L^p – convergence of Rees-Stanojevic sum

Nawneet Hooda
Department of Mathematics,
DCR University of Sci& Tech, Murthal , Sonepat-131039 (INDIA)

Abstract

We study L^p-convergence ($0<p<1$) of Rees-Stanojevic modified cosine sum[3] and deduce the result of Ul'yanov [4] as corollary from our result.

1. Introduction

Let us consider the series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.1}$$

with coefficients $a_k \downarrow 0$ or even satisfying the conditions $a_k \to 0$ as $k\to\infty$ and $\sum_{k=1}^{\infty} |\Delta a_k| < \infty$. Riesz [Cf .1] showed that the function f(x) defined by the series (1.1) for $a_k \downarrow 0$ can be non-summable. However, they are summable to any degree p provided $0 < p < 1$.

Theorem A.[4] If the sequence $< a_k >$ satisfies the condition $a_k \to 0$ and $\Sigma |\Delta a_k| < +\infty$, then for any p, $0 < p < 1$, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(x) - S_n(x) \right|^p dx = 0,$$

where $S_n(x)$ is the partial sum of the series (1.1) .

Rees and Stanojevic [3] (see also Garrett and Stanojevic [2]) introduced a modified cosine sum

$$h_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx. \tag{1.2}$$

Regarding the convergence of (1.2) in L-metric, Garrett and Stanojevic [2] proved the following result :

Theorem B. If $\{a_k\}$ is a null quasi-convex sequence . Then

$$\|h_n(x) - f(x)\| = o(1),\ n\to\infty,$$

where f(x) is the sum of cosine series (1.1).

In this paper ,we study the L^p–convergence of this modified sum (1.2) and deduce Theorem A as corollary of our theorem.

2000 AMS Mathematics Subject Classification : 42A20, 42A32.

2. Results.

Theorem . If the sequence $\{a_k\}$ satisfies the conditions $a_k \to 0$ and $\Sigma|\Delta a_k| < \infty$, then for any p, $0 < p < 1$, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - h_n(x)|^p \, dx = 0 \ .$$

Proof. We have

$$h_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \ .$$

Using Abel's transformation,

$$h_n(x) = \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x)$$

$$= \sum_{k=0}^{n} \Delta a_k D_k(x) \ .$$

Since $D_n(x) = O(1/x^2)$ for $x \neq 0$, and $a_n \to 0$, right side tends to zero, where

$$D_n(x) = (1/2) + \cos x + \ldots + \cos nx$$

represents Dirichlet's kernel.

Now,

$$f(x) - h_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x)$$

This means

$$|f(x) - h_n(x)|^p \leq \left(\frac{2}{|x|}\right)^p \left[\sum_{k=n+1}^{\infty} |\Delta a_k|\right]^p ,$$

and therefore,

$$\int_{-\pi}^{\pi} |f(x) - h_n(x)|^p \, dx \leq 2^p \left[\sum_{k=n+1}^{\infty} |\Delta a_k|\right]^p \int_{-\pi}^{\pi} \frac{dx}{x^p}$$

$$\to 0 \text{ as } n \to \infty \ .$$

Corollary 1. If the sequence $\{a_k\}$ satisfies $a_k \to 0$ and $\Sigma|\Delta a_k| < \infty$, then for any $0 < p < 1$, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - S_n(x)|^p \, dx = 0 \ .$$

We have

$$\int_{-\pi}^{\pi} |f(x) - S_n(x)|^p\,dx = \int_{-\pi}^{\pi} |f(x) - f_n(x) + f_n(x) - S_n(x)|^p\,dx$$

$$\leq \int_{-\pi}^{\pi} |f(x) - f_n(x)|^p\,dx + \int_{-\pi}^{\pi} |f_n(x) - S_n(x)|^p\,dx$$

$$= \int_{-\pi}^{\pi} |f(x) - f_n(x)|^p\,dx + \int_{-\pi}^{\pi} |a_{n+1} D_n(x)|^p\,dx$$

Now,

$$\int_{-\pi}^{\pi} |a_{n+1} D_n(x)|^p\,dx \leq \int_{-\pi}^{\pi} \left(\frac{2}{|x|}\right)^p |a_{n+1}|^p\,dx$$

$$= 2^p |a_{n+1}|^p \int_{-\pi}^{\pi} (dx/x^p) \to 0 \quad \text{as } n\to\infty,$$

also $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - h_n(x)|^p\,dx = 0$ by our theorem . Hence the corollary follows.

References

[1] N.K. Bari, *A Treatise on trigonometricseries* (London : Pregamon Press) Vol. II (1964).

[2] J.W.Garrett and C.V.Stanojevic, On L^1─convergence of certain cosine sums, *Proc. Amer. Math. Soc.*, **54** (1976) 101─105.

[3] C.S. Rees and Č.V. Stanojević, Necessary and sufficient condition for Integrability of certain cosine sums, *J. Math.* Anal. Appl. **43**(1973), 579-586.

[4] P.L. Ul'yanov, Application of A-integration to a class of trigonometric series (in Russian) *MC* **35** (1954), 469-490.