ISSN (e): 2250 – 3005 || Volume, 15 || Issue, 4|| April – 2025 ||
International Journal of Computational Engineering Research (IJCER)

Minimal Sets Representing Davenport Constant over $\mathbb{Z}_3^4$

Xingliang Yi Lu Zhao*

School of Mathematical Sciences, Tiangong University, Tianjin, 300387

Abstract

Let G be a finite abelian group. Let$\mathcal{B}(G)$ be the monoid consisting of all zero-sum sequences over G, and let $\mathcal{A}(G)$ be the set consisting of all irreducible elements of the monoid $\mathcal{B}(G)$. For $\Omega \subset \mathcal{B}(G)$, the universal zero-sum invariant $d_\Omega(G)$ is defined to be the smallest positive integer ℓ such that every sequence T over G of length ℓ has a subsequence in Ω. If Ω is equal to $\mathcal{A}(G)$, then $d_\Omega(G)$ reduces to the well-known Davenport constant $D(G)$. A set $\Omega \subset \mathcal{B}(G)$ is called a minimal set (to represent the Davenport constant) if$d_\Omega(G) = D(G)$ and $d_{\Omega'} \neq D(G)$ for every proper subset Ω' of Ω. In [8], for any finite abelian group G with $exp(G) \neq 3$, G. Wang determined all classes of groups in which $\mathcal{A}(G)$ is a minimal set, and furthermore conjectured that for the case $exp(G) = 3$, i.e., $G \cong \mathbb{Z}_3^r (r \geq 1)$, then $\mathcal{A}(G)$ is a minimal set. In this paper, we confirm $the conjecture\ for\ the\ group \mathbb{Z}_3^r with r \leq 4$.

Key Words: *Davenport constant; the minimal set representing Davenport constant; Universal zero-sum invariant; zero-sum; elementary 3-groups*

Date of Submission: 13-04-2025 Date of acceptance: 26-04-2025

I. Introduction

Let G be a finite abelian group. The Davenport constant $\mathrm{D}(G)$ of G is defined as the smallest positive integer ℓ such that, every sequence of ℓ terms from G contains some terms with sum being the identity element. This invariant was first formulated by K. Rogers [7], and popularized by H. Davenport in the 1960's, notably for its link with algebraic number theory (as reported in [6]), and has been investigated extensively in the past 60 years. A lot of researches were motivated by the Davenport constant together with the celebrated EGZ Theorem obtained by P. Erdős, A. Ginzburg and A. Ziv [1] in 1961 on additive properties of sequences in finite abelian groups, which have been developed into a branch, called zero-sum theory (see [2] for a survey). To generalize the Davenport constant and some other zero-sum invariants and to understand their common properties, Gao, Li, etc. [5], defined the universal zero-sum invariant of a finite abelian group as follows.

Let $\mathcal{B}(G)$ be the free commutative monoid, multiplicatively written, with basis G, i.e., consisting of all finite zero-sum sequences of terms from the group G. For any nonempty set $\Omega \subset \mathcal{B}(G)$, we define the *universal zero-sum invariant*$\mathsf{d}_\Omega(G)$ to be the smallest positive integer ℓ (if it exists, otherwise $\mathsf{d}_\Omega(G) = \infty$) such that every sequence over G of length ℓ has a subsequence in Ω.

The universal zero-sum invariant motivates some reseraches (see [3,4,8] for example). Among which, the minimal set to represent the Davenport constant is the basic question on the universal zero-sum invariant, the definition for which is proposed as follows.

Definition [5] Let G be a finite abelian group, and let $t > 0$ be an integer. A set $\Omega \subset \mathcal{B}(G)$ is called minimal with respect to t provided that $\mathsf{d}_\Omega(G) = t$ and $\mathsf{d}_{\Omega'}(G) \neq t$ for any subset $\Omega' \subsetneq \Omega$. In particular, if $t = \mathrm{D}(G)$ then we just call Ω a minimal set for short.

Let $\mathcal{A}(G)$ be the set consisting of all minimal zero-sum sequences over G. It is easy to show that $\mathsf{d}_{\mathcal{A}(G)}(G) = \mathrm{D}(G)$ (see [8]). The following basic question on the minimality of $\mathcal{A}(G)$ to represent the Davenport constant is proposed.

Question[5] *Is $\Omega = \mathcal{A}(G)$ minimal with respect to $\mathsf{d}_\Omega(G) = \mathrm{D}(G)$, i.e., does there exist $\Omega' \subsetneq \mathcal{A}(G)$ such that $\mathsf{d}_{\Omega'}(G) = \mathrm{D}(G)$?*

Among other results, very recently Wang proved the following theorem on the minimal set with respect to the Davenport constant.

Theorem[8] Let G be a finite nonzero abelian group with $\exp(G) \neq 3$. Then $\mathcal{A}(G)$ is a minimal set with respect to Davenport constant $\mathrm{D}(G)$ if and only if one of the following conditions holds: $(i) G \cong \mathbb{Z}_4$; $(ii) G \cong \mathbb{Z}_5$; $(iii) G \cong \mathbb{Z}_2^r$ for $r \geq 1$.
However, for the case of $\exp(G) = 3$, i.e., $G \cong \mathbb{Z}_3^r (r \geq 1)$ this question remains unsolved. Hence, Wang [8] proposed the following conjecture.

Conjecture [8] $\mathcal{A}(G)$ is a minimal set with respect to $\mathrm{D}(G)$ if $G \cong \mathbb{Z}_3^r$ where $r \geq 1$.

In this paper, we prove that this conjecture holds true for the case that $r \leq 4$, which is stated as Theorem 1 in Section 3.

Notation

Let $\mathbb{N}$ denote the set of positive integers. For $n, r \in \mathbb{N}$, let $\mathbb{Z}_n$ be the cyclic group of order n, and $\mathbb{Z}_n^r = \underbrace{\mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n}_{r}$ the direct sum of r copies of $\mathbb{Z}_n$. A set $\{e_1, \cdots, e_r\} \subseteq \mathbb{Z}_n^r$ is called a basis of $\mathbb{Z}_n^r$ if $\mathbb{Z}_n^r = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$. Let G be a finite abelian group, $\mathcal{F}(G)$ the free abelian monoid with basis G, whose operation is denoted by '$\cdot$'. Denote $[x, y] = \{z \in \mathbb{Z}: x \leq z \leq y\}$ for integers $x, y \in \mathbb{Z}$. By $T \in \mathcal{F}(G)$, we mean T is a sequence of terms from G which is unordered, repetition of terms allowed. We write $T = \prod_{a \in G} a^{\mathrm{v}_a(T)}$, where $\mathrm{v}_a(T)$ denotes the multiplicity of element a in T. By $|T|$, we denote the length of T and $|T| = \sum_{a \in T} v_a(T)$. By T', we donote a subsequence of T (written $T' \mid T$) if $\mathrm{v}_a(T') \leq \mathrm{v}_a(T)$ for all $a \in T$. Let $\sigma(T) = \sum a \mid Ta$ be the sum of all terms of the sequence T. A sequence T is called zero-sum if $\sigma(T) = 0_G$ (where 0_G is the identity element of G), and minimal zero-sum if none of its proper subsequences is zero-sum. The Davenport constant $\mathrm{D}(G)$ is defined as the smallest positive integer such that every sequence T over G with $|T| \geq \mathrm{D}(G)$ contains a non-empty zero-sum subsequence. Let $\mathcal{A}(G) \subset \mathcal{F}(G)$ denote the set consisting of all minimal zero-sum sequences over G.

Minimal sets representing Davenport constant over $\mathbb{Z}_3^4$

Lemma 1[9] $\mathcal{A}(G)$ is a minimal set to represent the Davenport constant in the groups $\mathbb{Z}_3^r$ with $r \leq 3$.

Lemma 2[6] If G is a finite abelian p-group, i.e., $G \cong \mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_r}}$, then $\mathrm{D}(G) = 1 + \sum_{i=1}^{r} (p^{n_i} - 1)$.

Lemma 3[8] Let G be a finite abelian group and V a non-empty zero-sum sequence over G. Then V belongs to every minimal set contained in $\mathcal{A}(G)$ if and only if there exists a sequence $T \in \mathcal{F}(G)$ of length $\mathrm{D}(G)$ such that every minimal zero-sum subsequence of T is equal to V.

Now we are in a position to prove the main theorem of this paper.

Theorem 1 Let $G \cong \mathbb{Z}_3^r (r \leq 4)$. $\mathcal{A}(G)$ is a minimal set representing Davenport constant for G.

Proof By Lemma 1, it suffices to consider the case of $r = 4$, i.e, $G \cong \mathbb{Z}_3^4$. Then it follows from Lemma 2 that $\mathrm{D}(G) = 9$. To prove $\mathcal{A}(G)$ is a minimal set, by Lemma 3 it suffices to show that for each $S \in \mathcal{A}(G)$, there exists a sequence $T \in \mathcal{F}(G)$ of length 9 such that every minimal zero-sum subsequence of T is equal to S. Since $|S| \leq 9 = \mathrm{D}(G)$. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of G.

Claim. If the cardinality of the maximal linearly independent subset of $supp(S)$ is no more than 4 then the conclusion holds.

Proof of the claim. We can assume without loss of generality that S is a sequence of terms from the subgroup $H =< e_1, e_2, e_3 >\cong \mathbb{Z}_3^3$. Then it follows from Lemma 1 that we can find a sequence T' over H with $|T'| = \mathrm{D}(H) = 7$ such that every minimal zero-sum sequences of T' is equal to S. Then we take $T = T' \cdot e_4^2$ to be a sequence over G with length $|T| = 9$. It is easy to check that every minimal zero-sum subsequence is equal to S, we are done. □

If $|S| = 9 = \mathrm{D}(G)$ we can just take $T = S$ and have the conclusion proved. Hence, combined with the above claim, we need only to consider the case of $|S| \in [5,8]$. Then we shall distinguish some cases by the length of $|S|$ as follows.

Case 1. Suppose $|S| = 5$. Since S contains four linearly independent elements, we can assume without loss of generality that $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (2e_1 + 2e_2 + 2e_3 + 2e_4)$. Then we check that $T = S \cdot e_1 \cdot e_2 \cdot e_3 \cdot e_4$ is the desired sequence.

Case 2. Suppose $|S| = 6$.

If S contains a pair of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (e_1 + 2e_2 + 2e_3 + 2e_4))$. Then we take $T = S \cdot e_2 \cdot e_3 \cdot e_4$.

Otherwise, S contains no duplicate elements. Then there are four possible subcases (1)$S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (e_1 + e_2) \cdot (e_1 + e_2 + 2e_3 + 2e_4)$; (2) $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (e_1 + 2e_2) \cdot (e_1 + 2e_3 + 2e_4)$; (3) $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (e_1 + e_2 + e_3) \cdot (e_1 \cdot e_2 \cdot e_3 \cdot 2e_4)$; (4) $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot (2e_1 + e_2 + e_3) \cdot (e_2 + e_3 + 2e_4)$. Then we can construct T has the forms (1) $T = S \cdot e_2 \cdot e_3 \cdot e_4$; (2) $T = S \cdot e_2 \cdot e_3 \cdot e_4 \cdot$; (3)$T = S \cdot e_2 \cdot e_3 \cdot e_4 \cdot$; (4)$T = S \cdot e_1 \cdot e_2 \cdot e_4$, respectively.

Case 3. Suppose $|S| = 7$. Then various constructions based on elements repetitions and linear combinations.

If S contains four linearly independent elements and 2 pairs of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2^2 \cdot e_3 \cdot e_4 \cdot (e_1 + e_2 + 2e_3 + 2e_4)$, then take $T = S \cdot e_3 \cdot e_4$.

If S contains a pair of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2 \cdot e_3 \cdot e_4 \cdot x \cdot y$. Then there are five possible subcases (1)$x = e_2 + e_3$, $y = e_1 + e_2 + e_3 + 2e_4$; (2)$x = e_2 + 2e_3$, $y = e_1 + e_2 + 2e_4$; (3)$x = e_1 + e_2 + e_3 + e_4$, $y = e_2 + e_3 + e_4$; (4)$x = e_2 + 2e_3 + e_4$, $y = e_1 + e_2 + e_4$; (5)$x = e_2 + 2e_3 + 2e_4$, $y = e_1 + e_2$. Then we can construct T has the forms (1)$T = S \cdot e_2 \cdot e_4$;(2)$T = S \cdot x \cdot y$; (3)$T = S \cdot e_2 \cdot e_3$; (4)$T = S \cdot e_2 \cdot e_3$; (5)$T = S \cdot e_3 \cdot e_4$, respectively.

If S contains no duplicate element, we can show that under the isomorphism of G, $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot x \cdot y \cdot z$. Then there are ten possible subcases (1)$x = e_1 + e_2$, $y = e_3 + e_4$, $z = e_1 + e_2 + e_3 + e_4$; (2)$x = e_1 + 2e_2$, $y = 2e_2 + e_3 + e_4$, $z = e_1 + e_2 + e_3 + e_4$; (3)$x = e_1 + 2e_2 + 2e_3$, $y = 2e_2 + 2e_3 + e_4$, $z = e_1 + e_2 + e_3 + e_4$; (4)$x = e_1 + e_2$, $y = 2e_2 + e_3 + e_4$, $z = e_1 + 2e_2 + e_3 + e_4$; (5)$x = e_1 + 2e_2$, $y = e_2 + e_3 + e_4$, $z = e_1 + 2e_2 + e_3 + e_4$; (6)$x = e_2 + 2e_3 + 2e_4$, $y = e_1 + 2e_2 + 2e_3 + 2e_4$, $z = e_1 + 2e_2 + e_3 + e_4$; (7)$x = e_2 + 2e_4$, $y = e_1 + 2e_2 + e_3 + 2e_4$, $z = e_1 + 2e_2 + e_3 + e_4$; (8)$x = e_1 + 2e_2$, $y = e_2 + 2e_3 + e_4$, $z = e_1 + 2e_2 + 2e_3 + e_4$; (9)$x = e_2 + e_3 + e_4$, $y = e_1 + 2e_2 + 2e_3$, $z = e_1 + 2e_2 + 2e_3 + e_4$; (10)$x = e_2 + e_3 + 2e_4$, $y = e_1 + 2e_2 + 2e_3 + 2e_4$, $z = e_1 + 2e_2 + 2e_3 + e_4$. Then we can construct T has the forms (1)$T = S \cdot e_1 \cdot e_3$; (2)$T = S \cdot e_1 \cdot e_3$; (3)$T = S \cdot e_2 \cdot e_3$; (4)$T = S \cdot e_1 \cdot e_3$; (5)$T = S \cdot e_3 \cdot e_4$; (6)$T = S \cdot e_2 \cdot e_3$; (7)$T = S \cdot e_3 \cdot e_4$; (8)$T = S \cdot e_3 \cdot e_4$; (9)$T = S \cdot e_3 \cdot e_4$; (10)$T = S \cdot e_3 \cdot e_4$, respectively.

Case 4. Suppose $|S| = 8$. If S contains 3 pairs of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2^2 \cdot e_3^2 \cdot e_4 \cdot (e_1 + e_2 + e_3 + 2e_4)$, then take $T = S \cdot e_4$.

If S contains 2 pairs of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2^2 \cdot e_3 \cdot e_4 \cdot x \cdot y$. If $x = e_3 + e_4$, $y = e_1 + e_2 + e_3 + e_4$, take $T = S \cdot e_3$; if $x = e_3 + 2e_4$, $y = e_1 + e_2 + e_3$, take $T = S \cdot e_3$.

If S contains a pair of same elements, we can show that under the isomorphism of G, $S = e_1^2 \cdot e_2 \cdot e_3 \cdot e_4 \cdot x \cdot y \cdot z$. Then there are six possible subcases (1)$x = e_2 + e_4,\ y = e_3 + e_4,\ z = e_1 + e_2 + e_3$; (2)$x = e_1 + e_2,\ y = e_3 + e_4,\ z = e_2 + e_3 + e_4$; (3)$x = 2e_2 + e_4,\ y = e_1 + e_2 + e_3,\ z = 2e_2 + e_3 + e_4$; (4)$x = e_2 + e_3,\ y = e_1 + 2e_2 + e_4,\ z = 2e_2 + e_3 + e_4$; (5)$x = e_2 + 2e_3,\ y = 2e_2 + e_3 + e_4,\ z = e_1 + 2e_2 + 2e_3 + e_4$; (6)$x = e_1 + e_2 + 2e_3,\ y = 2e_2 + e_3 + e_4,\ z = 2e_2 + 2e_3 + e_4$. Then T has the form $T = S \cdot e_3$ is the desired.

Otherwise, S contains no duplicate element and we can show that under the isomorphism of G, $S = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot x \cdot y \cdot z \cdot v$. Then there are twenty-three possible subcases (1)$x = e_1 + e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_2 + e_3,\ v = e_2 + e_4$; (2)$x = e_1 + e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_1 + 2e_2,\ v = e_1 + e_2 + e_3$; (3)$x = e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = 2e_1 + 2e_2 + e_3,\ v = e_2 + e_4$; (4)$x = e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = e_1 + 2e_2 + e_3,\ v = e_1 + e_2 + e_4$; (5)$x = 2e_1 + e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = e_1 + 2e_2 + e_3,\ v = e_1 + e_2 + e_4$; (6)$x = 2e_1 + e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_1 + 2e_2 + e_3,\ v = e_2 + e_4$; (7)$x = 2e_1 + e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_2 + e_4,\ v = 2e_1 + e_2 + e_3$; (8)$x = 2e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = e_1 + 2e_2 + e_4,\ v = e_2 + e_3$; (9)$x = 2e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = 2e_2 + e_3,\ v = e_1 + e_2 + e_4$; (10)$x = 2e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = e_1 + 2e_2 + e_3,\ v = e_2 + e_4$; (11)$x = 2e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + e_2 + e_3,\ z = e_3 + e_4,\ v = e_1 + 2e_3$; (12)$x = 2e_1 + e_2 + e_3 + e_4,\ y = 2e_1 + 2e_2 + e_3,\ z = e_1 + e_2 + 2e_4,\ v = e_1 + 2e_4$; (13)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = e_1 + e_2 + e_3,\ v = e_1 + e_2 + e_4$; (14)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_1 + e_2 + e_3,\ v = e_2 + e_4$; (15)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = e_1 + e_2,\ z = 2e_1 + e_3,\ v = 2e_2 + e_4$; (16)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = e_1 + e_2,\ v = e_2 + e_3$; (17)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = 2e_1 + e_2,\ z = e_2 + e_3,\ v = e_1 + e_2 + e_4$; (18)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = 2e_1 + e_2 + e_3,\ z = e_1 + e_2 + 2e_4,\ v = e_2 + 2e_4$; (19)$x = 2e_1 + 2e_2 + e_3 + e_4,\ y = 2e_1 + 2e_2 + e_3,\ z = e_1 + 2e_4,\ v = e_2 + 2e_4$; (20)$x = 2e_1 + 2e_2 + 2e_3 + e_4,\ y = e_1 + e_2,\ z = e_1 + e_3,\ v = e_1 + 2e_2 + 2e_3 + e_4$; (21)$x = 2e_1 + 2e_2 + 2e_3 + e_4,\ y = e_1 + e_2,\ z = e_1 + 2e_3,\ v = e_1 + 2e_2 + e_3 + e_4$; (22)$x = 2e_1 + 2e_2 + 2e_3 + e_4,\ y = e_1 + e_2,\ z = e_1 + 2e_3 + 2e_4,\ v = e_1 + 2e_2 + e_3 + 2e_4$; (23)$x = 2e_1 + 2e_2 + 2e_3 + e_4,\ y = e_1 + e_3,\ z = e_1 + 2e_2 + e_3 + e_4,\ v = e_1 + e_2 + e_3$. Then we construct $T = S \cdot e_4$ for the subcases (19) and (21), and $T = S \cdot e_1$ for other subcases, respectively. This completes the proof of the Theorem 1. □

References

[1] P. Erdős, A. Ginzburg and A. Ziv, *Theorem in additive number theory*, Bull. Res. Council Israel, **10F** (1961) 41–43.

[2] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math., **24** (2006) 337–369.

[3] W. Gao, S. Hong, W. Hui, X. Li, Q. Yin and P. Zhao, *Representation of zero-sum invariants by sets of zero-sum sequences over a finite abelian group*, Period. Math. Hungar., **85** (2022) 52–71.

[4] W. Gao, S. Hong, W. Hui, X. Li, Q. Yin and P. Zhao, *Representation of zero-sum invariants by sets of zero-sum sequences over a finite abelian group II*, J. Number Theory, **241** (2022) 738–760.

[5] W. Gao, Y. Li, J. Peng and G. Wang, *A unifying look at zero-sum invariants*, Int. J. Number Theory, **14** (2018) 705–711.

[6] J.E. Olson, *A Combinatorial Problem on Finite Abelian Groups, I*, J. Number Theory, **1** (1969) 8–10.

[7] K. Rogers, *A combinatorial problem in Abelian groups*, Math. Proc. Cambridge Philos. Soc., **59** (1963) 559–562.

[8] G. Wang, *The universal zero-sum invariant and weighted zero-sum for infinite abelian groups*, Communications in algebra, **53** (2025) 1581–1599.

[9] X. Zhao, X. Wu, *Representation of Minimal Set of Davenport Constants in the Elementary 3-group*, Journal of Luoyang Normal University, **42** (2023) 1–4.