Laplace Substitution Method for Solving Partial Differential Equations Involving Mixed Partial Derivatives

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Abstract:
In this paper we introduced a new method, named Laplace substitution method (LSM), which is based on Lapalce transform. This new method with a convenient way to find exact solution with less computation as compared with Method of Separation of Variables(MSV) and Variational iteration method (VIM). The proposed method solves linear partial differential equations involving mixed partial derivatives.

Keywords: Approximate solution, Adomian decomposition method, Laplace decomposition method, Nonlinear partial differential equations.

Introduction:
Nonlinear ordinary or partial differential equations involving mixed partial derivatives arise in various fields of science, physics and engineering. The wide applicability of these equations is the main reason why they have gained so much attention from many mathe- maticians and scientists. Unfortunately they are sometimes very difficult to solve, either Numerically or theoretically. There are many methods to obtain approximate solutions of these kinds of equations. Partial differential equations (PDE) are differential equations that contain unknown multi variable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. The Method of separation of variables [8] and Variational iteration method [2-4] has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti -Se kine- Mura method [5], JI-Huan He [4] developed the Variational iteration method (VIM) in 1999. The variational iteration method , has been widely applied to solve nonlinear problems, more and more merits have been discovered and some modifications are suggested to overcome the demerits arising in the solution procedure. For example, T.A.Abassy. et al [6,7] also proposed further treatments of these modification results by using Pade approximants and the Laplace transform. The Laplace transform is a widely used integral transform. The Laplace transform has the useful property that many relationships and operations over the originals f(t) correspond to simpler relationships and operations over the images F(s). It is named for Pierre-Simon Laplace (1749-1827) [1], who introduced the transform in his work on probability theory.

The main goal of this paper is to describe new method for solving linear partial Differential equations involving mixed partial derivatives. This powerful method will be proposed in section 2; in section 3 we will apply it to some examples and in last section we give some conclusion.

2. Laplace Substitution Method:

The aim of this section is to discuss the Laplace substitution method. We consider the general form of non homogeneous partial differential equation with initial conditions is given below

$$Lu(x,y) + Ru(x,y) = h(x,y) \qquad (2.1)$$

$$u(x,0) = f(x), \qquad u_y(0,y) = g(y) \qquad (2.2)$$

Where $L = \frac{\partial}{\partial x\, \partial y}$, $Ru(x,y)$ is the remaining linear terms in which contains only first order partial derivatives of u(x, y) with respect to either x or y and h(x, y) is the source term. We can write equation (2.1) in the following form

$$\frac{\partial^2 u}{\partial x\, \partial y} + Ru(x,y) = h(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + Ru(x,y) = h(x,y) \qquad (2.3)$$

Substituting $\frac{\partial u}{\partial y} = U$ in equation (2.3), we get

$$\frac{\partial U}{\partial x} + Ru(x,y) = h(x,y) \qquad (2.4)$$

Taking Laplace transform of equation (2.4) with respect to x, we get

$$U(s,y) = \frac{1}{s}U(0,y) + \frac{1}{s}L_x[h(x,y) - Ru(x,y)]$$

$$U(s,y) = \frac{1}{s}u_y(0,y) + \frac{1}{s}L_x[h(x,y) - Ru(x,y)]$$

$$U(s,y) = \frac{1}{s}g(y) + \frac{1}{s}L_x[h(x,y) - Ru(x,y)] \qquad (2.5)$$

Taking inverse Laplace transform of equation (2.5) with respect to x, we get

$$U(x,y) = g(y) + L_x^{-1}\{L_x[h(x,y) - Ru(x,y)]\} \qquad (2.6)$$

Re-substitute the value of U(x, y) in equation (2.6), we get

$$\frac{\partial u(x,y)}{\partial y} = g(y) + L_x^{-1}\{L_x[h(x,y) - Ru(x,y)]\} \qquad (2.7)$$

This is the first order partial differential equation in the variables x and y. Taking the Laplace transform of equation (2.7) with respect to y, we get

$$su(x,s) = f(x) + L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y)]\right]\right]$$

$$u(x,s) = \frac{1}{s}f(x) + \frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y)]\right]\right] \qquad (2.8)$$

Taking the inverse Laplace transform of equation (2.8) with respect to y, we get

$$u(x,y) = f(x) + L_y^{-1}\left\{\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y)]\right]\right]\right\} \qquad (2.9)$$

The last equation (2.9) gives the exact solution of initial value problem (1.1).

3. Applications:

To illustrate this method for coupled partial differential equations we take three examples in this section.

Example 1: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-y}\cos x \qquad (3.10)$$

with initial conditions $u(x,0) = 0$, $u_y(0,y) = 0$ (3.11)

In the above initial value problem $Lu(x,y) = \frac{\partial^2 u}{\partial x \partial y}$, $h(x,y) = e^{-y}\cos x$ and general linear term Ru(x,y) is zero.

Equation (3.10) we can write in the following form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = e^{-y}\cos x \qquad (3.12)$$

Substituting $\frac{\partial u}{\partial y} = U$ in equation (3.12), we get

$$\frac{\partial U}{\partial x} = e^{-y}\cos x \qquad (3.13)$$

This is the non homogeneous partial differential equation of first order. Taking Laplace transform on both sides of equation (3.13) with respect to x, we get

$$sU(s,y) - U(0,y) = L_x[e^{-y}\cos x]$$

$$U(x,y) = e^{-y}\frac{1}{s}\frac{s}{(1+s^2)}$$

Taking inverse Laplace transform of equation (3.13) with respect to x, we get

$U(x,y) = e^{-y}\cos x$

$$\frac{\partial u(x,y)}{\partial y} = e^{-y}\cos x \qquad (3.14)$$

This is the partial differential equation of first order in the variables x and y. Taking Laplace transform of equation (3.14) with respect to y, we get

$su(x,s) - u(x,0) = \sin x\frac{1}{1+s}$

$$u(x,s) = \sin x\frac{1}{s(1+s)} \qquad (3.15)$$

Taking inverse Laplace transform of equation (3.15) with respect to y, we get

$$u(x,s) = \sin x(1 - e^{-y}) \qquad (3.16)$$

This is the required exact solution of equation (3.10). Which can be verify through the substitution. Which is same the solution obtained by (MSV) and (VIM).

Example 2: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial y\,\partial x} = \sin x \sin y \qquad (3.17)$$

with the inital conditions

$$u(x,0)=1+\cos x \ , \quad u_y(0,y)= -2\sin y \qquad (3.18)$$

In the above example assume that ux(x, y) and uy(x, y) both are differentiable in the domain of definition of function u(x, y) [Young's Theorem]. This implies that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$. Given initial conditions (3.18) force to write the equation (3.17) in following form and use the substitution $\frac{\partial u}{\partial y} = U$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \sin x \sin y \qquad (3.19)$$

$$\frac{\partial U}{\partial x} = \sin x \sin y \qquad (3.20)$$

Taking Laplace transform of equation (3.20) with respect to x, we get

$$U(x,y) = \frac{-2\sin y}{s} + \sin y\left[\frac{1}{s} - \frac{s}{1+s^2}\right] \qquad (3.21)$$

Taking inverse Laplace transform of equation (3.21) with respect to x, we get

$$U(x,y) = -2\sin y + \sin y[1 - \cos x]$$

$$\frac{\partial u(x,y)}{\partial x} = -2\sin y + \sin y[1 - \cos x] \qquad (3.22)$$

Taking Laplace transform of equation (3.22) with respect to y, we get

$$su(x,s) - u(x,0) = \frac{1}{1+s^2}[-1 - \cos x]$$

$$su(x,s) = (1+\cos x) - \frac{1}{1+s^2}[1+\cos x]$$

$$u(x,s) = (1+\cos x)\left[\frac{1}{s} - \frac{1}{s(1+s^2)}\right] \qquad (3.23)$$

Taking inverse Laplace transform of equation (3.23) with respect to y, we get

$$u(x,y)=(1+\cos x)\cos y \qquad (2.24)$$

This is the required exact solution of equation (3.17). Which can be verify through the substitution. Which is same the solution obtained by (MSV) and (VIM).

Example 3: Consider the following partial differential equation with $Ru(x,y) \neq 0$

$$\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial x} + u = 6x^2 y \qquad (3.26)$$

With initial conditions

$$u(x,0)=1, \qquad u(0,y)=y\ , \quad u_y(0,y)=0 \qquad (3.26)$$

In the above example Ru(x, y) = $\frac{\partial^2 u}{\partial x \partial y} + u(x,y)$. Use the substitution $\frac{\partial u}{\partial y} = U(x,y)$ in equation (3.25), we get

$$\frac{\partial U}{\partial x} + \frac{\partial u}{\partial x} + u = 6x^2 y \qquad (3.27)$$

Taking Laplace transform of equation (3.27) with respect to x, we get

$$sU(s,y) - U(0,y) + su(s,y) - u(0,y) + L_x[u(x,y)]\frac{12y}{s^2}$$

$$U(s,y) = -u(s,y) - \frac{1}{s}L_x[u(x,y)] + \frac{12y}{s^4} \qquad (3.28)$$

Taking inverse Laplace transform of equation (3.28) with respect to x, we get

$$U(x,y) = -u(x,y) - L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right] + 2yx^3$$

$$\frac{\partial u(x,y)}{\partial x} = -u(x,y) - L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right] + 2yx^3 \qquad (3.\ 29)$$

Taking Laplace transform of equation (3.29) with respect to y, we get

$$su(x,s) - u(x,0) = -L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right] + 2x^3\frac{1}{s^2}$$

$$u(x,s) = \frac{1}{s} - L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right] + 2x^3\frac{1}{s^2} \qquad (3.30)$$

Taking inverse Laplace transform of equation (3.30) with respect to y, we get

$$u(x,y) = 1 - L_y^{-1}\left[\frac{1}{s}L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right]\right] + x^3y^2 \qquad (3.31)$$

We cannot solve the equation (3.31) because our goal u(x, y) is appeared in both sides of equation (3.31). Therefore the equation (3.25) we cannot solve by using LSM because of $Ru(x,y) \neq 0$.

3. Conclusion

In this paper, we proposed Laplace Substitution Method (LSM) is applicable to solve partial differential equations in which involves mixed partial derivatives and general linear term Ru(x, y) is zero. The result of first two examples compared with (MSV) and (VIM), tell us that these methods can be use alternatively for the solution of higher order initial value problem in which involves the mixed partial derivatives with general linear term Ru(x, y) is zero. But the result of example number three tell us that (LSM) is not applicable for those partial differential equations in which R u(x, y) $\neq$ 0.Consequently the (LSM) is promising and can be applied for other equations that appearing in various scientific fields.

References

[1] Joel L. Schiff, The Laplace Transform: Theory and Applications, Springer

[2] J. H. He, Approximate analytical solution for seepage ow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167(1998)57-68.

[3] J. H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg. 167(1998)69-73.

[4] J.-H. He, Variational iteration method- a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear. Mech. 34 (1999) 699-708.

[5] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, Variational Methods in the Mechanics of Solids, Pergamon Press, New York, 1978, pp. 156162.

[6] T. A. Abassy , M.A.El-Tawil ,H.El-Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration Pade technique ,Journal of Computational and Applied Mathematics 207 (1)(2007)73-91.

[7] T. A. Abassy, M.A.El-Tawil ,H.El-Zoheiry, Exact solutions of some nonlinear partial diferential using the variational iteration method linked with Laplace transforms and the Pade technique, Computers and Mathematics with Applications,54 (7-8))(2007)940-954.

[8] Richard Jozsa, DAMTP Cambridge, PDEs on bounded domains: Separation of vari-ables, Part II, (2011).