Paired Triple Connected Domination Number of a Graph

[1]G. Mahadevan, [2] Selvam Avadayappan, [3]A.Nagarajan,
[4]A.Rajeswari, [5] T. Subramanian

[1]Dept. of Mathematics, Anna University, Chennai -25 Tirunelveli Region
[2] Dept. of Mathematics, V.H.N.S.N College, Virudhunagar-626 001
[3]Dept. of Mathematics, V.O.Chidambaram College, Tuticorin-628 008
[4]Dept. of Mathematics, Aarupadai Veedu Institute of Technology Paiyanoor, Chennai- 603 104
[5]Research Scholar Anna University, Chennai -25 Tirunelveli Region

Abstract

The concept of triple connected graphs with real life application was introduced in [10] by considering the existence of a path containing any three vertices of G and also they studied their properties. In [2, 4], the authors introduced the concept of triple connected domination number and complementary triple connected domination number of a graph. In this paper, we introduce another new concept called paired triple connected domination number of a graph. A subset S of V of a nontrivial connected graph G is said to be paired triple connected dominating set, if S is a triple connected dominating set and the induced subgraph $<S>$ has a perfect matching. The minimum cardinality taken over all paired triple connected dominating sets is called the paired triple connected domination number and is denoted by γ_{ptc}. We determine this number for some standard classes of graphs and obtain some bounds for general graph. Its relationship with other graph theoretical parameters are investigated.

Key words: Domination Number, Triple connected graph, Paired triple connected domination number

AMS (2010): *05C69*

1. Introduction

By a ***graph*** we mean a finite, simple, connected and undirected graph $G(V, E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has p vertices and q edges. ***Degree*** of a vertex v is denoted by $d(v)$, the ***maximum degree*** of a graph G is denoted by $\Delta(G)$. We denote a ***cycle*** on p vertices by C_p, a ***path*** on p vertices by P_p, and a ***complete graph*** on p vertices by K_p. A graph G is ***connected*** if any two vertices of G are connected by a path. A maximal connected subgraph of a graph G is called a ***component*** of G. The number of components of G is denoted by $\omega(G)$. The ***complement*** $\bar{G}$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A ***tree*** is a connected acyclic graph. A ***bipartite graph*** (or ***bigraph***) is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A ***complete bipartite graph*** is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. The complete bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, is denoted $K_{m,n}$. A ***star***, denoted by $K_{1,p-1}$ is a tree with one root vertex and p – 1 pendant vertices. A ***bistar***, denoted by $B(m,n)$ is the graph obtained by joining the root vertices of the stars $K_{1,m}$ and $K_{1,n}$. The ***friendship graph***, denoted by F_n can be constructed by identifying n copies of the cycle C_3 at a common vertex. A ***wheel graph***, denoted by W_p is a graph with p vertices, formed by connecting a single vertex to all vertices of an $(p-1)$ cycle. A ***helm graph***, denoted by H_n is a graph obtained from the wheel W_n by joining a pendant vertex to each vertex in the outer cycle of W_n by means of an edge. A graph G is said to be ***semi – complete*** if and only if it is simple and for any two vertices u, v of G there is a vertex w of G such that w is adjacent to both u and v in G i.e., uwv is a path in G. Let G be a finite graph and $v \in V(G)$, then $N(N[v]) – N[v]$ is called the ***consequent neighbourhood set*** of v, and its cardinality is called the ***consequent neighbourhood number*** of v in G. ***Corona*** of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$ is the disjoint union of one copy of G_1 and $|V_1|$ copies of G_2 ($|V_1|$ is the number of vertices in G_1) in which i^{th} vertex of G_1 is joined to every vertex in the i^{th} copy of G_2. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. If S is a subset of V, then $<S>$ denotes the vertex induced subgraph of G induced by S. The ***open neighbourhood*** of a set S of vertices of a graph G, denoted by $N(S)$ is the set of all vertices adjacent to some vertex in S and $N(S) \cup S$ is called the ***closed neighbourhood*** of S, denoted by $N[S]$. The ***diameter*** of a connected graph is the maximum distance between two verties in G and is denoted by $diam(G)$. A ***cut – vertex (cut edge)*** of a graph G is a vertex(edge) whose removal increases the number of components. A ***vertex cut***, or ***separating set*** of a connected graph G is a set of vertices whose removal renders G disconnected. The ***connectivity*** or ***vertex connectivity*** of a graph G, denoted by $\kappa(G)$ (where G is not complete) is the size of a smallest vertex cut. A connected subgraph H of a connected graph G is called a **H -cut** if $\omega(G – H) \geq 2$. The ***chromatic number*** of a graph G, denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colour. Terms not defined here are used in the sense of [1].

A subset S of V is called a ***dominating set*** of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The ***domination number***$\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a ***connected dominating set*** of G if the induced sub graph <S> is connected. The minimum cardinality taken over all connected dominating sets is the ***connected domination number*** and is denoted by γ_c. A dominating set S of a connected graph G is said to be a ***tree dominating set*** of G if the induced sub graph <S> is a tree. The minimum cardinality taken over all tree dominating sets is the ***tree domination number*** and is denoted by γ_{tr}.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set[8,11]. Recently the concept of triple connected graphs was introduced by Paulraj Joseph J. et. al.,[10] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graph and established many results on them. A graph G is said to be ***triple connected*** if any three vertices lie on a path in G. All paths and cycles, complete graphs and wheels are some standard examples of triple connected graphs.

In [2, 4], the authors introduced the concept of triple connected domination number and complementary triple connected domination number of a graph.

A dominating set S of a connected graph G is said to be a ***triple connected dominating set*** of G if the induced sub graph <S> is triple connected. The minimum cardinality taken over all triple connected dominating sets is the ***triple connected domination number*** and is denoted by γ_{tc}. A dominating set S of a connected graph G is said to be a ***complementary triple connected dominating set*** of G if S is a dominating set and the induced subgraph <V – S> is triple connected. The minimum cardinality taken over all complementary triple connected dominating sets is the ***complementary triple connected domination number*** and is denoted by γ_{ctc}.

In this paper we use this idea to develop another new concept called paired triple connected dominating set and paired triple connected domination number of a graph.

Theorem 1.1 [10] A tree T is triple connected if and only if $T \cong P_p$; $p \geq 3$.

Theorem 1.2 [10] A connected graph $G$is not triple connected if and only if there exists a H -cut with $\omega(G - H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components C_1, C_2, and C_3 of $G - H$.

Theorem 1.3 [6]G is semi – complete graph with $p \geq 4$ vertices. Then G has a vertex of degree 2 if and only if one of the vertices of G has consequent neighbourhood number $p - 3$.

Theorem 1.4 [6]G is semi – complete graph with $p \geq 4$ vertices such that there is a vertex with consequent neighbourhood number p – 3. Then $\gamma(G) \leq 2$.

Notation 1.4 Let G be a connected graph with m vertices $v_1, v_2, \ldots, v_m$. The graph $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ *where* $n_i, l_i \geq 0$ and $1 \leq i \leq m$, is obtained from G by pasting n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex $v_2$and so on.

Example 1.5 Let v_1, v_2,v_3, v_4, be the vertices of K_4, the graph $K_4(2P_2, P_3, 3P_2, P_2)$ is obtained from K_4 by pasting 2 times a pendant vertex of P_2 on v_1, 1 times a pendant vertex of P_3on v_2, 3 times a pendant vertex of P_2 on v_3 and 1 times a pendant vertex of P_2 on v_4 and the graph shown below in $G_1$of Figure 1.1.

G_1:

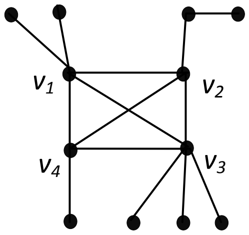

Figure 1.1

2 Paired Triple connected domination number

Definition 2.1 A subset S of V of a nontrivial graph G is said to be a *paired triple connected dominating set*, if S is a triple connected dominating set and the induced subgraph <S> has a perfect matching. The minimum cardinality taken over all paired triple connected dominating sets is called the *paired triple connected domination number* and is denoted by γ_{ptc}. Any paired triple connected dominating set with γ_{ptc} vertices is called a γ_{ptc}-set of G.

Example 2.2 For the graph $C_5 = v_1v_2v_3v_4v_5v_1$,$S = \{v_1, v_2, v_3, v_4\}$ forms a paired triple connected dominating set. Hence$\gamma_{ptc}(C_5) = 4$.

Observation 2.3 Paired triple connected dominating set does not exists for all graphs and if exists, then $\gamma_{ptc}(G) \geq 4$.

Example 2.4 For the graph G_2 in Figure 2.1, we cannot find any paired triple connected dominating set.

G_2:

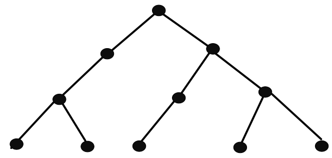

Figure 2.1

Remark 2.5 Throughout this paper we consider only connected graphs for which paired triple connected dominating set exists.

Observation 2.6 The complement of the paired triple connected dominating set need not be a paired triple connected dominating set.

Example 2.7 For the graph G_3 in Figure 2.2, $S = \{v_1, v_2, v_3, v_4\}$ is a paired triple connected dominating set of G_3. But the complement $V - S = \{v_5, v_6, v_7, v_8\}$ is not a paired triple connected dominating set.

G_3:

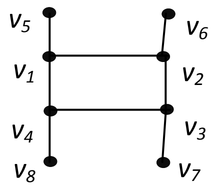

Figure 2.2

Observation 2.8 Every paired triple connected dominating set is a dominating set but not the converse.

Observation 2.9 For any connected graph G, $\gamma(G) \leq \gamma_c(G) \leq \gamma_{tc}(G) \leq \gamma_{ptc}(G)$ and the inequalities are strict and for a connected graph G with $p \geq 5$ vertices, $\gamma_c(G) \leq \gamma_{tr}(G) \leq \gamma_{tc}(G) \leq \gamma_{ptc}(G)$.

Example 2.10 For the graph G_4 in Figure 2.3, $\gamma(G_4) = \gamma_c(G_4) = \gamma_{tc}(G_5) = \gamma_{ptc}(G) = 4$ and for C_6 $\gamma_c(C_6) = \gamma_{tr}(C_6) = \gamma_{tc}(C_6) = \gamma_{ptc}(C_6) = 4$.

G_4:

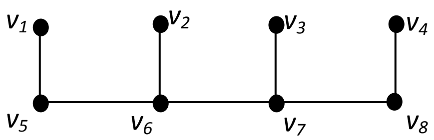

Figure 2.3

Theorem 2.11 If the induced subgraph of all connected dominating set of G has more than two pendant vertices, then G does not contains a paired triple connected dominating set.

Proof This theorem follows from *Theorem 1.2.*

Example 2.12 For the graph G_5 in Figure 2.4, S = $\{v_4, v_5, v_6, v_7, v_8, v_9\}$ is a minimum connected dominating set so that $\gamma_c(G_7) = 6$. Here we notice that the induced subgraph of S has three pendant vertices and hence G does not contains a paired triple connected dominating set.

G_5:

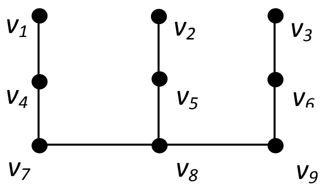

Figure 2.4

Paired Triple connected domination number for some standard graphs are given below

1) For any path of order $p \geq 4$, $\gamma_{ptc}(P_p) = \begin{cases} 4 & \text{if } p = 4 \\ p-1 & \text{if } p \text{ is odd} \\ p-2 & \text{if } p \text{ is even.} \end{cases}$

2) For any cycle of order $p \geq 4$, $\gamma_{ptc}(C_p) = \begin{cases} 4 & \text{if } p = 4 \\ p-1 & \text{if } p \text{ is odd} \\ p-2 & \text{if } p \text{ is even.} \end{cases}$

3) For the complete bipartite graph of order $p \geq 4$, $\gamma_{ptc}(K_{m,n}) = 4$. (where $m, n \geq 2$ and $m + n = p$).

4) For any complete graph of order $p \geq 4$, $\gamma_{ptc}(K_p) = 4$.

5) For any wheel of order $p \geq 4$, $\gamma_{ptc}(W_p) = 4$.

6) For any helm graph of order $p \geq 9$, $\gamma_{ptc}(H_n) = \begin{cases} \frac{p-1}{2} & \text{if } n \text{ is odd} \\ \frac{p-1}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$

(where $2n - 1 = p$).

7) For any bistar of order $p \geq 4$, $\gamma_{ptc}(B(m, n)) = 4$ *(where $m, n \geq 1$ and $m + n + 2 = p$).*

Observation 2.13 If a spanning sub graph H of a graph G has a paired triple connected dominating set then G also has a paired triple connected dominating set.

Observation 2.14 Let G be a connected graph and H be a spanning sub graph of G. If H has a paired triple connected dominating set, then $\gamma_{ptc}(G) \leq \gamma_{ptc}(H)$ and the bound is sharp.

Example 2.15 Consider C_6 and its spanning subgraph P_6, $\gamma_{ptc}(C_6) = \gamma_{ptc}(P_6) = 4$.

Observation 2.16 For any connected graph G with p vertices, $\gamma_{ptc}(G) = p$ if and only if $G \cong P_4, C_4, K_4, C_3(P_2), K_4 - \{e\}$.

Theorem 2.17 For any connected graph G with $p \geq 5$, we have $4 \leq \gamma_{ptc}(G) \leq p - 1$ and the bounds are sharp.

Proof The lower and upper bounds follows from *Definition 2.1* and *Observation 2.3* and *Observation 2.16.* For P_5, the lower bound is attained and for C_7 the upper bound is attained.

Theorem 2.18 For a connected graph G with 5 vertices, $\gamma_{ptc}(G) = p - 1$ if and only if G is isomorphic to P_5, C_5, W_5, K_5, $K_{2,3}$, F_2, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, $P_4(0, P_2, 0, 0)$ or any one of the graphs shown in Figure 2.5.

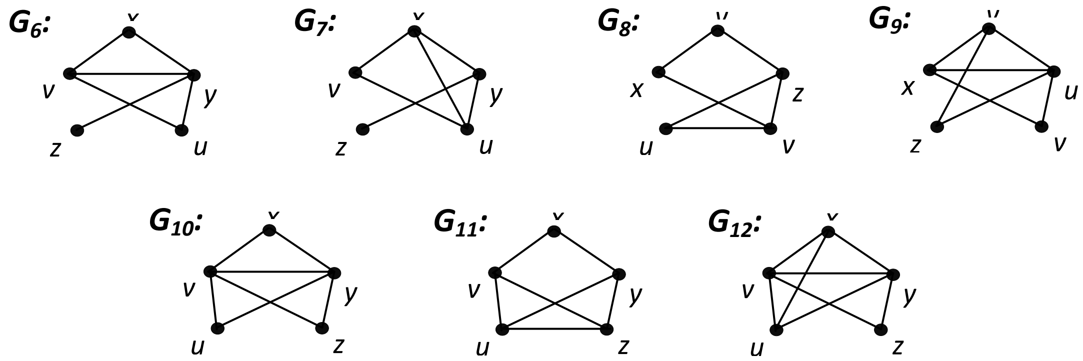

Figure 2.5

Proof Suppose G is isomorphic to P_5, C_5,W_5, K_5,$K_{2,3}$, F_2,$K_5 - \{e\}$,$K_4(P_2)$, $C_4(P_2)$,$C_3(P_3)$,$C_3(2P_2)$,$C_3(P_2, P_2, 0)$,$P_4(0, P_2, 0, 0)$ or any one of the graphs $G_6$to G_{12} given in Figure 2.5., then clearly $\gamma_{ptc}(G) = p - 1$. Conversely, Let G be a connected graph with 5 vertices and $\gamma_{ptc}(G) = p - 1$.Let $S = \{w, x, y, z\}$ be a paired triple connected dominating set of G. Let $V - S = V(G) - V(S) = \{ v\}$.

Case (i) $<S>$is not a tree.

Then $<S>$ contains a cycle C. Let $C = wxyw$ and let z be adjacent to w. Since S is a paired triple connected dominating set, there exists a vertex say w or x (or y) or z is adjacent to v. Let w be adjacent to v. If $d(w) = 4$, $d(x) = d(y) = 2$, $d(z) = 1$, then $G \cong C_3(2P_2)$.Let x be adjacent to v. If $d(w) = d(x) = 3$, $d(y) = 2$, $d(z) = 1$, then $G \cong C_3(P_2, P_2, 0)$.Let z be adjacent to v. If $d(w) = 3$, $d(x) = d(y) = d(z) = 2$, then $G \cong C_3(P_3)$.Now by adding edges to $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, and $C_3(P_3)$, we have G $\cong W_5$, K_5, $K_{2,3}$, F_2,$K_5 - \{e\}$, $K_4(P_2)$ or any one of the graphs G_6 to G_{12}given in Figure 2.5.

Case (ii) $<S>$is a tree.

Since S is a paired triple connected dominating set. Therefore by *Theorem 1.1*, we have $<S> \cong P_{p-1}$. Since S paired triple connected dominating set, there exists a vertex say w (or z) or x (or y) is adjacent to v. Let w be adjacent to v. If $d(w) = d(x) = d(y) = 2$, $d(z) = 1$, then $G \cong P_5$.Let w be adjacent to v and let z be adjacent to v. If $d(w) = d(x) = d(y) = d(z) = 2$, then $G \cong C_5$.Let w be adjacent to v and let y be adjacent to v. If $d(w) = d(x) = 2$, $d(y) = 3$, $d(z) = 1$, then $G \cong C_4(P_2)$. Let x be adjacent to v. If $d(w) = d(z) = 1$, $d(x) = 3$, $d(y) = 2$, then $G \cong P_4(0, P_2, 0, 0)$. In all the other cases, no new graph exists.

Theorem 2.19 If G is a graph such that G and $\bar{G}$ have no isolates of order $p \geq 5$, then $\gamma_{ptc}(G) + \gamma_{ptc}(\bar{G}) \leq 2(p - 1)$and the bound is sharp.

Proof The bound directly follows from the *Theorem 2.17*. For the cycle C_5, $\gamma_{ptc}(G) + \gamma_{ptc}(\bar{G}) = 2(p - 1)$.

Theorem 2.20 If G is a semi – complete graph with p ≥ 4 vertices such that there is a vertex with consequent neighbourhood number p – 3, then $\gamma_{ptc}(G) = 4$.

Proof By *Theorem 1.3*, it follows that there is a vertex say v of degree 2 in G. Let $N(v) = \{v_1, v_2\}$ (say). Let $u \in V(G) - N[v]$. Since G is semi – complete and $u, v \in V(G)$, there is a $w \in V(G)$ such that $\{u, w, v\}$ is a path in G. Clearly $w \in N(v)$. Therefore the vertices not in $N[v]$ is dominated by either $v_1$or v_2. Here the $S = \{u, v_1, v, v_2\}$ is a paired triple connected dominating set of G. Hence $\gamma_{ptc}(G) = 4$.

3 Paired Triple Connected Domination Number and Other Graph Theoretical Parameters

Theorem3.1 For any connected graph G with $p \geq 5$ vertices, $\gamma_{ptc}(G) + \kappa(G) \leq 2p - 2$and the bound is sharp if and only if $G \cong K_5$.

Proof Let G be a connected graph with $p \geq 5$vertices. We know that $\kappa(G) \leq p - 1$andby *Theorem 2.17*, $\gamma_{ptc}(G) \leq p - 1$.Hence $\gamma_{ptc}(G) + \kappa(G) \leq 2p - 2$.Suppose G is isomorphic to K_5.Then clearly $\gamma_{ptc}(G) + \kappa(G) = 2p - 2$.Conversely, Let $\gamma_{ptc}(G) + \kappa(G) = 2p - 2$. This is possible only if $\gamma_{ptc}(G) = p - 1$and $\kappa(G) = p - 1$.But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{ptc}(G) = 4 = p - 1$ so that $p = 5$. Hence $G \cong K_5$.

Theorem 3.2 For any connected graph G with $p \geq 5$ vertices, $\gamma_{ptc}(G) + \chi(G) \leq 2p - 1$ and the bound is sharp if and only if $G \cong K_5$.

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\chi(G) \leq p$ and by *Theorem 2.17*, $\gamma_{ptc}(G) \leq p - 1$.Hence $\gamma_{ptc}(G) + \chi(G) \leq 2p - 1$.Suppose G is isomorphic to K_5. Then clearly

$\gamma_{ptc}(G) + \chi(G) = 2p - 2$.Conversely,Let $\gamma_{ptc}(G) + \chi(G) = 2p - 1$. This is possible only if $\gamma_{ptc}(G) = p - 1$and$\chi(G) = p$. But $\chi(G) = p$, and so G is isomorphic to K_p for which $\gamma_{ptc}(G) = 4 = p - 1$ so that $p = 5$.Hence $G \cong K_5$.

Theorem3.3 For any connected graph G with $p \geq 5$ vertices, $\gamma_{ptc}(G) + \Delta(G) \leq 2p - 2$and the bound is sharp.

ProofLet G be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p - 1$andby *Theorem 2.17*, $\gamma_{ptc}(G) \leq p - 1$.Hence $\gamma_{ptc}(G) + \Delta(G) \leq 2p - 2$.For $K_5$the bound is sharp

References

[1.] John Adrian Bondy.; MurtyU.S.R.: *Graph Theory*Springer, 2008.

[2.] Mahadevan G.; SelvamAvadayappan.; Paulraj Joseph. J.; Subramanian T.: *Triple connected domination number of a graph – Preprint (2012).*

[3.] Mahadevan G.; A. Nagarajan.; A.Selvam.; A.Rajeswari.: *A new characterization of Paired domination number of a graph* , ICMMSC, CCIS 283, pp.86-96, Springer Verlag, Berlin Heidelbberg (2012).

[4.] Mahadevan G.; SelvamAvadayappan.; Paulraj Joseph. J.; Subramanian T.: Complementary *Tripleconnected domination number of a graph – Preprint (2012).*

[5.] Mahadevan G.; SelvamAvadayappan.; MydeenBibi A.: *Extended results on two domination number and chromatic number of a graph*, International Journal of Mathematics and Soft Computing, Vol. 1, pp.77-87 (2011).

[6.] Naga Raja Rao I. H.; Siva Rama Raju S. V.: *Semi - complete graphs*, International Journal of Computational Cognition, Vol. 7 No. 3: 50–54 (2009).

[7.] Nordhaus E. A.; Gaddum J. W.: *On complementary graphs*, Amer. Math.

[8.] Monthly, 63: 175–177 (1956).

[9.] Paulraj Joseph J.; Mahadevan G.: *On complementary perfect domination number of a graph*, ActaCienciaIndica, Vol. XXXI M, No. 2.: 847–853 (2006).

[10.] Paulraj Joseph J.; and Arumugam. S.: *Domination and coloring in graphs*, InternationalJournal of Management Systems, 8 (1): 37–44 (1997).

[11.] Paulraj Joseph J.; Angel Jebitha M.K.; Chithra Devi P.; Sudhana G.: *Triple connected graphs*, *Indian Journal of Mathematics and Mathematical Sciences*- to appear (2012).

[12.] Teresa W. Haynes.; Stephen T.; Hedetniemi.; Peter J. Slater.: *Domination in graphs*, Advanced Topics, Marcel Dekker, New York (1998).

[13.] Teresa W. Haynes.; Stephen T. Hedetniemi.; Peter J.Slater.: *Fundamentals of domination in graphs*, Marcel Dekker, New York (1998).