix with eigen values having negative real parts and equation (10) reduces to

$$\left.\begin{aligned} \dot{e}_1(t) &= -e_1(t) \\ \dot{e}_2(t) &= -2e_2(t) \\ \dot{e}_3(t) &= -3e_3(t) \\ \dot{e}_4(t) &= -4e_4(t) \end{aligned}\right\} \quad (12)$$

Thus, by Lyapunov stability theory, the error dynamical system (12) is stable.

4 . Numerical Simulation For Synchronization

For the parameters involved in system under investigation, $T1 = 1$, $T2 = 2$, $\ell 1 = 1$, $\ell 2 = 2$, $m1 = 1$, $m2 = 1$ and $g = 9.8$m/s2 with the initial conditions for master system and slave system

$[x1(0), x2(0), x3(0), x4(0)] = [3.5, 0.4, -3.5, 0.4]$

And

$[y1(0), y2(0), y3(0), y4(0)] = [1.5, 0.2, -0.5, 0.8]$

respectively. We have simulated the system under consideration using mathematica. Phase portraits and time series analysis of master and slave system are the witness of irregular behavior of system (see figures 2, 3, 4 and 5). For

$[e1(0), e2(0), e3(0), e4(0)] = [-2, 0.2, 3, 0.4]$

convergence diagrams of errors are the witness of achieving synchronization between master and slave systems (see figure 6).

5 Anti-Synchronization Via Active Control

In order to formulate the active controllers for anti-synchronization we need to redefine the error functions as, $e_1(t) = y1 + x1$, $e_2(t) = y2 + x2$, $e_3(t) = y3 + x3$, $e_4(t) = y4 + x4$.

Accordingly, error dynamics are:

$$\dot{e}_1(t) = \dot{y}_1 + \dot{x}_1 \quad \dot{e}_3(t) = \dot{y}_3 + \dot{x}_3$$
$$\dot{e}_2(t) = \dot{y}_2 + \dot{x}_2, \quad \dot{e}_4(t) = \dot{y}_4 + \dot{x}_4$$

from (4) and (5), error dynamics can be written as:

$$\begin{cases} \dot{e}_1(t) = e_2(t) + u_1(t) \\ \dot{e}_2(t) = -\dfrac{T_1(y_1 - y_3) + gm_1y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \dfrac{(y_1-y_3)^2}{2}\ell_1 m_1} \\ -\dfrac{T_1(x_1 - x_3) + gm_1x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \dfrac{(x_1-x_3)^2}{2}\ell_1 m_1} + u_2(t) \\ \dot{e}_3(t) = e_4(t) + u_3(t) \\ \dot{e}_4(t) = -\dfrac{\left[gm_2 - T_2\left(1+\dfrac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot{y}_2 + \ell_2 m_2\left(y_4^2 + \dfrac{y_4^2 y_3^2}{2}\right)\right]}{\ell_2 m_2 y_3} \\ \quad -\dfrac{\left[gm_2 - T_2\left(1+\dfrac{x_3^2}{2}\right) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot{x}_2 + \ell_2 m_2\left(x_4^2 + \dfrac{x_4^2 x_3^2}{2}\right)\right]}{\ell_2 m_2 x_3} \\ \quad + u_4(t). \end{cases} \tag{13}$$

In order to express (13) as only linear terms in $e1(t)$, $e2(t)$, $e2(t)$ and $e4(t)$, we redefine the control functions as follows:

$$\begin{cases} u_1(t) = v_1(t) \\ u_2(t) = \dfrac{T_1(y_1 - y_3) + gm_1y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \dfrac{(y_1-y_3)^2}{2}\ell_1 m_1} \\ \quad +\dfrac{T_1(x_1 - x_3) + gm_1x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \dfrac{(x_1-x_3)^2}{2}\ell_1 m_1} + v_2(t) \\ u_3(t) = v_3(t) \\ u_4(t) = \dfrac{gm_2 - T_2\left(1+\dfrac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot{y}_2 + \ell_2 m_2\left(y_4^2 + \dfrac{y_4^2 y_3^2}{2}\right)}{\ell_2 m_2 y_3} \\ \quad +\dfrac{gm_2 - T_2\left(1+\dfrac{x_3^2}{2}\right) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot{x}_2 + \ell_2 m_2\left(x_4^2 + \dfrac{x_4^2 x_3^2}{2}\right)}{\ell_2 m_2 x_3} \\ \quad + v_4(t). \end{cases} \tag{14}$$

Using (13) and (14), we have

$$\begin{cases} \dot{e}_1(t) = e_2(t) + v_1(t) \\ \dot{e}_2(t) = v_2(t) \\ \dot{e}_3(t) = e_4(t) + v_3(t) \\ \dot{e}_4(t) = v_4(t) \end{cases} \tag{15}$$

Furthermore, as in the previous case we choose $v1(t)$, $v2(t)$, $v3(t)$ and $v4(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{pmatrix} = A \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix},$$

where A is 4×4 constant matrix to be determined. Equation (15) reduces to

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{pmatrix} = B \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix},$$

where B is given by (11) whose eigen values have negative real parts. Thus by Lyapunov stability theory, the error dynamical system (15) is stable.

6 Numerical Simulation For Anti-Synchronization

For the parameters involved in system under investigation, $T1 = 1$, $T2 = 2$, $\ell 1 = 1$, $\ell 2 = 2$, $m1 = 1$, $m2 = 1$ and $g = 9.8$m/s2 with the initial conditions for master and slave systems

$[x1(0), x2(0), x3(0), x4(0)] = [3.5, 0.4, -3.5, 0.4]$
and

$[y1(0), y2(t), y3(0), y4(0)] = [1.5, 0.2, -0.5, 0.8]$
respectively. We have simulated the system under consideration using mathematica. Phase portraits and time series analysis of master system and slave system are the witness of irregular behavior of system (see figures 2, 3, 4 and 5). For

$[e1(0), e2(0), e3(0), e4(0)] = [5, 0.6, 4.0, 1.2]$

convergence diagram of errors are the witness of achieving anti-synchronization between master and slave system (see figure 7).

7 Hybrid Synchronization Via Active Control

The idea of the hybrid synchronization is to use the output of the master system to control the slave system so that the odd outputs of the two systems are completely synchronized, while the even outputs of the two systems are anti-synchronized so that both complete synchronization and anti-synchronization persist in the synchronization of master and slave systems. In order to formulate the active controllers for "hybrid synchronization" we are redefining the error functions in the following three ways:

$$\begin{array}{ccc} \begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 + x_2 \\ e_3 &= y_3 + x_3 \\ e_4 &= y_3 + x_4 \end{aligned} & \begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 + x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 + x_4 \end{aligned} & \begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 + x_4 \end{aligned} \\ \textbf{(I)} & \textbf{(II)} & \textbf{(III)} \end{array}$$

Accordingly, the error dynamics are:

$$\begin{array}{ccc} \begin{aligned} \dot e_1 &= \dot y_1 - \dot x_1 \\ \dot e_2 &= \dot y_2 + \dot x_2 \\ \dot e_3 &= \dot y_3 + \dot x_3 \\ \dot e_4 &= \dot y_4 + \dot x_4 \end{aligned} & \begin{aligned} \dot e_1 &= \dot y_1 - \dot x_1 \\ \dot e_2 &= \dot y_2 + \dot x_2 \\ \dot e_3 &= \dot y_3 - \dot x_3 \\ \dot e_4 &= \dot y_4 + \dot x_4 \end{aligned} & \begin{aligned} \dot e_1 &= \dot y_1 - \dot x_1 \\ \dot e_2 &= \dot y_2 - \dot x_2 \\ \dot e_3 &= \dot y_3 - \dot x_3 \\ \dot e_4 &= \dot y_4 + \dot x_4 \end{aligned} \\ \textbf{(I)} & \textbf{(II)} & \textbf{(III)} \end{array}.$$

Now first we take any one of above equation, let us take (II),

$$\begin{aligned} \dot e_1 &= \dot y_1 - \dot x_1 \\ \dot e_2 &= \dot y_2 + \dot x_2 \\ \dot e_3 &= \dot y_3 - \dot x_3 \\ \dot e_4 &= \dot y_4 + \dot x_4 \end{aligned}$$

from (II), (4) and (5) we have,

$$\begin{cases} \dot e_1(t) = e_2(t) + u_1(t) \\ \dot e_2(t) = -\dfrac{T_1(y_1 - y_3) + g m_1 y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \dfrac{(y_1-y_3)^2}{2}\ell_1 m_1} \\ \qquad -\dfrac{T_1(x_1 - x_3) + g m_1 x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \dfrac{(x_1-x_3)^2}{2}\ell_1 m_1} + u_2(t) \\ \dot e_3(t) = e_4(t) + u_3(t) \\ \dot e_4(t) = \dfrac{-\left[g m_2 - T_2\left(1+\dfrac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot y_2 + \ell_2 m_2\left(y_4^2 + \dfrac{y_4^2 y_3^2}{2}\right)\right]}{\ell_2 m_2 y_3} \\ \qquad \dfrac{-\left[g m_2 - T_2\left(1+\dfrac{x_3^2}{2}\right) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot x_2 + \ell_2 m_2\left(x_4^2 + \dfrac{x_4^2 x_3^2}{2}\right)\right]}{\ell_2 m_2 x_3} \\ \qquad + u_4(t). \end{cases} \tag{16}$$

The error dynamical system (16) to be controlled must be a linear system with control inputs. Therefore we redefine the control functions such as to eliminate non-linear terms in $e1(t)$, $e2(t)$, $e3(t)$ and $e4(t)$ of (16) as follows:

$$\begin{cases} u_1(t) = v_1(t) \\ u_2(t) = \dfrac{T_1(y_1 - y_3) + gm_1y_3 - \ell_1 m_1 y_1 y_2^2}{\ell_1 m_1 + \dfrac{(y_1 - y_3)^2}{2}\ell_1 m_1} \\ \qquad + \dfrac{T_1(x_1 - x_3) + gm_1x_3 - \ell_1 m_1 x_1 x_2^2}{\ell_1 m_1 + \dfrac{(x_1 - x_3)^2}{2}\ell_1 m_1} + v_2(t) \\ u_3(t) = v_3(t) \\ u_4(t) = \dfrac{\left[gm_2 - T_2\left(1 + \dfrac{y_3^2}{2}\right) + \ell_1 m_2 y_2^2 + \ell_1 m_2 y_1 \dot{y}_2 + \ell_2 m_2 \left(y_4^2 + \dfrac{y_4^2 y_3^2}{2}\right)\right]}{\ell_2 m_2 y_3} \\ \qquad + \dfrac{\left[gm_2 - T_2\left(1 + \dfrac{x_3^2}{2}\right) + \ell_1 m_2 x_2^2 + \ell_1 m_2 x_1 \dot{x}_2 + \ell_2 m_2 \left(x_4^2 + \dfrac{x_4^2 x_3^2}{2}\right)\right]}{\ell_2 m_2 x_3} \\ \qquad + v_4(t). \end{cases} \tag{17}$$

From (16) and (17), error dynamics are:

$$\begin{cases} \dot{e}_1(t) = e_2(t) + v_1(t) \\ \dot{e}_2(t) = v_2(t) \\ \dot{e}_3(t) = e_4(t) + v_3(t) \\ \dot{e}_4(t) = v_4(t). \end{cases} \tag{18}$$

Similarly, we get same error-dynamical systems in all three cases. Furthermore, as in the above cases we choose $v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{pmatrix} = A \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix}$$

and (18) reduces to

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{pmatrix} = B \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix},$$

where B is given by (11), whose eigen-values have negative real parts. Thus, by Lyapunov stability theory, the error dynamical system (18) is stable.

8 Numerical Simulation For Hybrid Synchronization

For the parameters involved in system under investigation, $T1 = 1$, $T2 = 2$, $\ell 1 = 1$, $\ell 2 = 2$, $m1 = 1$, $m2 = 1$ and $g = 9.8m/s2$ with the initial conditions of master and slave systems

$[x1(0), x2(0), x3(0), x4(0)] = [3.5, 0.4, -3.5, 0.4]$,

and

$[y1(0), y2(0), y3(0), y4(0)] = [1.5, 0.2, -0.5, 0.8]$

respectively.

We have simulated the system under consideration by using mathematica. Phase portraits and time series analysis of master and slave system are the witness of irregulars behavior of the system (see figures 2, 3, 4 and 5) and for

$[e1(0), e2(0), e3(0), e4(0)] = [-2, 0.6, 3, 1.2]$

convergence diagram of errors are the witness of achieving hybrid synchronization between master and slave system (see figure 8). Figures are given below:

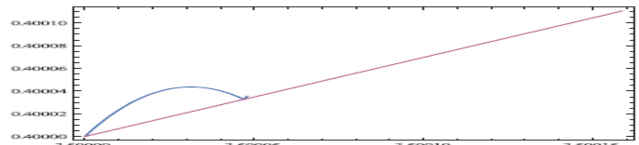

Fig. 2 Phase portrait of master system.

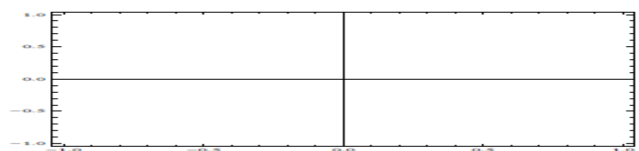

Fig. 3 Phase portrait of slave system.

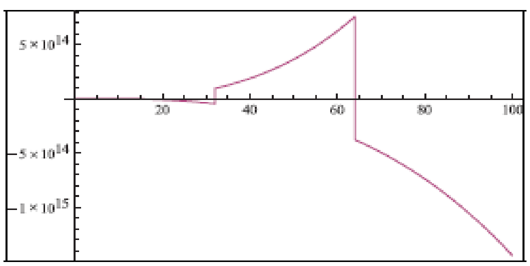

Fig. 4 Time series analysis of $x(t)$

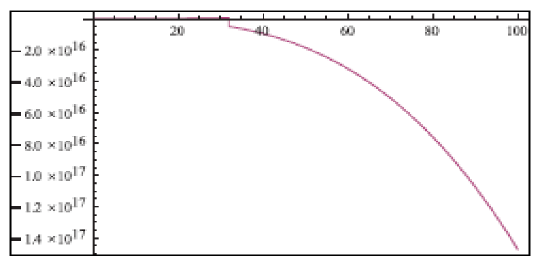

Fig. 5 Time series analysis of $y(t)$

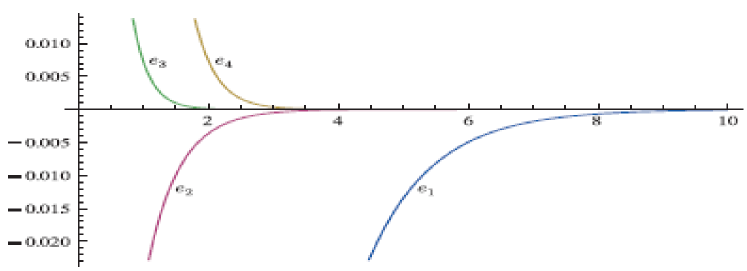

Fig. 6 Convergence of errors e_1, e_2, e_3 and e_4 for synchronization and $t = [0, 10]$

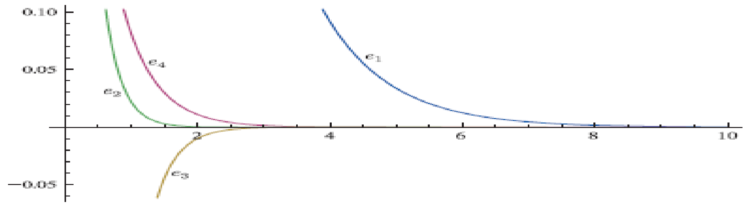

Fig. 7 Convergence of errors e_1, e_2, e_3 and e_4 for anti-synchronization and $t = [0, 10]$

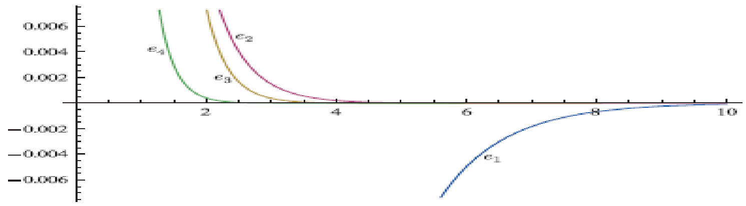

Fig. 8 Convergence of errors e_1, e_2, e_3 and e_4 for hybrid synchronization and $t = [0, 10]$

9 Conclusion

An investigation on synchronization, anti-synchronization and hybrid synchronization of the double pendulum under the effect of external forces via active control technique based on Lyapunov stability theory and Routh-Hurwitz criteria have been made. The results are validated by numerical simulations using mathematica.

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