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Confidence interval for a hazard function of a two parameter Weibull distribution

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Precise estimation of the failure rate, also known as the hazard function, is fundamental in reliability engineering, especially for systems where components experience failure due to aging or wear-out. The two-parameter Weibull distribution is frequently applied to model such behaviours due to its versatility in capturing both early-life and wear-out failure patterns. However, confidence interval (CI) estimation for the hazard function in this context has been relatively underrepresented in research. This paper proposes a new method for constructing CI for the Weibull hazard function using the generalized variable (GV) approach, designed for both complete samples and Type-II right-censored data. Reliable interval estimates are critical in fields such as aircraft engine maintenance and equipment servicing, where they inform decisions regarding maintenance scheduling, safety measures, and cost optimization. The proposed CI provides a range for the failure rate function, facilitating better planning for component wear, resource management, and regulatory adherence. Through a comprehensive simulation study, this method was assessed across different sample sizes, levels of censoring, and parameter values. Results indicate that the proposed method achieves accurate and focused coverage probabilities, even under conditions with small sample sizes or high censoring levels, highlighting its value for reliability engineers in applications where accurate failure rate estimation is essential for effective operational planning and safety.
Keywords: *Hazard function, Weibull distribution, Confidence interval, Generalized Variable technique, Type II Censoring.*

I. Introduction

The two-parameter Weibull distribution is widely recognized for its versatility and is applied across various scientific fields. Its ability to model data effectively makes it invaluable in disciplines such as biology, environmental science, health, physical sciences, and social sciences. Moreover, the Weibull distribution is integral to meteorology, hydrology, and reliability engineering, serving as a fundamental framework for analysing time-dependent failure data. Its significance in reliability theory arises from its capability to capture diverse failure patterns, from early-life failures to those occurring due to wear and tear, based on its parameterization. This flexibility allows researchers and practitioners to predict component lifetimes and assess system reliability, which is crucial for maintenance strategies and risk management.

In the realm of reliability engineering, the Weibull distribution has become a standard tool for estimating the time to failure of components and systems. Its broad applicability is supported by foundational studies by researchers such as Grace and Eagleson (1966), Crow (1982), and Nathan and McMahon (1990), among others. Recent advancements in this area by authors like Lun and Lam (2000), Yang et al. (2007), Krishnamoorthy and Lin (2010), Kulkarni and Powar (2011) and Jamdade and Jamdade (2012) further validate its usefulness in practical applications. Contributions from J.I. McCool (2012) and Powar and Kulkarni (2015) have also enhanced its relevance in contemporary reliability and risk analysis.

A continuous random variable (RV) X is said to follow a Weibull distribution with scale parameter α and shape parameter β if its probability density function (pdf) is given by,

$$f_X(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right); x > 0, \alpha > 0, \beta > 0.$$

We denote it as $X \rightarrow$ Weibull (α, β). The failure rate function or hazard function at t, $h(t)$, for Weibull (α, β) distribution is,

$$h(t,\alpha,\beta) = \beta t^{\beta-1}/\alpha^{\beta}; t > 0, \alpha > 0, \beta > 0.$$

The estimation of CI for $h(t,\alpha,\beta)$ for Weibull (α, β) is vital in assessing failure rates, aiding decision-making across numerous fields. By providing a range of values likely to encompass the true hazard rate, this estimation informs strategies for maintenance, resource allocation, and risk management.

In reliability engineering, accurately predicting when components or systems may fail is essential for minimizing downtime and controlling maintenance expenses. Interval estimation of the hazard function enables engineers to gain insights into the possible variation in failure rates over time. For example, a company producing electronic devices estimates the hazard function for a critical component and determines that the CI for the failure rate after 500 hours of operation is between 0.02 and 0.05 failures per hour. This information allows the engineering team to strategically schedule maintenance. If the upper limit indicates a heightened risk of failure, the team may opt for pre-emptive maintenance to prevent unexpected disruptions, ensuring smoother operations.

In healthcare, understanding the hazard function can significantly impact treatment strategies for diseases, particularly for those with high recurrence rates. For example, during a clinical trial for a new cancer treatment, researchers analyse patient data to estimate the hazard function regarding cancer recurrence. If a 95% CI for the hazard rate at one-year post-treatment is found to be between 0.10 and 0.20, oncologists can better inform patients about their recurrence risks and formulate follow-up care plans accordingly. A higher upper bound may indicate a need for more intensive monitoring or additional therapies.

In the aviation industry, ensuring the reliability of aircraft components is critical for passenger safety. Interval estimation of the hazard function plays a key role in maintenance decision-making. For example, an airline analyses engine failure data and uses the two-parameter Weibull distribution to estimate the hazard function. If a 90% CI for the hazard rate at 1,000 flight hours is calculated to be between 0.01 and 0.03 failures per flight hour, the airline can utilize this information to establish maintenance schedules that pre-empt potential failures, thereby enhancing safety and adhering to industry regulations.

Interval estimation is also instrumental in quality control processes, allowing manufacturers to assess product reliability effectively. For example, a battery manufacturer evaluates the reliability of a new battery model by fitting a two-parameter Weibull distribution to their failure time data. If the 95% CI for the hazard function at 300 cycles is found to be between 0.005 and 0.015 failures per cycle, the manufacturer gains crucial insights. A lower bound that indicates reliability may prompt the company to market the product more aggressively, while a higher upper bound could lead to design improvements.

In environmental studies, evaluating the risk of failure in critical infrastructure, such as levees and dams, is essential for public safety. For example, Engineers tasked with assessing the integrity of a levee system might apply interval estimation of the hazard function based on historical flood data. If they find a 95% CI for the hazard rate suggesting a failure rate of between 0.0005 and 0.002 failures per year, this information can guide maintenance strategies, investment decisions, or even emergency response plans during extreme weather events.

These examples emphasize that interval estimation of the hazard function for the two-parameter Weibull distribution is a fundamental tool for managing risk and improving decision-making across diverse sectors, including engineering, healthcare, aviation, manufacturing, and environmental science. By offering a reliable range of possible values for failure rates, stakeholders can develop effective maintenance strategies, allocate resources judiciously, and enhance safety measures.

In various reliability and life-testing studies, researchers often encounter challenges when trying to gather complete data on failure times for all experimental units. For instance, in clinical trials, constraints such as limited funding can lead to participant dropout before the study concludes. Likewise, in industrial settings, units may either experience unforeseen failures or be intentionally withdrawn prior to failure to save time and minimize costs. The resulting data from such studies are classified as censored data, which can complicate statistical analyses and the resulting interpretations.

Among the various types of censoring, Type-I and Type-II are the most recognized. Type-I censoring occurs when the experiment has a fixed duration, denoted as T, while the number of failures can vary. In contrast, Type-II censoring is defined by a predetermined number of failures, referred to as r, with the duration of the experiment being variable. This article emphasizes the GV method, which is well-suited for analysing Type-II singly right-censored samples. In this scenario, the pivotal quantities employed for maximum likelihood estimators (MLEs) remain applicable, allowing for effective estimation of the hazard function even when data is censored.

While the Weibull distribution is extensively utilized in various domains, the literature has largely overlooked the estimation of CI for its hazard function. This gap is especially significant in the context of small sample sizes, where the challenges associated with accurately estimating CI can be heightened. Filling this void is essential for improving the reliability of hazard function evaluations, particularly in real-world applications where data may be limited.

This article presents an empirical analysis that reveals a strong correlation between the estimated coverage probabilities and the nominal coverage probabilities for the proposed method of CI estimation for the hazard function $h(t,\alpha,\beta)$. This correlation is especially noticeable when assessing various t values in uncensored samples, particularly in situations involving small sample sizes. Such circumstances are often encountered in healthcare research, where the high costs of laboratory testing for contaminants can limit sample sizes.

As regulatory frameworks may require the estimation of the hazard function at larger *t* values using small to moderate sample sizes, addressing this issue is vital. Thus, the objective of this article is to introduce a new method for estimating CI for the hazard function of the commonly used Weibull distribution. The proposed approach aims to maintain a close alignment of coverage probabilities with nominal values, even when faced with small sample sizes and in both uncensored and censored data contexts for all values of *t*.

This study tackles the statistical challenge of constructing CI for the hazard function of the Weibull distribution through the GV approach, originally developed by Tsui and Weerahandi (1989) and further advanced by Weerahandi (1993). For a more in-depth understanding of the GV approach and its wide-ranging applications, Weerahandi's works (1995, 2004) provide valuable insights. Hannig et al. (2006) also offer illustrative examples that highlight the practical use of this method. The GV approach facilitates the creation of a generalized pivotal quantity (GPQ), which is instrumental in deriving CIs for various parametric functions.

A GPQ is distinct from traditional pivotal quantities, as it is derived from observed statistics combined with random variables, without depending on unknown parameters. A major advantage of the GV method is its flexibility in forming a GPQ for parameter functions by substituting GPQs associated with each parameter individually (Krishnamoorthy et al., 2009). This study presents a GV-based technique for constructing two-sided CI for the hazard function in distributions with defined GPQs for their parameters. The performance of this method is examined through numerical simulations for the Weibull distribution, incorporating both uncensored data and Type-II singly right-censored samples.

The layout of this paper is structured as follows: Section 2 reviews the foundational concepts related to GPQs and introduces the proposed approach. Section 3 describes the methodology for constructing CI for the Weibull distribution's hazard function. Section 4 outlines the simulation studies conducted to evaluate the CI coverage probabilities of the proposed method and section 5 concludes with a summary of the main findings and implications.

II. A GPQ Method for Reliable CI Estimation:

A GPQ, symbolized as G_θ for a parameter θ, is defined through a random variable $T_\theta(X; x)$. In this context, *X* is a random variable whose distribution depends on both the primary parameter θ and an additional nuisance parameter δ. The observed value of *X* is represented by x, and the GPQ $T_\theta(X; x)$ is structured to fulfill two main conditions:

1. At $X = x$, the GPQ $G_\theta = T_\theta(X; x)$ remains unaffected by the nuisance parameter δ. Often, this means $G_\theta = \theta$.
2. The distribution of $G_\theta = T_\theta(X; x)$, conditioned on $X = x$, does not involve any unknown parameters.

These characteristics make GPQ a useful tool for parameter estimation that is independent of nuisance parameters.

2.1 An Innovative Method for CI Estimation of Population Hazard Function:

Let $X_1, X_2, \ldots, X_n$ represent a random sample of size *n* from a distribution with the probability density function $f_X(x; \underline{\theta})$, where $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ denotes a vector of unknown parameters. For each component of $\underline{\theta} \in \Theta \subseteq \mathfrak{R}^k$, a GPQ is assumed to be available, denoted by $G_{\underline{\theta}} = \left(G_{\theta_1}, G_{\theta_2}, \ldots, G_{\theta_k}\right)$. Let $h\left(t, \underline{\theta}\right)$ be the hazard function of *X*. Although $h\left(t, \underline{\theta}\right)$ may not always have a straightforward analytical form, it can be computed numerically for specific values of *t* and θ. A GPQ for $h\left(t, \underline{\theta}\right)$ can be expressed as:

$$G_{h_t} = h\left(t, G_{\underline{\theta}}\right) \quad (1)$$

where G_{h_t} follows a distribution independent of θ. To construct a two-sided CI for $h\left(t, \underline{\theta}\right)$ at a confidence level of $(1-\delta) \times 100$, based on the GPQ G_{h_t}, the following procedure can be applied:

1. For observed data $\underline{x}$ and maximum likelihood estimates (or other appropriate estimators) $\widehat{\underline{\theta}_0}$ of $\underline{\theta}$, repeat the steps below N times (e.g. N=100,000):
   i. Compute GPQs $G_{\underline{\theta}} = \left(G_{\theta_1}, G_{\theta_2}, \ldots, G_{\theta_k}\right)$ for $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, possibly using the method outlined by Iyer and Patterson (2002).
   ii. Calculate G_{h_t} using the above expression (1).
2. The percentiles $(100\times\delta/2)$ and $100\times(1-\delta/2)$ of the N generated values of G_{h_t} provide the lower (L) and upper (U) bounds, respectively, of the two-sided $(1-\delta) \times 100\%$ CI for $h\left(t, \underline{\theta}\right)$, denoted as [L, U]. This interval serves as the "Generalized Confidence Interval (GCI)" for $h\left(t, \underline{\theta}\right)$.

The GPQ-based inference technique is known for yielding precise results, as discussed, for example, by Roy and Bose (2009).

III. Proposed CI for the Weibull Hazard Function $h(t, \alpha, \beta)$:

For a complete sample, the maximum likelihood estimator (MLE) for the parameter β, denoted as $\hat{\beta}$, is determined by solving the following equation:

$$\frac{1}{\hat{\beta}} - \frac{\sum_{i=1}^{n} x_i^{\hat{\beta}} \log(x_i)}{\sum_{i=1}^{n} x_i^{\hat{\beta}}} + \frac{1}{n}\sum_{i=1}^{n} \log(x_i) = 0 \qquad (2)$$

Alongside this, the estimator for α is given by:

$$\hat{\alpha} = \left(\sum_{i=1}^{n} x_i^{\hat{\beta}} /n\right)^{1/\hat{\beta}}$$

In the context of a Type-II singly right-censored sample, where only the smallest r observations are available in an ordered form $x_{(1)} \le x_{(2)} \le \cdots \le x_{(r)}$, the MLE for β can be computed by solving the following equation:

$$\frac{1}{\hat{\beta}} - \frac{\sum_{i=1}^{n} x_{iu}^{\hat{\beta}} \log(x_{iu})}{\sum_{i=1}^{n} x_{iu}^{\hat{\beta}}} + \frac{1}{r}\sum_{i=1}^{r} \log(x_{iu}) = 0 \qquad (3)$$

The estimator for α remains:

$$\hat{\alpha} = \left(\sum_{i=1}^{n} x_{iu}^{\hat{\beta}} /n\right)^{1/\hat{\beta}}$$

In this case, $x_{iu} = x_{(i)}$ denotes the observed values in ordered form for i =1, 2,…,r and $x_{iu} = x_{(r)}$ for i=r+1,…,n.

To solve equations (2) and (3) iteratively, the Newton–Raphson method can be applied. Additionally, statistical software such as R and MINITAB offers functionalities for directly estimating these parameters, streamlining the analysis process.

3.1 Exploring GPQs for the Parameters α, β and Hazard Function $h(t, \alpha, \beta)$:

Krishnamoorthy et al. (2009) introduced the concept of GPQs for the parameters α and β. Denote $\widehat{\alpha_0}$ and $\widehat{\beta_0}$ as the observed maximum likelihood estimates (MLEs) for α and β, respectively. The GPQs for these parameters can be formulated as follows:

$$G_\alpha = \widehat{\alpha_0} \left(\frac{\alpha}{\tilde{\alpha}}\right)^{\tilde{\beta}/\widehat{\beta_0}} = \widehat{\alpha_0} \left(\frac{1}{\tilde{\alpha}}\right)^{\tilde{\beta}/\widehat{\beta_0}} \qquad (4)$$

and

$$G_\beta = \frac{\beta}{\tilde{\beta}}\, \widehat{\beta_0} = \frac{\widehat{\beta_0}}{\tilde{\beta}} \qquad (5)$$

In these expressions, $\tilde{\alpha}$ and $\tilde{\beta}$ are the MLEs for α and β derived from either censored or uncensored samples from a Weibull (1,1) distribution. Using equation (1) as a reference, the GPQ for the hazard function $h(t, \alpha, \beta)$ can be expressed as:

$$G_{h_t} = h(t, G_\alpha, G_\beta) = G_\beta t^{G_\beta - 1}/G_\alpha{}^{G_\beta} = \tilde{\alpha}\frac{\widehat{\beta_0}}{\tilde{\beta}}(t)^{\frac{\widehat{\beta_0}}{\tilde{\beta}}-1}(\widehat{\alpha_0})^{-\frac{\widehat{\beta_0}}{\tilde{\beta}}} \qquad (6)$$

To construct a two-sided (1−δ)×100% GCI for $h(t, \alpha, \beta)$ when $t > 0, \alpha > 0, \beta > 0$ and using a complete sample, the following algorithm may be applied. This procedure can also be adapted for Type-II singly right-censored samples by substituting the relevant MLEs and GPQs.

Algorithm Steps:

1. Determine the MLEs $\widehat{\alpha_0}$ and $\widehat{\beta_0}$ for the parameters α and β from a sample $x_1, x_2, \ldots, x_n$ of size n, assuming a Weibull (α, β) distribution.
2. Using the values $\widehat{\alpha_0}$ and $\widehat{\beta_0}$, repeat the following steps N times (e.g. N=100,000):
   i. Generate n independent random values $x_{1_{11}}, x_{2_{11}}, x_{3_{11}}, \ldots, x_{n_{11}}$ from a Weibull (1,1) distribution, then compute $\tilde{\alpha}$ and $\tilde{\beta}$, the MLEs for α and β based on this simulated sample.
   ii. Using the computed values in Equations (4) and (5), determine the GPQs G_α and G_β.
   iii. Substitute these GPQs into Equation (6) to calculate G_{h_t}, the GPQ for $h(t, \alpha, \beta)$.
3. The (1−δ)×100% GCI for $h(t, \alpha, \beta)$ at t >0 is then constructed as:

$$\left[G_{h_t;\delta/2},\ G_{h_t;1-\delta/2}\right] \qquad (7)$$

where $G_{h_t;\delta}$ represents the (100×δ)th percentile of the GPQ values G_{h_t} obtained from the simulations.

IV. Simulation study

The two-parameter Weibull distribution is widely applied in areas such as manufacturing, healthcare, and technology, with its effectiveness demonstrated in studies by Lun and Lam (2000), Krishnamoorthy and Lin

(2010), and Jamdade and Jamdade (2012). This study carries out a simulation-based evaluation of the GCI proposed, focusing on both complete and type-II censored sample cases.

In this analysis, a significance level of 0.05 is set, generating 10,000 samples from a Weibull distribution with scale parameters α=0.5,1,2,…,10 and shape parameters β=0.3,0.5,1.5,2,3,5,7,9,10 considering values of t such that hazard function takes the values 10, 25, 50 and 100 across sample sizes n=5,15,25,50. The lower and upper bounds, Li and Ui (for i=1,2,…,10,000), for the two-sided CI are calculated based on Equation (7), and coverage probability is then determined, showing how often the true value of h(t,α,β) is included within each interval.

The study's goal is to evaluate the performance of the CI estimator over various sample sizes (n) and t values. Figure 1 presents boxplots of coverage probabilities across all combinations of n, α, β, and h(t,α,β). The results highlight that the GV method consistently produces coverage probabilities near the nominal level with minimal variation, confirming its strong performance. As noted by Roy and Bose (2009), the GV method achieves exact results, distinguishing this approach as particularly precise, and we recommend its use in applied settings.

In type-II censored samples, the proportion of censored observations, denoted by PC=P(X > X(r)), is tested at levels of 0.3, 0.5, and 0.7. Figures 2 ,3, 4 and 5 provide graphical representations of these cases for n=5, 15, 25 and 50 respectively. Visuals from Figures 1 to 5 demonstrate that the proposed method maintains coverage probabilities close to 0.95, even for small uncensored sample sizes like n=5, and continues to achieve accurate results for censored samples as long as the proportion of censored observations is up to 0.70.

V. Overall conclusion

This paper introduces a novel approach for constructing CI for the hazard function of a two-parameter Weibull distribution using the generalized variable technique. The method is designed to handle both complete and Type-II censored data and is easy to implement. Simulation results indicate that the proposed CI maintain coverage probabilities close to the intended values, even when sample sizes are small (as low as five observations) in uncensored cases, and for Type-II right-censored samples with up to 70% censoring. The practical utility of the method is demonstrated through the analysis of real-world datasets, illustrating its effectiveness in assessing health risks associated with environmental exposure to chemicals and microbes.

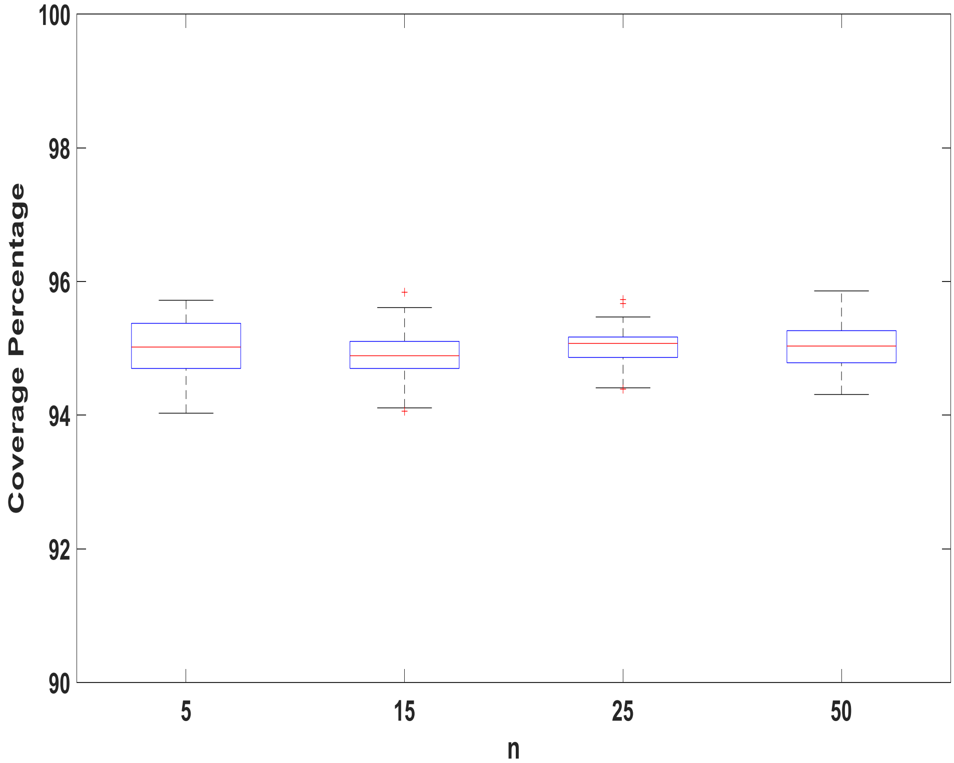

Fig.1 Box plots of simulated expected coverage probabilities (in percentage) for 95% CI based upon proposed GV method for sample sizes n=5,15,25 and 50 over the range of α = 0.5,1,2,…,10, β=0.3,0.5,1.5,2,3,5,7,9,10 and h(t) = 10, 25, 50 and 100.

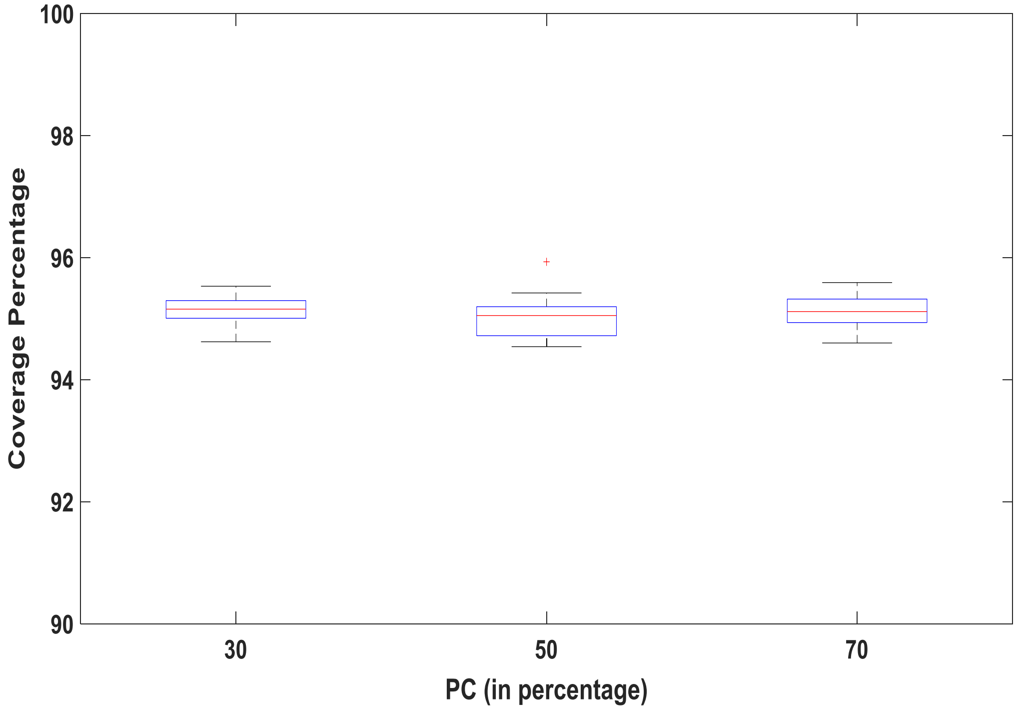

Fig.2 Box plots of simulated expected coverage probabilities (in percentage) for 95% CI based upon proposed GV method for sample size n=5 over the range of α = 0.5,1,2,…,10, β=0.3,0.5,1.5,2,3,5,7,9,10 and h(t) = 10, 25, 50 and 100 for Type-II censored samples with proportion of censoring (in percentage) PC= 30%, 50%, 70%.

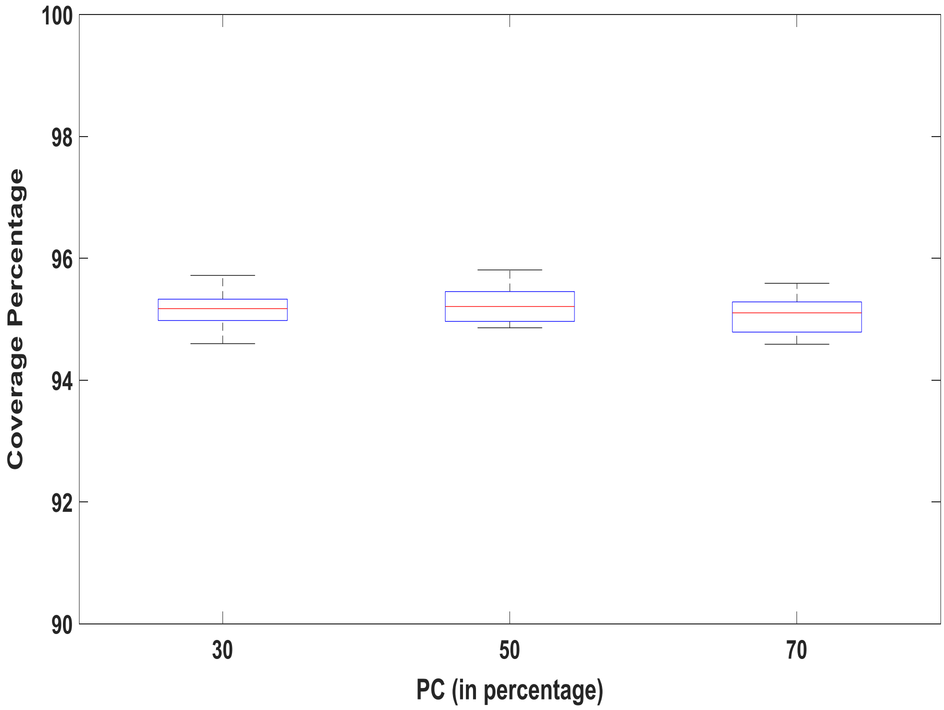

Fig.3 Box plots of simulated expected coverage probabilities (in percentage) for 95% CI based upon proposed GV method for sample size n=15 over the range of α = 0.5,1,2,…,10, β=0.3,0.5,1.5,2,3,5,7,9,10 and h(t) = 10, 25, 50 and 100 for Type-II censored samples with proportion of censoring (in percentage) PC= 30%, 50%, 70%.

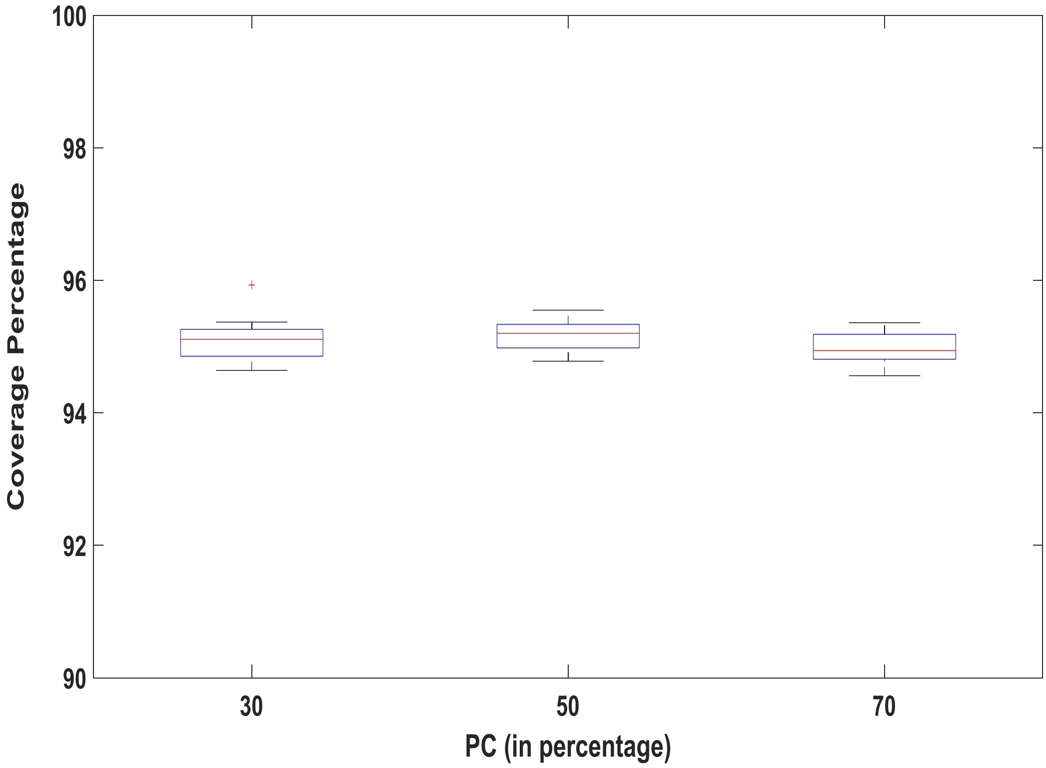

Fig.4 Box plots of simulated expected coverage probabilities (in percentage) for 95% CI based upon proposed GV method for sample size n=25 over the range of α = 0.5,1,2,…,10, β=0.3,0.5,1.5,2,3,5,7,9,10 and h(t) = 10, 25, 50 and 100 for Type-II censored samples with proportion of censoring (in percentage) PC= 30%, 50%, 70%.

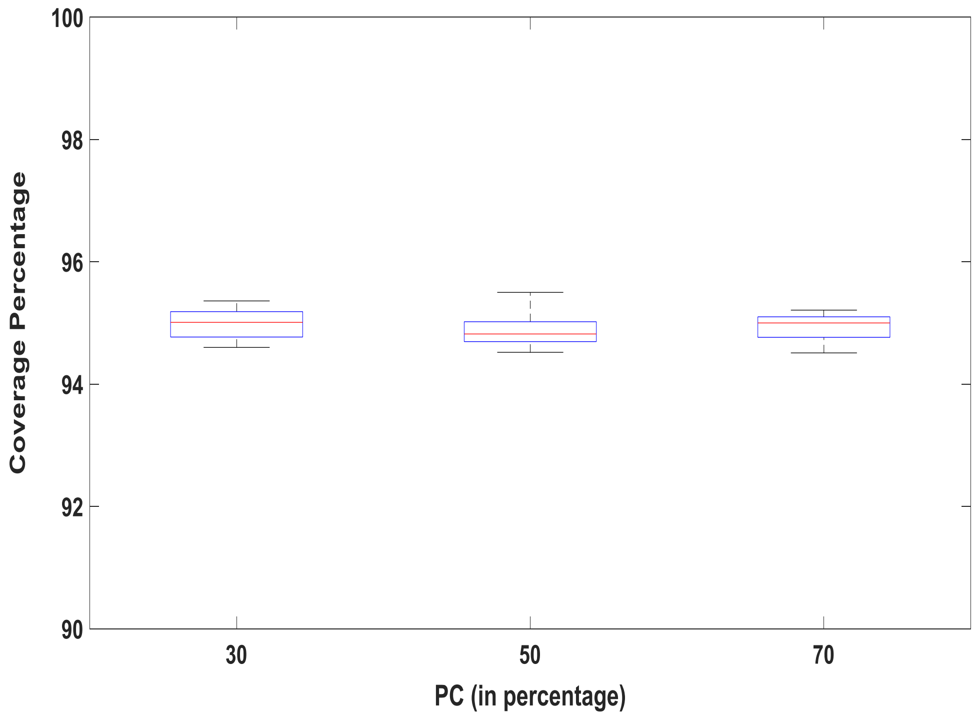

Fig.5 Box plots of simulated expected coverage probabilities (in percentage) for 95% CI based upon proposed GV method for sample size n=50 over the range of α = 0.5,1,2,…,10, β=0.3,0.5,1.5,2,3,5,7,9,10 and h(t) = 10, 25, 50 and 100 for Type-II censored samples with proportion of censoring (in percentage) PC= 30%, 50%, 70%.

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