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Total Prime Graph

M.Ravi (a) Ramasubramanian[1], R.Kala[2]
1 Dept.of Mathematics, Sri Shakthi Institute of Engineering & Technology, Coimbatore – 641 062.
2 Dept. of Mathematics, Manonmaniam Sundaranar University, Tirunelveli – 627 012.

Abstract:
We introduce a new type of labeling known as "Total Prime Labeling". Graphs which admit a Total Prime labeling are called "Total Prime Graph". Properties of this labeling are studied and we have proved that Paths P_n, Star $K_{1,n}$, Bistar, Comb, Cycles C_n where n is even, Helm H_n, $K_{2,n}$, $C_3^{(t)}$ and Fan graph are Total Prime Graph. We also prove that any cycle C_n where n is odd is not a Total Prime Graph.

Keywords: Prime Labeling, Vertex prime labeling, Total Prime Labeling, Total Prime Graph

1. Introduction

By a graph G = (V,E) we mean a finite, simple and undirected graph. In a Graph G, V(G) denotes the vertex set and E(G) denotes the edge set. The order and size of G are denoted by 'p' and 'q' respectively. The terminology followed in this paper is according to [1]. A labeling of a graph is a map that carries graph elements to numbers. A complete survey of graph labeling is in [2]. Prime labeling and vertex prime labeling are introduced in [4] and [6]. Combining these two, we define a total prime labeling.

Two integers 'a' and 'b' are said to be relatively prime if their greatest common divisor is 1, (i.e.) $(a,b)=1$. If $(a_i,a_j)=1$, for all $i \neq j$ $(1 \le i, j \le n)$ then the numbers $a_1, a_2, a_3, \ldots, a_n$ are said to be relatively prime in pairs. Relatively prime numbers play a vital role in both analytic and algebraic number theory.

Definition 1.1 [4] Let G=(V,E) be a graph with 'p' vertices. A bijection $f: V(G) \to \{1,2,3,\ldots,p\}$ is said to be as "Prime Labeling" if for each edge e=xy the labels assigned to x and y are relatively prime. A graph which admits prime labeling is called "Prime Graph".

Definition 1.2 [6] Let G=(V,E) be a graph with 'p' vertices and 'q' edges. A bijection $f: E(G) \to \{1,2,3,\ldots,q\}$ is said to be a "Vertex Prime Labeling", if for each vertex of degree at least two, the greatest common divisor of the labels on its incident edges is 1.

2. Total Prime Graph:

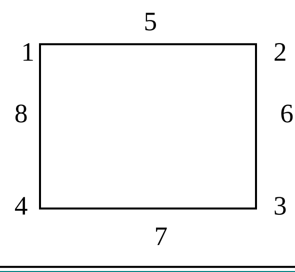
Definition 2.1 Let G=(V,E) be a graph with 'p' vertices and 'q' edges. A bijection $f: V \cup E \to \{1,2,3,\ldots,p+q\}$ is said to be a "Total Prime Labeling" if

(i) for each edge e=uv, the labels assigned to u and v are relatively prime.
(ii) for each vertex of degree at least 2, the greatest common divisor of the labels on the incident edges is 1.

A graph which admits "Total Prime Labeling" is called "Total Prime Graph".

Example 2.2

(1) C_4 is a Total Prime Graph.



(2) C_3 (or) K_3 is not a Total Prime Graph, because we can assign only one even label to an edge and one more even label to a vertex. But we have totally three even labels and the third even label can be assigned neither to any vertex not to any edge. Note that C_3 has Prime Labeling as well as Vertex Prime Labeling.

Notations 2.3

(1) Δ and δ denotes the maximum and minimum degree of a vertex respectively.

(2) $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n.

(3) $\lceil n \rceil$ denotes the least integer greater than or equal to n.

(4) g.c.d denotes greatest common divisor.

Theorem 2.4 The path P_n is a Total Prime Graph.

Proof Let $P_n = v_1 v_2 v_3 \ldots v_n$. P_n has 'n' vertices and 'n-1' edges.

We define $f : V \cup E \to \{1, 2, 3, \ldots, (2n-1)\}$ as follows.

$$f(v_i) = i, 1 \le i \le n$$

$$f(e_j) = n + j, 1 \le j < n$$

Clearly f is a bijection.

According to this pattern, the vertices are labeled such that for any edge e=uv $\in$ G, gcd $[f(u), f(v)] = 1$.

Also the edges are labeled such that for any vertex v_i, the g.c.d of all the edges incident with v_i is 1.

Hence P_n is a Total Prime graph.

Definition 2.5 K_1 with 'n' pendent edges incident with $V(K_1)$ is called a Star Graph and is denoted by $K_{1,n}$.

Theorem 2.6 $K_{1,n}, (n > 1)$ is a Total Prime Graph.

Proof Let $V(K_1) = \{u\}$ and $v_i, 1 \le i \le n$ be the vertices adjacent to u.

Therefore $K_{1,n}$ has 'n+1' vertices and 'n' edges.

Now we define $f : V \cup E \to \{1, 2, 3, \ldots, (2n+1)\}$ as follows.

$$f(u) = 1$$

$$f(v_i) = 2i, 1 \le i \le n$$

$$f(e_j) = 2j + 1, 1 \le j \le n$$

Clearly f is a bijection.

According to this pattern, $K_{1,n}$ is a Total Prime Graph.

Definition 2.7 The graph obtained from $K_{1,n}$ and $K_{1,m}$ by joining their centers with an edge is called a Bistar and is denoted by B(n,m)

Theorem 2.8 Bistar B(n,m) is a Total Prime Graph.

Proof Let $V(K_2) = \{u, v\}$ and $u_i, 1 \le i \le n$; $v_i, 1 \le i \le m$ be the vertices adjacent to u and v respectively.

For $1 \le j \le n$, $e_j = uu_j$; $e_{n+1} = uv$; for $(n+2) \le j \le (n+m+1)$, $e_j = vv_j$.

Therefore B(n,m) has 'n+m+2' vertices and 'n+m+1' edges.

Now we define $f : V \cup E \to \{1, 2, 3, \ldots, (2n+2m+3)\}$ as follows.

$$f(u) = 1$$
$$f(v) = 2$$
$$f(u_i) = 2(i+1), 1 \le i \le n$$
$$f(v_i) = 2i+1, 1 \le i \le m$$
$$f(e_j) = (n+m+2) + j, 1 \le j \le (n+m+1)$$

Clearly f is a bijection.

According to this pattern, clearly B(n,m) is a Total Prime Graph.

Definition 2.9 A graph obtained by attaching a single pendent edge to each vertex of a path $P_n = v_1 v_2 v_3 \ldots v_n$ is called a Comb.

Theorem 2.10 Comb is a Total Prime Graph.

Proof Let G be a Comb obtained from the path by joining a vertex u_i to v_i, $1 \le i \le n$.

The edges are labeled as follows:

For $1 \le i \le n$, $e_{2i-1} = v_i u_i$ and $e_{2i} = v_i v_{i+1}$

Therefore G has '2n' vertices and '2n-1' edges.

Now define $f : V \cup E \to \{1, 2, 3, \ldots, (4n-1)\}$ as follows.

$$f(v_i) = 2i-1, 1 \le i \le n$$
$$f(u_i) = 2i, 1 \le i \le n$$
$$f(e_j) = 2n + j, 1 \le j \le (2n-1)$$

Clearly f is a bijection.

According to this pattern, Comb is a Total Prime Graph.

Theorem 2.11 Cycle C_n, n is even, is a Total Prime Graph.

Proof Let $C_n = (v_1 e_1 v_2 e_2 v_3 \ldots v_n e_n v_1)$

Therefore C_n has 'n' vertices and 'n' edges.

Now we define $f : V \cup E \to \{1, 2, 3, \ldots, 2n\}$ as follows.

$$f(v_i) = i, 1 \le i \le n$$
$$f(e_j) = n + j, 1 \le i \le n$$

Clearly f is a bijection.

According to this pattern, clearly Cycle C_n, n is even, is a Total Prime Graph.

Theorem 2.12 Cycle C_n, n is odd, is not a Total Prime Graph.

Proof Let $C_n = (v_1 e_1 v_2 e_2 v_3 \ldots v_n e_n v_1)$

Therefore C_n has 'n' vertices and 'n' edges.

Define $f : V \cup E \to \{1, 2, 3, \ldots, 2n\}$

Now, no. of even labels available is 'n'.

For any 3 consecutive vertices, we can assign at most one even label and so, number of vertices with even labels is at most $\left\lceil \frac{n}{3} \right\rceil$.

Also, out of 3 consecutive edges, we can assign at most one even label and so the number of edges with even labels is at most $\left\lceil \frac{n}{3} \right\rceil$.

Therefore the necessary condition for existence of total Prime Graph is $2\left\lceil \frac{n}{3} \right\rceil$=n.

Case 1: $n \equiv 0(\bmod 3)$

(i.e.) n is a multiple of 3.

Therefore, in this case $2\left(\frac{n}{3}\right) = n$

(i.e.) $2n = 3n$

(i.e.) $2 = 3$

Which is a contradiction.

Case 2: $n \equiv 1(\bmod 3)$

In this case $2\left(\frac{n+2}{3}\right) = n$

(i.e.) $2n + 4 = 3n$

(i.e.) $n = 4$

But n is odd, so it's not possible.

Case 3: $n \equiv 2(\bmod 3)$

In this case $2\left(\frac{n+1}{3}\right) = n$

(i.e.) $2n + 2 = 3n$

(i.e.) $n = 2$

But n is odd, so it's not possible.

Therefore Cycle C_n, n is odd, is not a Total Prime Graph.

Definition 2.13 Helm H_n is a graph obtained from wheel by attaching a pendent edge at each vertex of n-cycle.

Theorem 2.14 Helm H_n is a Total Prime Graph.

Proof Here center vertex will be labeled as u and all the vertices on the cycle are labeled as $u_1, u_2, \ldots, u_n$. The corresponding pendent vertices are labeled as $v_1, v_2, \ldots, v_n$. The edges are labeled as $e_1, e_2, \ldots, e_{2n}$ starting from the pendent edge incident at vertex u_1 and with labeling the edge on the cycle alternatively in clockwise direction $e_1, e_2, \ldots, e_{2n}$ and the spokes of the wheels are labeled as $e_{2n+1}, e_{2n+2}, \ldots, e_{3n}$ starting from the edge uu_1 and proceeding in the clockwise direction.

Therefore Helm H_n has '2n+1' vertices and '3n' edges.

Now we define $f: V \cup E \rightarrow \{1, 2, 3, \ldots, (5n+1)\}$ as follows.

$$f(u) = 1$$

$$f(u_i) = 2i + 1, 1 \le i \le \left\lfloor \frac{2n+1}{2} \right\rfloor$$

$$f(v_i) = 2i, 1 \le i \le \left\lfloor \frac{2n+1}{2} \right\rfloor$$

$$f(e_j) = (2n+1) + j, 1 \le j \le 3n$$

Clearly f is a bijection.

According to this pattern, clearly Helm H_n is a Total Prime Graph.

Definition 2.15 $K_{m,n}$ is a complete bipartite graph with bipartition X and Y, in which any two vertices in X as well as any two vertices in Y are non-adjacent. Also every vertex of X is adjacent to every vertex of Y.

Theorem 2.16 $K_{2,n}$, is a Total Prime Graph.

Proof $K_{m,n}$ have 'm+n' vertices and 'mn' edges. Here m=2. Therefore $K_{2,n}$ has '2+n' vertices and '2n' edges.

Let $X=\{u_1,u_2\}$ and $Y=\{v_1,v_2,v_3,\ldots,v_n\}$. The edges are labeled in a continuous manner starting from $e_1=v_1u_1$ to $e_{2n-1}=v_nu_1$ and the last edge $e_{2n}=v_1u_2$.

Now we define $f:V\cup E\rightarrow\{1,2,3,\ldots,(3n+2)\}$ as follows:

$$f(u_i)=2i-1, 1\le i\le 2$$

The vertices $Y=\{v_1,v_2,v_3,\ldots,v_n\}$ are partitioned into $\left\lfloor\frac{n}{2}\right\rfloor$ sets as follows:

for $j=even$ and $0\le j\le\left\lceil\frac{n}{2}\right\rceil$, let $S_j=\{v_{j+1},v_{j+2}\}$

$$f(v_i)=2i+j, 1\le i<n \text{ and } v_i\in S_j$$

$$f(v_n)=\begin{cases}\text{3n+1, n is odd}\\ \text{3n+2, n is even}\\ \text{3n+1, n is of the form 10r-2 and r=1,2,3,}\ldots\end{cases}$$

The edges are labeled as follows:-

Case 1: n is odd

(i) for $0\le k\le\left\lceil\frac{n}{2}\right\rceil-1$, $e_{4k+1}=u_1v_{\left\lceil\frac{4k+1}{2}\right\rceil}$

$$e_{4k+2}=u_1v_{2k+2}$$

$$e_{4k+3}=u_2v_{\left\lceil\frac{4k+3}{2}\right\rceil}$$

(ii) for $0\le k\le\left\lceil\frac{n}{2}\right\rceil-2$, $e_{4k+4}=u_2v_{2k+3}$

(iii) $e_{2n-2}=u_2v_n$, $e_{2n-1}=u_1v_n$, $e_{2n}=u_2v_1$

Case 2: n is even

(i) for $0\le k\le\frac{n}{2}-1$, $e_{4k+1}=u_1v_{\left\lceil\frac{4k+1}{2}\right\rceil}$

$$e_{4k+2}=u_1v_{2k+2}$$

$$e_{4k+3}=u_2v_{\left\lceil\frac{4k+3}{2}\right\rceil}$$

(ii) for $0\le k\le\frac{n}{2}-2$, $e_{4k+4}=u_2v_{2k+3}$

(iii) $e_{2n}=u_2v_1$

The unassigned labels given in their order to the edges in the order $\{e_1, e_2, e_3, \ldots, e_{2n}\}$.

Clearly f is a bijection.

According to this pattern, clearly $K_{2,n}$, is a Total Prime Graph.

Definition 2.17 $C_3^{(t)}$ denotes the one-point union of 't' cycles of length 3. $C_3^{(t)}$ is also called as Friendship Graph (or) Dutch t-windmill.

Theorem 2.18 $C_3^{(t)}$ is a Total Prime Graph.

Proof $C_3^{(t)}$ has '2t+1' vertices and '3t' edges.

Let the vertex set be $\{v_0, v_1, v_2, \ldots, v_{2t}\}$ with centre vertex v_0. Let the edge set be $\{e_1, e_2, e_3, \ldots, e_{3t}\}$ with $e_1 = v_0 v_1$ and label the edges in clockwise direction.

Now we define $f : V \cup E \to \{1, 2, 3, \ldots, (5t+1)\}$ as follows:

$$f(v_i) = i+1, 0 \le i \le 2t+1$$

$$f(e_j) = (2t+1) + j, 1 \le j \le 3t$$

Clearly f is a bijection.

According to this pattern, clearly $C_3^{(t)}$ is a Total Prime Graph.

Definition 2.19 The fan graph F_n is defined as $K_1 + P_n$, P_n is a path of n vertices.

Theorem 2.20 Fan graph $F_n, n \ge 3,$ is a Total Prime Graph.

Proof F_n has $n+1$ vertices and '2n-1' edges.

We define $f : V \cup E \to \{1, 2, 3, \ldots, 3n\}$ as follows.

$$f(v_i) = i, 1 \le i \le n$$

For $\mathrm{e}_i = v_i v_{i+1},\ 1 \le i \le n$

$$f(e_i) = n+1+i$$

For $\mathrm{e}_{n+j-2} = v_1 v_j,\ 3 \le j \le n+1$

$$f(e_{n+j-2}) = 3n+3-j$$

Clearly f is a bijection.

According to this pattern, Fan Graph F_n is a Total Prime Graph.

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