Triple Connected Two Domination Number of a Graph

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Abstract:

The concept of triple connected graphs with real life application was introduced in [7] by considering the existence of a path containing any three vertices of a graph G. In[3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. In this paper, we introduce a new domination parameter, called triple connected two domination number of a graph. A subset *S* of *V* of a nontrivial graph *G* is called a *dominating set* of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S*. The *domination number* $\gamma(G)$ of *G* is the minimum cardinality taken over all dominating sets in *G.* A subset *S* of *V* of a nontrivial graph *G* is said to be triple connected dominating set, if *S* is a dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by γ_{tc}. A dominating set is said to be two dominating set if every vertex in V – S is adjacent to atleast two vertices in S. The minimum cardinality taken over all two dominating sets is called the two domination number and is denoted by γ_2. A subset *S* of *V* of a nontrivial graph *G* is said to be triple connected two dominating set, if *S* is a two dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all triple connected two dominating sets is called the triple connected two domination number and is denoted by γ_{2tc}. We determine this number for some standard graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters are also investigated.

Key words: Domination Number, Triple connected graph, Triple connected domination number, Triple connected two domination number.

AMS (2010): *05C69*

1. Introduction

By a ***graph*** we mean a finite, simple, connected and undirected graph *G(V, E),* where *V* denotes its vertex set and *E* its edge set. Unless otherwise stated, the graph *G* has *p* vertices and *q* edges. ***Degree*** of a vertex *v* is denoted by *d(v)*, the ***maximum degree*** of a graph *G* is denoted by $\Delta(G)$. We denote a ***cycle*** on *p* vertices by C_p, a ***path*** on *p* vertices by P_p, and a ***complete graph*** on *p* vertices by K_p. A graph *G* is ***connected*** if any two vertices of *G* are connected by a path. A maximal connected subgraph of a graph *G* is called a ***component*** of *G*. The number of components of *G* is denoted by $\omega(G)$. The ***complement*** $\bar{G}$ of *G* is the graph with vertex set *V* in which two vertices are adjacent if and only if they are not adjacent in *G*. A ***tree*** is a connected acyclic graph. A ***bipartite graph*** (or ***bigraph***) is a graph whose vertex set can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A ***complete bipartite graph*** is a bipartite graph where every vertex of V_1 is adjacent to every vertex in V_2. The complete bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, is denoted by $K_{m,n}$. A ***star***, denoted by $K_{1,p-1}$ is a tree with one root vertex and p – 1 pendant vertices. The ***open neighbourhood*** of a set *S* of vertices of a graph G, denoted by *N(S)* is the set of all vertices adjacent to some vertex in *S* and $N(S) \cup S$ is called the ***closed neighbourhood*** of *S*, denoted by *N[S]*. A ***cut – vertex (cut edge)*** of a graph G is a vertex (edge) whose removal increases the number of components. A ***vertex cut***, or ***separating set*** of a connected graph *G* is a set of vertices whose removal results in a disconnected. The ***connectivity*** or ***vertex connectivity*** of a graph G, denoted by $\kappa(G)$ (where G is not complete) is the size of a smallest vertex cut. A connected subgraph *H* of a connected graph *G* is called a **H -cut** if $\omega(G – H) \geq 2$. The ***chromatic number*** of a graph *G*, denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph *G* in which adjacent vertices receive different colours. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. A ***Nordhaus -Gaddum-type*** result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2]. A subset *S* of *V* is called a ***dominating set*** of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S*. The ***domination number*** $\gamma(G)$ of *G* is the minimum cardinality taken over all dominating sets in *G*. A dominating set *S* of a connected graph G is said to be a ***connected dominating set*** of *G* if the induced sub graph <*S*> is connected. The minimum cardinality taken over all connected dominating sets is the ***connected domination number*** and is denoted by γ_c.A dominating set is said to be **two dominating set** if every vertex in V – S is adjacent to atleast two vertices in S. The minimum cardinality taken over all two dominating sets is called the **two domination number** and is denoted by γ_2.Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [8, 9]. Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph J. et. al.,[7] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be ***triple connected*** if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs.In[3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. A subset S of V of a nontrivial graph G is said to be a ***triple connected dominating set***, if S is a dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the ***triple connected domination number*** of G and is denoted by $\gamma_{tc}(G)$. Any triple connected dominating set with γ_{tc} vertices is called a γ_{tc} -set of G. In[4, 5, 6] G. Mahadevan et. al., was introduced **complementary triple connected domination number, complementary perfect triple connected domination number** and **paired triple connected domination number of a graph** and investigated new results on them.

In this paper, we use this idea to develop the concept of triple connected two dominating set and triple connected two domination number of a graph.

Theorem 1.1 [7] A tree T is triple connected if and only if $T \cong P_p;\ p \geq 3$.

Theorem 1.2 [7] A connected graph G is not triple connected if and only if there exists a H -cut with $\omega(G - H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components C_1, C_2, and C_3 of $G - H$.

Notation 1.3 Let G be a connected graph with m vertices $v_1, v_2, \ldots., v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots., n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Example 1.4 Let v_1, v_2, v_3, v_4, be the vertices of K_4. The graph $K_4(P_2, P_3, 2P_4, P_3)$ is obtained from K_4 by attaching 1 time a pendant vertex of P_2 on v_1, 1 time a pendant vertex of P_3 on v_2, 2 times a pendant vertex of P_4 on v_3 and 1 time a pendant vertex of P_3 on v_4 and is shown in Figure 1.1.

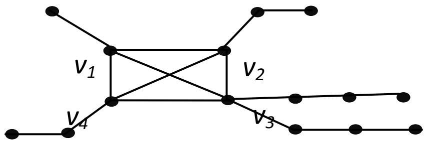

Figure 1.1 : $K_4(P_2, P_3, 2P_4, P_3)$

2. Triple connected two domination number

Definition 2.1 A subset S of V of a nontrivial graph G is said to be a *triple connected two dominating set*, if S is a two dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all triple connected two dominating sets is called the *triple connected two domination number* of G and is denoted by $\gamma_{2tc}(G)$. Any triple connected two dominating set with γ_{2tc} vertices is called a γ_{2tc} -set of G.

Example 2.2 For the graph G_1 in Figure 2.1, $S = \{v_1, v_2, v_3\}$ forms a γ_{2tc} -set of G_1. Hence $\gamma_{2tc}(G_1) = 3$.

G_1:

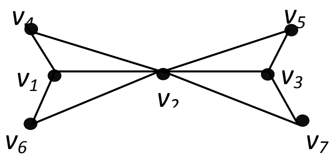

Figure 2.1 : Graph with $\gamma_{2tc} = 3$.

Observation 2.3 Triple connected two dominating set does not exists for all graphs and if exists, then $\gamma_{2tc}(G) \geq 3$.

Example 2.4 For the graph G_2 in Figure 2.2, any minimum two dominating set must contain the vertices v_1, v_4, v_5. But any two dominating set contains these vertices is not triple connected and hence γ_{2tc} does not exists.

G_3:

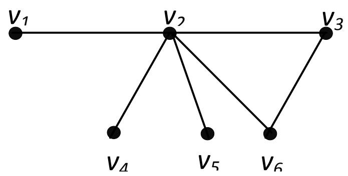

Figure 2.2 : Graph with no tctd set

Throughout this paper we consider only connected graphs for which triple connected two dominating set exists.

Observation 2.5 The complement of the triple connected two dominating set need not be a triple connected two dominating set.

Example 2.6 For the graph G_3 in Figure 2.3, $S = \{v_3, v_4, v_5\}$ forms a triple connected two dominating set of G_3. But the complement $V - S = \{v_1, v_2, v_6\}$ is not a triple connected two dominating set.

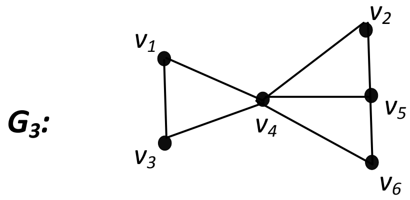

Figure 2.3 : Graph in which V – S is not a tctd set

Observation 2.7 Every triple connected two dominating set is a dominating set but not conversely.

Theorem 2.8 If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a triple connected two dominating set.

Proof The proof follows from *Theorem 1.2*.

Exact value for some standard graphs:

1) For any cycle of order $p \geq 4$, $\gamma_{2tc}(C_p) = p - 1$.
2) For any complete graph of order $p \geq 4$, $\gamma_{2tc}(K_p) = 3$.
3) For any complete bipartite graph of order $p \geq 4$, $\gamma_{2tc}(K_{m,n}) = 3$.
   (where $m, n \geq 2$ and $m + n = p$).

Observation 2.9 If a spanning sub graph H of a graph G has a triple connected two dominating set, then G also has a triple connected two dominating set.

Observation 2.10 Let G be a connected graph and H be a spanning sub graph of G. If H has a triple connected two dominating set, then $\gamma_{2tc}(G) \leq \gamma_{2tc}(H)$ and the bound is sharp.

Example 2.11 Consider the graph G_4 and its spanning subgraph H_4 and G_5 and its spanning subgraph H_5 shown in Figure 2.4.

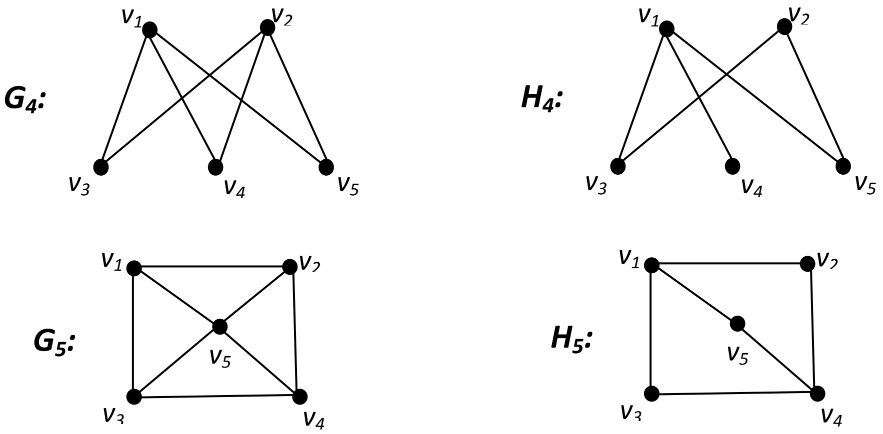

Figure 2.4

For the graph G_4, $S = \{v_1, v_2, v_4\}$ is a triple connected two dominating set and so $\gamma_{2tc}(G_4) = 3$. For the spanning subgraph H_4 of G_4, $S = \{v_1, v_3, u_4, v_2\}$ is a triple connected two dominating set so that $\gamma_{2tc}(H_4) = 4$. Hence $\gamma_{2tc}(G_4) < \gamma_{2tc}(H_4)$. For the graph G_5, $S = \{v_1, v_5, v_4\}$ is a triple connected two dominating set and so $\gamma_{2tc}(G_5) = 3$. For the spanning subgraph H_5 of G_5, $S = \{v_1, v_5, v_4\}$ is a triple connected two dominating set so that $\gamma_{2tc}(H_5) = 3$.Hence $\gamma_{2tc}(G_5) = \gamma_{2tc}(H_5)$.

Theorem 2.12 For any connected graph G with $p \geq 4$, we have $3 \leq \gamma_{2tc}(G) \leq p - 1$ and the bounds are sharp.

Proof The lower and bounds follows from *Definition 2.1*. For K_4, the lower bound is attained and for C_5 the upper bound is attained.

Observation 2.13 For any connected graph G with 4 vertices, $\gamma_{2tc}(G) = p - 1$ if and only if $G \cong K_4$, W_4, C_4, $C_3(P_2)$, $K_4 - e$.

Theorem 2.14 For any connected graph G with 5 vertices, $\gamma_{2tc}(G) = p - 1$ if and only if $G \cong C_5$, $C_4(P_2)$, $C_3(P_3)$, $C_3(P_2, P_2, 0)$ or the graph G_6 shown in Figure 2.5 .

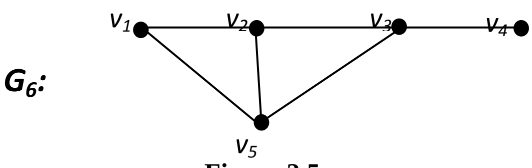

Figure 2.5

Proof Suppose G is isomorphic to C_5, $C_4(P_2)$, $C_3(P_3)$, $C_3(P_2, P_2, 0)$ or the graph G_6 given in Figure 2.5, then clearly $\gamma_{2tc}(G) = p - 1$. Conversely, Let G be a connected graph with 5 vertices, and $\gamma_{2tc}(G) = 4$. Let $S = \{v_1, v_2, v_3, v_4\}$ be the $\gamma_{2tc}(G)$ -set of G. Hence <S> = P_4, C_4, $C_3(P_2)$, $K_4 - e$.

Case(i): Let <S> = $P_4 = v_1v_2v_3v_4$. Since S is a triple connected two dominating set, v_5 is adjacent to atleast two vertices of S. If v_5 is adjacent to v_1 and v_2, then $G \cong C_3(P_3)$. If v_5 is adjacent to v_1 and v_3, then $G \cong C_4(P_2)$. If v_5 is adjacent to v_1 and v_4, then $G \cong C_5$. If v_5 is adjacent to v_2 and v_3, then $G \cong C_3(P_2, P_2, 0)$. If v_5 is adjacent to v_1, v_2 and v_3, then $G \cong G_6$.

Case(ii): Let <S> = $C_4 = v_1v_2v_3v_4v_1$.

Since S is a triple connected two dominating set, v_5 is adjacent to atleast two vertices of S. Either if v_5 is adjacent to v_1 and v_2 or v_1 and v_3, we can find a triple connected dominating set with less than four vertices, which is a contradiction.

Case(iii): Let <S> = $C_3(P_2)$.

Let $v_1v_2v_3v_1$ be the cycle and v_4 be adjacent to v_1. Since S is a triple connected two dominating set, v_5 is adjacent to atleast two vertices of S. For all the possibilities of v_5 is adjacent to two vertices of S, we can find a triple connected dominating set with less than four vertices, which is a contradiction.

Case(iv): Let <S> = $K_4 - e$. Since S is a triple connected two dominating set, v_5 is adjacent to atleast two vertices of S. For all the possibilities of v_5 is adjacent to two vertices of S, we can find a triple connected dominating set with less than four vertices, which is a contradiction.

The Nordhaus – Gaddum type result is given below:

Theorem 2.15 Let G be a graph such that G and $\bar{G}$ have no isolates of order $p \geq 4$. Then

(i) $\gamma_{2tc}(G) + \gamma_{2tc}(\bar{G}) \leq 2p - 2$.

(ii) $\gamma_{2tc}(G).\ \gamma_{2tc}(\bar{G}) \leq (p-1)^2$ and the bound is sharp.

Proof The bound directly follows from *Theorem 2.12*. For path C_4, both the bounds are attained.

3 Relation with Other Graph Theoretical Parameters

Theorem 3.1 For any connected graph G with $p \geq 4$ vertices, $\gamma_{2tc}(G) + \kappa(G) \leq 2p - 2$ and the bound is sharp if and only if $G \cong K_4$.

Proof Let G be a connected graph with $p \geq 4$ vertices. We know that $\kappa(G) \leq p - 1$ and by *Theorem 2.12*, $\gamma_{2tc}(G) \leq p - 1$. Hence $\gamma_{2tc}(G) + \kappa(G) \leq 2p - 2$. Suppose G is isomorphic to K_4. Then clearly $\gamma_{2tc}(G) + \kappa(G) = 2p - 2$. Conversely, Let $\gamma_{2tc}(G) + \kappa(G) = 2p - 2$. This is possible only if $\gamma_{2tc}(G) = p - 1$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{2tc}(G) = 3 = p - 1$. Hence $G \cong K_4$.

Theorem 3.2 For any connected graph G with $p \geq 4$ vertices, $\gamma_{2tc}(G) + \chi(G) \leq 2p - 1$ and the bound is sharp if and only if $G \cong K_4$.

Proof Let G be a connected graph with $p \geq 4$ vertices. We know that $\chi(G) \leq p$ and by *Theorem 2.12*, $\gamma_{2tc}(G) \leq p - 1$. Hence $\gamma_{2tc}(G) + \chi(G) \leq 2p - 1$. Suppose G is isomorphic to K_4. Then clearly $\gamma_{2tc}(G) + \chi(G) = 2p - 1$. Conversely, let $\gamma_{2tc}(G) + \chi(G) = 2p - 1$. This is possible only if $\gamma_{2tc}(G) = p - 1$ and $\chi(G) = p$. Since $\chi(G) = p$, G is isomorphic to K_p for which $\gamma_{2tc}(G) = 3 = p - 1$. Hence $G \cong K_4$.

Theorem 3.3 For any connected graph G with $p \geq 4$ vertices, $\gamma_{2tc}(G) + \Delta(G) \leq 2p - 2$ and the bound is sharp.

Proof Let G be a connected graph with $p \geq 4$ vertices. We know that $\Delta(G) \leq p - 1$ and by *Theorem 2.12*, $\gamma_{2tc}(G) \leq p - 1$. Hence $\gamma_{2tc}(G) + \Delta(G) \leq 2p - 2$. For K_4, the bound is sharp.

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