Route to Chaos in an Area-preserving System

Dr. Nabajyoti Das
Assistant Professor, Department of Mathematics Jawaharlal Nehru College, Boko -781123
District: Kamrup, State: Assam, Country: India

Abstract:
This paper highlights two important objectives on a two-dimensional area-preserving discrete dynamical system:
$E(x, y) = (-y + px - (1-p)x^2, x - y + px - (1-p)x^2)$,
where p is a tunable parameter. Firstly, by adopting suitable computer programs we evaluate period-doubling:
$period\ 1 \rightarrow period\ 2 \rightarrow period\ 4 \rightarrow \ldots \rightarrow period\ 2^k \rightarrow \ldots \rightarrow chaos$
bifurcations, as a universal route to chaos, for the periodic orbits when the system parameter p varied and obtain the Feigenbaum universal constant = 8.7210972…, and the accumulation point α = 7.533284771388…. beyond which chaotic region occurs.. Secondly, the periodic behaviors of the system are confirmed by plotting the time series graphs.

Key Words: Period-doubling bifurcations/ Periodic orbits / Feigenbaum universal constant / Accumulation point / Chaos / Area-preserving system
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1. Introduction

The initial universality discovered by Mitchell J. Feigenbaum in 1975 has successfully led to discover that large classes of nonlinear systems exhibit transitions to chaos which are universal and quantitatively measurable. If $\boldsymbol{X}$ be a suitable function space and $\boldsymbol{H}$, the hypersurface of co-dimension 1 that consists of the maps in $\boldsymbol{X}$ having derivative -1 at the fixed point, then the **Feigenbaum universality** is closely related to the doubling operator, $\boldsymbol{F}$ acting in $\boldsymbol{X}$ defined by

$$(\boldsymbol{F}\psi)(x) = -\alpha\psi(\psi(\alpha^{-1}x)) \qquad \psi \in \boldsymbol{X}$$

where α = 2.5029078750957… , a universal scaling factor. The principal properties of $\boldsymbol{F}$ that lead to universality are

(i) $\boldsymbol{F}$ has a fixed point x^*;

(ii) The linearised transformation $D\boldsymbol{F}(x^*)$ has only one eigenvalue δ greater than 1 in modulus; here δ = 4.6692016091029…

(iii) The unstable manifold corresponding to δ intersects the surface $\boldsymbol{H}$ transversally; In one dimensional case, these properties have been proved by Lanford [2, 10].

Next, let X be the space of two parameter family of area- preserving maps defined in a domain $U \subseteq \Re^2$, and Y, the space of two parameter family of maps defined in the same domain having not necessarily constant Jacobian. Then Y contains X. In area- preserving case, the Doubling operator $\boldsymbol{F}$ is defined by

$$\boldsymbol{F}\psi = \Sigma^{-1}\psi^2\Sigma,$$

where $\Sigma : \Re^2 \rightarrow \Re^2, \Sigma(x, y) = (\lambda x, \mu y)$ is the scaling transformation. Here λ and μ are the scaling factors; numerically we have $\lambda = -0.248875\ldots$ and $\mu = 0.061101\ldots$ In the area preserving case, Feigenbaum constant, δ = 8.721097200….. Furthermore, one of his fascinating discoveries is that if a family ψ presents period doubling bifurcations then there is an infinite sequence { μ_n } of bifurcation values such that $\alpha = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta$, where δ is a universal number already mentioned above. Moreover, his observation suggests that there is a universal size-scaling in the period doubling sequence designated as the Feigenbaum $\alpha -$value, $\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.5029\ldots$ where d_n is the size of the bifurcation pattern of period 2^n just before it gives birth to period 2^{n+1} [1, 6-8].

The birth and flowering of the Feigenbaum universality with numerous non-linear models has motivated me to wrire this paper.

2. Our Nonlinear Map and the Feigenbaum Universality:

Our concerned map:

$$E(x,y) = (-y + px - (1-p)x^2, x - y + px - (1-p)x^2) \tag{1.1}$$

where p is a tunable parameter. The Jacobian of E is the unity, so is area-preserving.
The map has one fixed point other than (0,0) whose coordinates is given by

$$x = \frac{3-p}{-1+p}, y = \frac{-6+2p}{1-p}$$

From this one finds that E has no fixed point if p = 1. In this context, we also wish to point out that the stability theory is intimately connected with the Jacobian matrix of the map, and that the trace of the Jacobian matrix is the sum of its eigenvalues and the product of the eigenvalues equal the Jacobian determinant. For a particular value of p, the map E depends on the real parameter p, and so a fixed point $\overline{x}_s$ of this map depends on the parameter value p, i.e. $\overline{x}_s = \overline{x}_s(p)$. Now, first consider the set, $U = (-\infty,3) \cup (7,+\infty)$.

The fixed point $\overline{x}_s$ remains stable for all values of p lying in U and a stable periodic trajectory of period-one appears around it. This means, the two eigenvalues

$$\lambda_1 = \frac{-5 + \sqrt{-7+p}\sqrt{-3+p}(-1+p) + 6p - p^2}{2(-1+p)},$$

$$\lambda_2 = \frac{-5 - \sqrt{-7+p}\sqrt{-3+p}(-1+p) + 6p - p^2}{2(-1+p)}$$

of the Jacobian matrix:

$$J = \begin{bmatrix} p - 2(1-p)x & -1 \\ 1 + p - 2(1-p)x & -1 \end{bmatrix}$$

at $\overline{x}_s$ remains less than one in modulus and consequently, all the neighbouring points (that is, points in the domain of attraction) are attracted towards $\overline{x}_s(p)$), p lying in U. If we now begin to increase the value of p, then it happens that one of the eigenvalues starts decreasing through –1 and the other remains less than one in modulus. When p = 7, one of the eigenvalues becomes –1 and then $\overline{x}_s$ loses its stability, i.e. $p_1 = 7$ emerging as **the first bifurcation value** of p. Again, if we keep on increasing the value of p the point $\overline{x}_s(p)$ becomes unstable and there arises around it two points, say, $\overline{x}_{21}(p)$ and $\overline{x}_{22}(p)$ forming a stable periodic trajectory of period-two. All the neighbouring points except the stable manifold of $\overline{x}_s(p)$ are attracted towards these two points. Since the period emerged becomes double, the previous eigenvalue which was –1 becomes +1 and as we keep increasing p, one of the eigenvalues starts decreasing from +1 to –1.Since the trace is always real, when eigenvalues are complex, they are conjugate to each other moving along the circle of radius $\sqrt{p_e}$, where $p_e = p^{2^n}$ is the effective Jacobian, in the opposite directions. When we reach a certain value of p, we find that one of the eigenvalues of the Jacobian of E^2 (because of the chain rule of differentiation, it does not matter at which periodic point one evaluates the eigenvalues) becomes -1, indicating the loss of stability of the periodic trajectory of period 2. Thus, the second bifurcation takes place at this value p_2 of p. We can then repeat the same arguments, and find that the periodic trajectory of period 2 becomes unstable and a periodic trajectory of period 4 appears in its neighbourhood. This phenomenon continues upto a particular value of p say p_3(p), at which the periodic trajectory of period 4 losses its stability in such a way that one of the eigenvalues at any of its periodic points become –1, and thus it gives the third bifurcation at $p_3(p)$. Increasing the value further and further, and repeating the same arguments we obtain a sequence $\{p_n(p)\}$ as bifurcation values for the parameter p such that at $p_n(p)$ a periodic trajectory of period 2^n arises and all periodic trajectories of period 2^m(m<n) remain unstable. The sequence $\{p_n(p)\}$ behaves in a universal manner such that $p_\infty(p)$- $p_n(p)\}$ ~ $c(p)\delta^{-n}$, where $c(p)$ is independent of n and δ and is the **Feigenbaum universal constant**. Since the map has constant Jacobian 1(unity), we have the conservative case, i.e. the preservation of area and in this case δ equals 8.721097200…. Furthermore, the Feigenbaum theory says that the our map E at parameter = $p_\infty(p)$

has an invariant set S of Cantor type encompassed by infinitely many unstable periodic orbits of period 2^n(n = 0, 1, 2,…), and that all the neighbouring points except those belonging to these unstable orbits and their stable manifolds are attracted to S under the iterations of the map *E*.

3. Numerical Method For Obtaining Periodic Points [2]:

Although there are so many sophisticated numerical algorithms available, to find a periodic fixed point, we have found that the Newton Recurrence formula is one of the best numerical methods with negligible error for our purpose. Moreover, it gives fast convergence of a periodic fixed point.

The Newton Recurrence formula is

$$\overline{x}_{n+1} = \overline{x}_n - Df(\overline{x}_n)^{-1} f(\overline{x}_n),$$

where n = 0,1,2,… and $Df(\overline{x})$ is the Jacobian of the map f at the vector $\overline{x}$. We see that this map *f* is equal to $E^k - I$ in our case, where *k* is the appropriate period. The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool in our map one needs a number of recurrence formulae which are given below.

Let the initial point be (x_0, y_0),

Then,

$$E(x_0, y_0) = (-y_0 + px_0 - (1-p)x_0^2, x_0 - y_0 + px_0 - (1-p)x_0^2) = (x_1, y_1)$$

$$E^2(x_0, y_0) = E(x_1, y_1) = (x_2, y_2)$$

Proceeding in this manner the following recurrence formula for our map can be established.

$x_n = -y_{n-1} + px_{n-1} - (1-p)x_{n-1}^2$, and $y_n = -y_{n-1} + px_{n-1} - (1-p)x_{n-1}^2$,

where n = 1,2,3…

Since the Jacobian of $\boldsymbol{E^k}$ (k times iteration of the map E) is the product of the Jacobian of each iteration of the map, we proceed as follows to describe our recurrence mechanism for the Jacobian matrix.

The Jacobian $\mathbf{J_1}$ for the transformation

$E(x_0, y_0) = (-y_0 + px_0 - (1-p)x_0^2, x_0 - y_0 + px_0 - (1-p)x_0^2)$ is

$$J_1 = \begin{pmatrix} p - 2(1-p)x_0 & -1 \\ 1 + p - 2(1-p)x_0 & -1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

where $A_1 = p - 2(1-p)x_0$, $B_1 = -1$, $C_1 = 1 + p - 2(1-p)x_0$, $D_1 = -1$.

Next the Jacobian J_2 for the transformation

$E^2(x_0, y_0) = (x_2, y_2)$, is the product of the Jacobians for the transformations

$E(x_1, y_1) = (-y_1 + px_1 - (1-p)x_1^2, x_1 - y_1 + px_1 - (1-p)x_1^2)$ and

$E(x_0, y_0) = (-y_0 + px_0 - (1-p)x_0^2, x_0 - y_0 + px_0 - (1-p)x_0^2)$.

So, we obtain

$$J_2 = \begin{pmatrix} p - 2(1-p)x_1 & -1 \\ 1 + p - 2(1-p)x_1 & -1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

where $A_2 = [p - 2(1-p)x_1]A_1 - C_1, B_2 = [p - 2(1-p)x_1]B_1 - D_1,$

$C_2 = [1 + p - 2(1-p)x_1]A_1 - C_1, D_2 = [1 + p - 2(1-p)x_1]B_1 - D_1.$

Continuing this process in this way, we have the Jacobian for E^m as

$$J_m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}$$

with a set of recursive formula as

$A_m = [p - 2(1-p)x_{m-1}]A_{m-1} - C_{m-1}, B_m = [p - 2(1-p)x_{m-1}]B_{m-1} - D_{m-1},$

$C_m = [p - 2(1-p)x_{m-1}]A_{m-1} - C_{m-1}, D_m = [p - 2(1-p)x_{m-1}]B_{m-1} - D_{m-1},$

(m = 2, 3, 4, 5…).

Since the fixed point of this map E is a zero of the map

$$G(x, y) = E(x, y) - (x, y),$$

the Jacobian of $G^{(k)}$ is given by

$$J_k - I = \begin{pmatrix} A_k - 1 & B_k \\ C_k & D_k - 1 \end{pmatrix}$$

Its inverse is $(J_k - I)^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_k - 1 & B_k \\ C_k & A_k - 1 \end{pmatrix}$,

where $\Delta = (A_k - 1)(D_k - 1) - B_k C_k$,

the Jacobian determinant. Therefore, Newton's method gives the following recurrence formula in order to yield a periodic point of E^k

$$x_{n+1} = x_n - \frac{(D_k - 1)(\hat{x}_n - x_n) - B_k(\hat{y}_n - y_n)}{\Delta}$$

$$y_{n+1} = y_n - \frac{(-C_k)(\hat{x}_n - x_n) + (A_k - 1)(\hat{y}_n - y_n)}{\Delta},$$

where $E^k(\overline{x}_n) = (\hat{x}_n, \hat{y}_n)$.

4. Numerical Methods For Finding Bifurcation Values [2, 4]:

First of all, we recall our recurrence relations for the Jacobian matrix of the map E^k described in the Newton's method and then the eigenvalue theory gives the relation $A_k + D_k = -1 + Det(J_k)$ at the bifurcation value. Again the Feigenbaum theory says that

$$p_{n+2} \approx p_{n+1} + \frac{p_{n-1} - p_n}{\delta} \qquad (1.2)$$

where $n = 1,2,3,\ldots$ and δ is the Feigenbaum universal constant.
In the case of our map, the first two bifurcation values p_1 and p_2 can be evaluated.
Furthermore, it is easy to find the periodic points for p_1 and p_2. We note that if we put $I = A_k + D_k = -1 + Det(J_k)$, then I turns out to be a function of the parameter p. The bifurcation value of p of the period k occurs when $I(p)$ equals zero. This means, in order to find a bifurcation value of period k, one needs the zero of the function $I(p)$, which is given by the Secant method,

$$p_{n+1} = p_n - \frac{I(p_n)(p_n - p_{n-1})}{I(p_n) - I(p_{n-1})}.$$

Then using the relation (1.2), an approximate value p_3' of p_3 is obtained. Since the Secant method needs two initial values, we use p_3' and a slightly larger value, say, $p_3' + 10^{-4}$ as the two initial values to apply this method and ultimately obtain p_3. In like manner, the same procedure is employed to obtain the successive bifurcation values $p_4, p_5, \ldots$ etc. to our requirement.

For finding periodic points and bifurcation values for the map E, above numerical methods are used and consequently, the following **Period-Doubling Cascade**: Table 1.1, showing bifurcation points and corresponding periodic points, are obtained by using suitable computer programs:

Table 1.1

Period	One of the Periodic points	Bifurcation Pt.
1	*(x=0.666666666666...., y= -1.333333333333....)*	*p_1=7.00000000000*
2	*(x= -0.500000000003..., y= -1.309016994376...)*	*p_2=7.47213595500*
4	*(x= -0.811061640408 ..., y= -1.273315586957...)*	*p_3=7.525683372…*
8	*(x=-0.813878975794 ..., y= -1.275108054848...)*	*p_4=7.531826966…*
16	*(x=-0.460474775277...,y= -1.273990103300...)*	*p_5=7.532531327…*
32	*(x=-0.46055735696 ..., y= -1.274198905689...)*	*p_6=7.532612093…*
64	*(x=-0.54742669886 ..., y= -1.356526357634...)*	*p_7=7.532621354…*
128	*(x=-0.547431479795 ..., y= -1.356530832564...)*	*p_8=7.532636823…*
…	… … …	… … ..

For the system (1.1), the values of δ are calculated as follows:

$$\delta_1 = \frac{p_2 - p_1}{p_3 - p_2} = 8.8171563807015\ldots, \qquad \delta_2 = \frac{p_3 - p_2}{p_4 - p_3} = 8.715976842041\ldots,$$

$$\delta_3 = \frac{p_4 - p_3}{p_5 - p_4} = 8.722215813427\ldots, \qquad \delta_4 = \frac{p_5 - p_4}{p_6 - p_5} = 8.721026780948\ldots,$$

and so on.
The ratios tend to a constant as k tends to infinity: more formally

$$\lim_{k\to\infty}\left[\frac{b_k - b_{k-1}}{b_{k+1} - b_k}\right] = \delta = 8.7210972\ldots$$

And the above table confirms that the 'universal' Feigenbaum constant δ = 8.7210972…

is also encountered in this area-preserving two-dimensional system.
The accumulation point p_∞ can be calculated by the formula

$$p_\infty = (p_2 - p_1)\frac{1}{\delta - 1} + p_2,$$

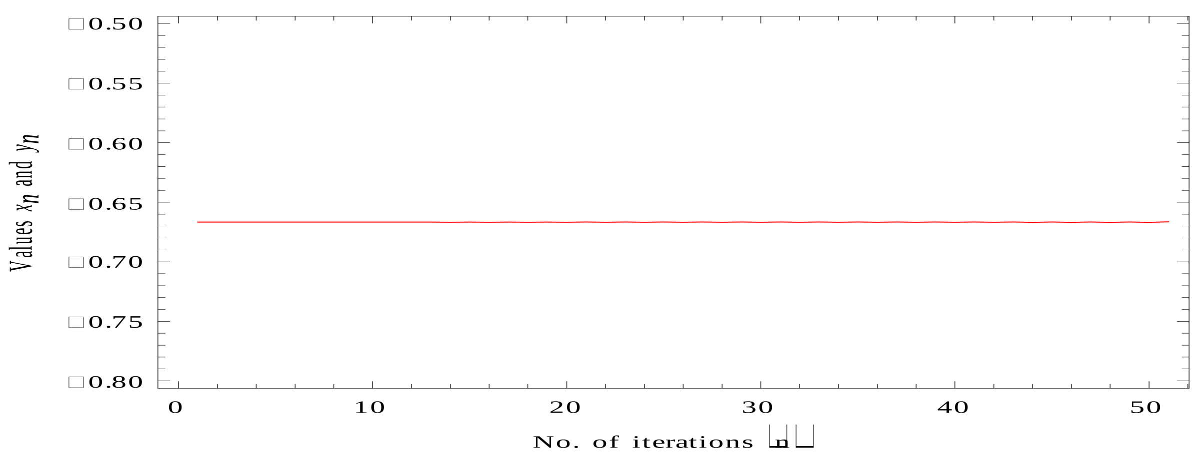
where δ is Feigenbaum constant, it is found to be **7.533284771388…. .,** beyond which the system (1.1) develops chaos.

5. Time Series Graphs [3]

The key theoretical tool used for quantifying chaotic behavior is the notion of a time series of data for the system [9]. A time series is a chronological sequence of observations on a particular variable. Usually the observations are taken at regular intervals. The system (1.1) giving the difference equations:

$$x_{n+1} = -y_n + px_n - (1-p)x_n^2,\; y_{n+1} = x_n - y_n + px_n - (1-p)x_n^2,\quad n = 0,1,2,\ldots \qquad (1.3)$$

On the horizontal axis the number of iterations ('time') are marked, that on the vertical axis the amplitudes are given for each iteration. The system (1.3) exhibis the following discrete time series graphs for the values of x_n and y_n , plotted together, showing the existences of periodic orbits of periods 2^k, k = 0, 1, 2,…, at different parameter



values.

Fig. 1 Showing period-1 behavior, parameter = 1st bifurcation point

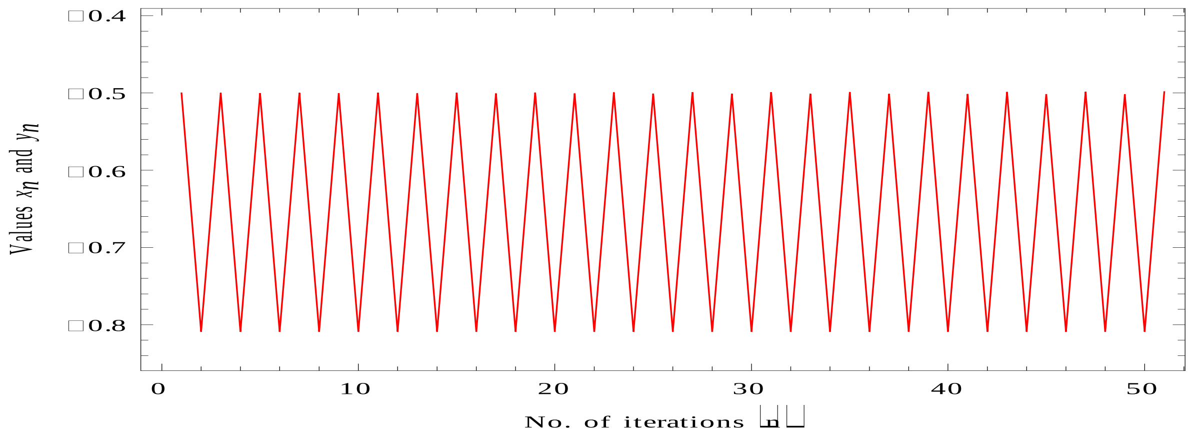


Fig. 1.2 Showing period-2 behavior, parameter = 2nd bifurcation point

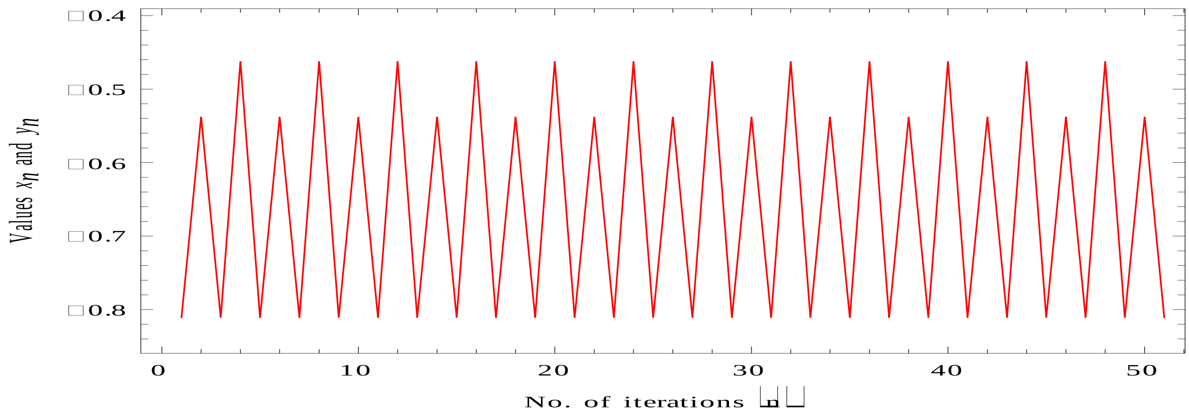


Fig. 1.3 Showing period-4 behavior, parameter = 3rd bifurcation point

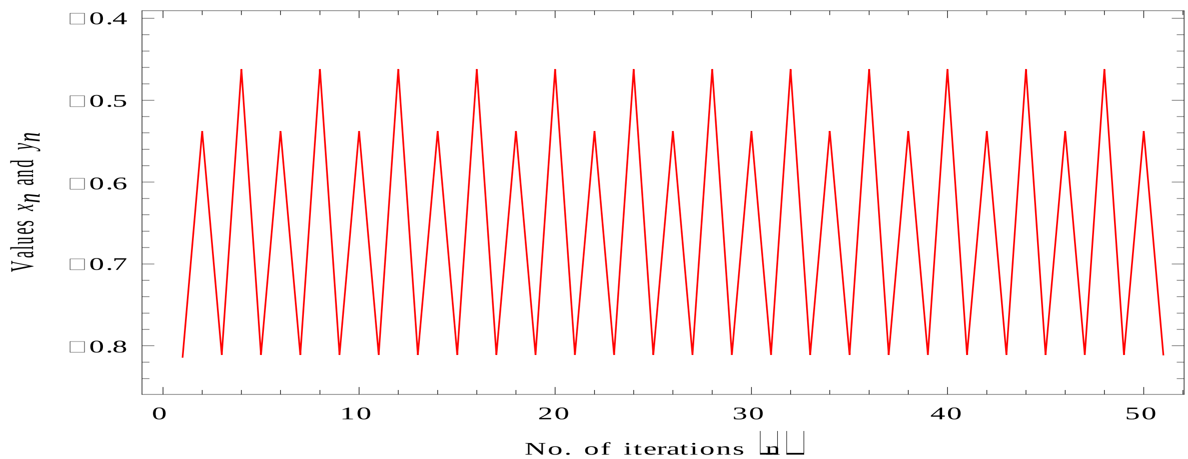


Fig. 1.4 Showing period-8 behavior, parameter = 4th bifurcation point

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