Smart Grid Load Forecasting

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Abstract

   Electrical load modeling and forecasting are critically important in the electrical network and smart grid. The sparse Bayesian Learning (SBL) algorithm can be utilized to model and forecast the electrical load behavior. The SBL algorithm can solve a sparse weight vector with respect to a kernel matrix for modeling electricity consumption. However, traditional SBL can only handle an electricity consumption record of one user at a time period. In this paper, we propose a joint SBL algorithm to model and forecast multi-users' electricity consumption at multiple time periods. The spatial and historical similarity in multi-users' electricity consumption records are exploited and integrated in the joint SBL algorithm for accurate prediction and good modeling. Experimental results based on real data show that the proposed joint SBL algorithm can produce much better prediction accuracy than the traditional SBL algorithm.

Keywords- Advanced metering infrastructure (AMI), communication technologies, quality-of-service (QoS), smart grid, standards

I. INTRODUCTION

   DURING the past several decades, the US electric utility industry has been trying to model the demand for various reasons. One of the primary reasons is load forecasting, of which the results can be further used in T&D planning, operations, demand side management (DSM), and energy purchase in the competitive energy markets. According to different forecast horizons and resolutions, load forecasts can be roughly grouped into long-term, mid-term, and short-term ones [1]. In most long-term load forecasts with T&D planning purposes, the questions of interest are *when*, *where*, and *how much* the load will be [2, 3]. In most short-term and mid-term load forecasts with energy purchase and DSM purposes, the core questions to be answered for these purposes are *when* and *how much* the demand will be. Nowadays, with the development of smart grid technologies, DSM becomes an emerging emphasis in the electric utility industry. Effective DSM requires utilities to understand the patterns of the consumption of electricity, and to perform peak load reduction accordingly. This emerging need leads to another question beyond *when* and *how much*: *why* is the electricity being consumed? A proper answer would not only be beneficial to the traditional forecasting

process, but also help the utilities decide whose load should be reduced or cut during the peak load hours, e.g., residential customers, office buildings, or industrial customers, etc. A large variety of pioneer research and practices have been devoted to modeling and forecasting the short-term electric load in *when* and *how much* level. Lots of these methods in the open literature can be roughly classified into two broad categories: artificial intelligence (AI) and statistical approaches. Among all the AI-based approaches, artificial neural networks (ANN) may have received the most attention by both academia and industry [4, 5]. Not only in short term load forecasting, ANN has also been tried in several other utility applications, such as distribution system failure forecast [6], with varying degrees of success. As a data-driven approach, ANN has been favored by several major vendors in the power industry due to its flexibility and the advantage of handling the nonlinearity of the data. Although empirically the ANN models can produce fairly acceptable forecast, as a black-box approach, ANN, together with most other AI-based approaches, such as fuzzy logic [7, 8], have not been completely convincing. The lack of interpretability and overfitting issue are still challenging problems in the research field of applying ANN to electric load modeling and forecasting. On the other hand, statistical approaches, such as similar day, time series, and regression,, are favored by the practitioners due to both accuracy and interpretability. Similar day approach, as one of the earliest methods for short term load forecasting, is still being used by many utilities due to the simplicity of implementation as well as acceptable results. The basic idea of this approach is to pick up the historical days with similar characteristics to the forecasted day as the major reference when performing the short term load forecasting [9]. Time series approaches have been extensively used by utilities in short term forecast [10]. Due to the involvement of autoregressive and/or moving average terms, time series models have the advantage on the short term accuracy, while their interpretability is not as straightforward as that of the regression models.

A. Hybrid intelligent method

   This paper proposes a hybrid intelligent method of the Regression Tree (RT) and Gaussian Processes (GP) for maximum daily temperature

forecasting. The prediction of maximum daily temperature forecasting plays a key role to handle DR [1] and PV [2] in smart grids as well as short-term load forecasting [3-6] and electricity price forecasting [7,8] in the conventional power systems. The prediction methods of maximum daily temperature forecasting may be classified as follows:

-Artificial Neural Networks (ANNs) [9]
-Kernel Machines [10]

As the conventional intelligent methods, ANNs and Kernel Machines are attractive due to the good model accuracy in comparison with other methods. Mori and Kanaoka clarified that RBFN (Radial Basis Function Network) of ANN gave better results than Multi-Layer Perceptron (MLP) [9]. They also showed that Support Vector Regression (SVR) outperformed RBFN in terms of the model accuracy [10]. Although the above two methods evaluate the predicted value in a point, it is required that the prediction method should deal with the uncertainties in a sense that the lower and upper bounds of the predicted value are estimated. In this paper, a new method is proposed to predict onestep ahead daily maximum load precisely. The proposed method consists of the regression tree (RT) [11] and Gaussian Process (GP) [12-13]. To improve the prediction accuracy, this paper uses the RT as the preconditioned technique. It is often used as a tool of data mining to extract *if-then* rules and to clarify the relationship between input and output variables easily [13]. Furthermore, RT has a function that explains the importance of input variables. Also, this paper makes use of GP of the kernel machine that evaluates the upper and lower limits of the predicted temperature as well as the average. It is based on the Bayesian approach that assumes the prior distribution and updates the posterior distribution by the Bayes' rule [12-13]. In other words, it evaluates the predicted value in distribution. The advantage is to evaluate the mean and the error bar that corresponds to the upper and the lower bounds of the predicted value at the same time. The preconditioned technique for the predictor allows improving the model accuracy by constructing GP at the terminal nodes of RT. The proposed method is successfully applied to daily maximum temperature of real data. The proposed method is compared with the conventional intelligent models.

B. Current trends

Nowadays the electricity market is very competitive. Electricity can be bought and sold at a flexible market price like any other commodity [19]. In consequence, the electrical price, reflecting the relationship of supply and demand, can fluctuate with the electric utilities and customers. The electrical network and smart grid is proposed to monitor the electricity consumption. Fig. I shows a typical smart grid, where electricity consumption of multi-users can be monitored in real time in an electrical network. Electrical load forecasting is important for both suppliers and users. On one hand, as the supplier, power producers have to utilize power load forecasts to plan and schedule generator maintenance, to make long-term and short-term investments in generation, and to plan the most cost-effective power distribution. Without a precise electrical load forecast, the amount of risk will increase substantially and the power company may lose profit. As a power user, on the other hand, it is necessary to understand the power load and price to schedule power usage or adopt a power storage system (such as batteries) to lower down the cost. To meet the demand of modeling the electrical load for economic generation of power, it requires a fair amount of statistical analysis for the electrical load. From the perspective of the power system, when supply and demand are fluctuating and electrical prices are spiking by a factor of 10 or more in a matter of hours, load forecasting is vitally important for all market participants and the power system. As shown in Fig. 1, given the electricity consumption of different users, we need to model and forecast electricity usage for all active market players. There are many approaches to model and forecast electrical load at a time period. A widely used method is the similar-day approach [1], which is based on searching Fig. 1: A typical smart grid network [1] historical data with similar characteristics for forecasting. It assumes that the history can be precisely repeated again based on some characteristics such as the day of the week, weather, and season. Other modeling and forecasting methods include neural networks, expert systems, time serials, and fuzzy logic [1]. Support Vector Machine (SVM) is a typical regression method for power usage prediction [14]. However, SVM makes unnecessarily liberal use of basis functions since the number of support vectors required typically grows linearly with the size of the training set [14]. The Sparse Bayesian Learning (SBL) [13] [15], or named as the relevance vector machine, incorporates Gaussian process, Bayes theorem and automatic relevance determination prior, which can achieve sparsity in a probabilistic Bayesian learning framework. The SBL algorithm can approach an accurate prediction which typically utilizes dramatically fewer basis functions than the comparable SVM algorithm while offering a number of additional advantages. Therefore, the SBL algorithm has been applied to many applications in [2] [3] [4] [5] [6] [7] [9] [10] [12]. All the above regression methods can model the electricity consumption for a single user. With the development of smart meters and communication technologies, we can easily monitor multi-uses' electricity consumption for a short term (e.g., in one day). In such a smart grid, it is very necessary to model and forecast electrical load for multi-users to improve

power efficiency. In a smart grid, we can monitor and collect multi-users' electricity consumption data at multiple time periods. In such an electrical network, the electricity consumption data of those users are very similar due to some common factors (such as weather, season, and holiday). We have two types of similarity.

C. Short-Term Load and Price Forecasting

Short-term load forecasting and short-term price forecasting have vital values for the operation of electricity markets and their participants [20]. In the traditional electricity markets these two operations are usually performed separately, because as discussed before in the present structure of electricity grids, the variation of electricity price has a negligible effect on the electricity demand. In a conventional load forecasting system, only the historical load and climatic data are employed as the inputs to forecast the short-term future demand of the power system. As mentioned before, by emerging the smart grids the response of consumers to the price signals of the electricity market will not be weak and the spot electricity price and demand will mutually affect each other. Therefore in this new electricity grid, load forecasting and price forecasting cannot be implemented independently. On the other hand, the penetration level of smart grid technologies like AMI systems, smart monitors and smart controllers and also the arrangement of different kinds of consumers (i.e. the ratio of industrial, commercial, and residential consumers) will affect the overall response of the demand-side to the price signals. Although, in long-term these factors will become fixed and the overall response of the demand-side will be stable, but they have great effect on DR in the transition period. The structure of the forecaster is also likely to be changed due to the dynamic relation between load and price. Fig 7, schematically shows possible inputs and outputs of the future load and price forecasting systems. It should be mentioned that the height of peaks and depth of valleys in the load profiles will be reduced, because the load management due to the smart grid technologies will flatten load profiles. Also small variations of demand will likely to be reduced, i.e. the load profiles may become smoother than today's load profiles. All of these possibilities can result in more precise forecasts of the future load and price forecasting system.

D. ERDF load measurements

For the analysis sake, ERDF provided seven real load measurements collected every 30 minutes during 10 months from 9th September 2009 to 30th June 2010, each one of which represents a specific MV/LV substation. Mainly, these substations are composed of domestic clients, service sector clients as well as industrial clients in different proportions. However, since these

substations are in the same region, they shared the same weather conditions during the measurements. As we can notice in Figure 1 and Figure 2, the energy consumption temporal pattern depends on the nature of the clients that the substation is dealing with. As a matter of fact, the substation denoted by Substation 3 only serves an industrial client. As a result, the power consumption level is cyclic and only depends on the time of the day as well as the day of the week. Temperatures showed in Figure 3 do not seem to influence its energy needs. Consequently, independant to the temperatures, Substation 3 provided stable electrical power all year long. While the substation denoted by Substation 5, in Figure 1, is composed of a mixture of 61% domestic clients, 23% service sector clients and 16% industrial clients, where the percentage refers to the total power supply provided for the clients in each category. In this case, we notice a significant increase of the clients' consumption when temperature drops, as shown in Figure 3. Thus, the substation's load level is sensitive to the temperature. Figure 1 and Figure 3 suggest a high correlation between the energy level of the substation and the temperature. The computed correlation coefficient between them gives a value of -0.831, where the minus sign indicates that the energy consumption evolves in an opposite way to the temperature. Finally, this preliminary analysis indicates that the designed forecasting model should take into account the diversity of the clients as well as the temperature and the time horizon. These observations to the data suggest the use of the time series model. In the next sequel, we remind the main time series related characteristics and define our considered time series based model. In this paper, only the weather sensitive substation is concerned. The industrial substations on the other hand, which are not influenced by the weather condition but rather the day type, should be treated differently. In the rest of the paper, we illustrate the results of the designed forecasting method on the Substation 5.

E. Time series based forecaster

Time series represents the evolution of a set of observations sampled at regular intervals along time. The specificity of time series models, compared to others statistic methods, is that it introduces "time" as one of its explicative variables. The considered time series model contains three components: the trend component, the cyclic component and the

II. ELECTRICAL LOAD MODELING AND FORECASTING BY MULTIPLE LINEAR REGRESSION

MLR is used to analyze the relationship between several independent variables and a

dependent variable (electric load) as a linear function. A general linear regression model with normal error terms can be defined as:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_{p-1} X_{i,p-1} + e_i, \quad (1)$$

where Y is the response variable, X_1 to X_{p-1} are predictor variables, Y_i and X_{i1} to $X_{i,p-1}$ are known constants, β's are the parameters to estimate, and e_i are independent $N(0,\sigma^2)$. A detailed discussion of multiple regression analysis, including general linear regression models, polynomial regression models and interaction regression models, can be found in the Part II of a classic statistic textbook [8]. 4 years (2005 – 2008) data of load (Fig. 2) and temperature history are used in this study. Cross validation is conducted by taking any three years of data in the modeling process and the other one year of data as the holdout sample to test the model. Since the results are not significant enough to affect the decision of selecting the winning model, for the simplicity of presentation, in the rest of this paper, 2005 – 2007 data are used for parameter estimation and 2008 data are used as the holdout sample. Adjusted R-square and Mean Absolute Percentage Error (MAPE) are used as the goodness-of-fit statistics, while MAPE, Mean Absolute Error (MAE), standard deviation of absolute percentage error (STDAPE), and standard deviation of absolute error (STDAE), are used as the accuracy statistics. In this section, we briefly introduce the development of the regression model to be used as the base model throughout the rest of this paper. This regression model has been deployed in the utility since 2009 as the benchmarking model. We first select the variables to be used in the model. We then build three candidate models according to different treatments to the temperature variables, and select the winning one based on the accuracy statistics. As shown in Fig. 2, an increasing trend and a seasonal pattern can be observed. Therefore, a linear trend to represent the progress of the system development and a class variable (Month) with 12 levels to represent the months of the year can be considered in the model. It is well known that the load curve has two other seasonal patterns: one is the 24 hours block, and the other one is the 7 days block. In this paper, we use a class variable (Day) with 4 levels to represent the days of a week: Sunday, Monday, Tuesday to Friday, and Saturday [9]. We then create the interaction effect between the hour of the day (Hour) and Day, which is equivalent to a class variable (Day×Hour) with 96 levels, to model the different load consumptions in different hours of different days of a week.

A. Selection of candidates for dispatchable demand response

The selection of candidates for dispatchable demand response is challenging in view of the wide possible realizations of uncertain events, in particular, the intermittent renewable generation such as wind and solar. To optimize the candidates for dispatchable demand response, our idea is to use online stochastic optimization in conjunction with the forecast of stochastic variables, including load, operating reserve, wind power production, solar power production, and electricity price, together with their confidence intervals. These forecasts, combined with unit commitment and economic dispatch decisions, will provide a cone of generation uncertainties, looking out one or two hours into the future with a time step of five minute. Similarly, a cone of demand uncertainties can be created, enlarged by the operating reserve requirements. Given these two cones, the situation when dispatchable demand response is needed (when the two cones intersect) can be easily identified and continually monitored. Online stochastic optimization [7] can then be performed in a moving window manner every five minutes. The goal is to find effective and robust solutions to manage dispatchable load The above, in combination with security-constrained unit commitment and economic dispatch, will shift demand response from a tool used during emergencies only to a tool used pro-actively to plan and dispatch the dispatchable load.

B. Approach for monitoring and forecasting of the expected operating conditions

The paper presents original intelligent methods and approaches to solve the problems of monitoring and forecasting of the expected EPS operating conditions [13], namely:

- . intelligent hybrid approach to short-term forecasting of the expected state variables (power flows, voltage levels, etc.) on the basis of combined application of artificial neural networks (ANN) and Hilbert-Huang transform (HHT) [14];

- intelligent approach to monitoring and forecasting of heavy load and/or emergency conditions on the basis of modern methods of adaptive clustering and factor analysis [15].

Thus, the prototype of a system of intelligent monitoring and forecasting of the expected state variables can be presented as follows (fig. 1). Central station, i.e. Wide Area Monitoring System (WAMS) collects data from PMUs allocated across the entire power system. The data include information on voltage, current, active power, frequency and phase angle. The information received is analyzed and processed in two system modules. In the module "Monitoring and forecasting of emergency situations" the algorithms of adaptive clustering (Kohonen maps, K-means method) and factor analysis make it possible to effectively follow changes in the current and expected EPS conditions

and forecast heavy load conditions and/or development of emergency situations in the modes close to the real-time mode. The intelligent approach to monitor expected EPS operating conditions is suggested as a modification of the current system for identification of emergency situations.

The hybrid "HHT-ANN" model is applied in the module "Short-term forecasting of expected state variables" to improve the forecast accuracy for different periods of time ahead. The accuracy is increased here by separation from the initial non-stationary time series of its particular components – empiric modes, instantaneous amplitudes and frequencies, which are input values for the specially chosen ANN model. It should be noted that the intelligent approach to monitoring of the expected conditions is suggested as a modification of the modern system for detection of emergency situations (Fig.1). The proposed modification supplements the procedure of monitoring of the current power system state with the effective tool of visualization and prediction of the expected power system operating conditions.

C. Module "Monitoring and elimination of emergency conditions"

The paper suggests an approach to monitoring and forecasting of heavy load and/or emergency conditions on the basis of modern clustering methods, (self-organizing Kohonen maps (SOM) and k-means method) [6], [7] and factor analysis [8]. At the first stage, the Kohonen maps are used to group databases into clusters in terms of different state variables of the considered electric network [15]. The obtained clusters correspond to certain operating conditions of the network. They are arranged in the form of a topological map which allows visualization of change in the network operating conditions. At parallel with this, the k-means method makes it possible to show a set of various components of operating conditions3. Thus, we obtain clusters that contain different sets of state variables, with different sensitivity to emergence of transients in electric network and as a result to development of heavy load or emergency conditions. The most sensitive clusters are identified and then are used to follow the extent to which various transients affect operation of power system. In the second stage of the application of factor analysis can be solved two important problems:

- identification the cluster-conditions obtained by the SOM and k-means method;
- from an array of condition information distinguish the main factors exerting the greatest influence on the development of heavy load and emergency conditions.

B. Module "Short-term forecasting of expected state variables"

In order to increase the accuracy of the operation conditions forecasting the "intelligent" neural approach has been developed in the Energy Systems Institute SB RAS. This approach is based on both neural network technologies and the Hilbert-Huang Transform (HHT) [5]. The Hilbert-Huang transform consists of two parts:

- Empirical mode decomposition (EMD)
- Hilbert transform (HT)

D. Empirical mode decomposition

According to EMD, the signal $x(t)$ is supposed to be decomposed into basis of special functions, called intrinsic mode functions (IMF) by special empirical algorithm. The data flow diagram is presented in Fig. 2. Thus, at the end of decomposition process, the original signal can be presented as follows:

E. Load forecasting

The system load profile is the result of a dynamic process composed of many individual components. The load profile is influenced by a number of factors, such as devices' operational characteristics, users' behaviors, economic factors, time of the day, day of the week, seasonal factors (i.e. weather), geographic patterns (influenced by weather but also external factors) and random effects. In the past, straight line extrapolations of historical load data served well the load forecasting purpose. However, with the appearance of novel technologies, Demand Side Management (DSM) options, changes in the lifestyle and energy consumption pattern etc., it becomes necessary to use alternative modeling techniques, to capture the effect of factors such as industry developments, environmental concerns, energy regulation, energy prices, per capita income, population segmentation and other variables. Different methods have been developed for forecasting the demand in the last decades. In [5] the authors classify load forecasting in terms of the planning horizon duration; up to one day for short-term load forecasting (STLF), one day to one year for medium-term load forecasting (MTLF), and one to ten years for long-term load forecasting (LTLF). For the purpose of this work, we extend the long term definition to include also the time period of ten years or more. This is well aligned with the notion of power system planning, since many assets of the grid (generators, power lines etc.) have a lifetime range that exceeds the ten year period. Most load forecasting techniques (i.e. Multiple Regression, Exponential Smoothing, Iterative Reweighted Least-squares, Adaptive Load Forecasting, Stochastic Time Series, Fuzzy Logic, Neural Networks), make use of historical and timeseries data in order to identify and correlate

patterns of load and temperature [4]. The implication of utilizing historical load data, within the smart grid concept, is that when referring to certain equipment, measured historical load data are most of the times not available, either because these devices have not yet applied in large scale or are still in the R&D phase. Thus, there is high degree of uncertainty about how these devices will be utilized (and to what extent) in the future. In [4], it is reported that forecasting methods such as Adaptive Load Forecasting, Stochastic Time Series, and Fuzzy Logic, perform better than classical deterministic models due to their ability to incorporate the intrinsic uncertainty of a process. Still, these models cannot cope with the high uncertainty involved in the future features of the power system, where radical innovations are expected to be integrated within existing infrastructures. Researchers have recognized the fact that, by utilizing time-series methods alone, it is not always possible to predict unique patterns of energy demand in fast developing areas [4]. In [13], the authors argue that unlike STLF, LTLF is mainly affected by economical factors rather than weather predictions. Still, only economical factors are not adequate to represent all the aspects that shape the load profile. Other identified inefficiencies with currently employed forecasting methods are that they do not discriminate between the specificity of individual customers (financial status, motives and needs). In [14], the authors incorporate in their models variables such as population and per capita gross domestic product, in order to exhibit the relation between population, economic development and demand for power. Still, these average indexes can roughly be considered representative of the diversity among the society. Other techniques utilize a basic classification between the customers served, such as residential, office, commercial or industrial. Still, this type of classification can hardly illustrate the large diversity of customers served in each of these load classes [15]. Forecasting load demand is a complex process that combines art with engineering; apart from scientific and technical knowledge it requires acquiring an insight into the way individuals express their needs, which consequently shape their demand for energy. Even though there is a wide range of tools for performing load research and demand side management, a key aspect in this process is the knowledge about the electricity consumer needs, and an understanding of the way individuals use electricity.

F. Generation forecasting

A main advantage of integrating decentralized generators in the current system is the potential energy loss reduction in the electricity network by reducing transportation losses [16]. In the future, this approach might also contribute to significantly less need for transmitting power over long distances. However, the incorporation of DGs, especially in the form ofintermittent renewable energy technologies (i.e. wind-turbines and photovoltaic modules) complicates the operation and planning of power systems. A supportive tool in this process is the employment of generation forecasting techniques. The accurate forecasting of power derived from renewable energy sources is essential for the power system's operation. Renewable energy generation forecasting is mainly a function of time of the day, season of the year, spatial characteristics and local weather. Compared to energy derived from the sun, wind is considered as one of the most difficult meteorological phenomena to forecast [13]. In addition, wind characteristics in off-shore or open space locations are significantly different than those within urban environments, where turbulence is a dominant effect affecting the output of small scale wind turbines. In [13], a comparison of various wind forecasting approaches is included.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$
\begin{cases}
(\dfrac{d}{dt}+l_i)V_i(t,r) = \displaystyle\sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\
\qquad\qquad\qquad +I_i^{ext}(r,t), \qquad t\ge 0, 1\le i \le p, \\
V_i(t,r)=\phi_i(t,r) \qquad\qquad t\in[-T,0]
\end{cases}
\tag{1}
$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d. The vector r and $\bar{r}$ represent points in Ω. The function $S: R \to (0,1)$ is the normalized sigmoid function:

$$
S(z)=\frac{1}{1+e^{-z}}
\tag{2}
$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i=1,...,p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i=1,...,p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p\times p$ matrix of functions $J=\{J_{ij}\}_{i,j=1,...,p}$ represents the

connectivity between populations i and j, see below. The p real values $h_i, i = 1, ..., p$, determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i = 1, ..., p$, determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i = 1, ..., p$, determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S : R^p \to R^p$, defined by $S(x) = [S(\sigma_1(x_1 - h_1)), ..., S(\sigma_p - h_p))]$, and the diagonal $p \times p$ matrix $L_0 = diag(l_1, ..., l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt} + l_i)$ is replaced by $(\frac{d}{dt} + l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt} + l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r, \bar{r})$ whose element $\tau_{ij}(r, \bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\overline{\Omega}^2, R_+^{p \times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r, \bar{r}) \neq \tau_{ij}(\bar{r}, r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T, 0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,\bar{r} \in \Omega \times \overline{\Omega})} \tau_{i,j}(r, \bar{r}) \qquad (3)$$

Hence we choose $T = \tau_m$

G. *Mathematical Framework*

A convenient functional setting for the non-delayed packet field equations is to use the space $F = L^2(\Omega, R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V, U \rangle_F = \sum_{i=1}^{p} \int_{\Omega} V_i(r) U_i(r) dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m, 0], F)$ with $\|\phi\| = \sup_{t \in [-\tau_m, 0]} \|\phi(t)\|_F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t + \theta), \theta \in [-\tau_m, 0]$, we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0 V(t) + L_1 S(V_t) + I^{ext}(t), \\ V_0 = \phi \in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1 : C \to F, \\ \phi \to \int_{\Omega} J(., \bar{r}) \phi(\bar{r}, -\tau(., \bar{r})) d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \leq \|J\|_{L^2(\Omega^2, R^{p \times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $\quad J \in L^2(\Omega^2, R^{p \times p})$,

2. $\quad$ The external current $I^{ext} \in C^0(R, F)$,

3. $\quad \tau \in C^0(\overline{\Omega}^2, R_+^{p \times p}), \sup_{\overline{\Omega}^2} \tau \leq \tau_m$.

Then for any $\phi \in C$, there exists a unique solution $V \in C^1([0, \infty), F) \cap C^0([-\tau_m, \infty, F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

H. **Boundedness of Solutions**

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t \in R^+} \|I^{ext}(t)\|_F < \infty$.

Proof :Let us defined $f : R \times C \to R^+$ as

$$f(t, V_t) \overset{def}{=} \langle -L_0 V_t(0) + L_1 S(V_t) + I^{ext}(t), V(t) \rangle_F = \frac{1}{2} \frac{d\|V\|_F^2}{dt}$$

We note $l = \min_{i=1,...p} l_i$

$$f(t, V_t) \leq -l\|V(t)\|_F^2 + (\sqrt{p|\Omega|}\|J\|_F + I)\|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F \geq 2\frac{\sqrt{p|\Omega|}.\|J\|_F + I}{l} \overset{def}{=} R, f(t,V_t) \leq -\frac{lR^2}{2} \overset{def}{=} -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \geq 0s$ and that $f < 0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C < R$ and set $T = \sup\{t \mid \forall s \in [0,t], V(s) \in \overline{B_R}\}$. Suppose that $T \in R$, then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have

$$\frac{d}{dt}\|V\|_F^2 \mid_{t=T} = f(T,V_T) \leq -\delta < 0$$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and small enough, $V(T+\varepsilon) \in \overline{B_R}$ which contradicts the definition of T. Thus $T \notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0) \in \partial B_R$ implies that $\forall t > 0, V(t) \in B_R$. Finally we consider the case $V(0) \in C\overline{B_R}$. Suppose that $\forall t > 0, V(t) \notin \overline{B_R}$, then

$$\forall t > 0, \frac{d}{dt}\|V\|_F^2 \leq -2\delta,$$ thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T > 0 \mid V(T) \in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on X. for $E\varepsilon M$, define

$$\phi(E) = \int_E s \, d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X (s+t)d\mu = \int_X s \, d\mu + \int_X t \, d\mu \qquad (2)$$

Proof : If s and if $E_1, E_2, \ldots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r)$$
$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \phi(E_r)$$

Also, $\varphi(\phi) = 0,$ so that φ is not identically ∞.

Next, let s be as before, let $\beta_1, \ldots, \beta_m$ be the distinct values of t,and let $B_j = \{x : t(x) = \beta_j\}$ If $E_{ij} = A_i \cap B_j,$ the

$$\int_{E_{ij}} (s+t)d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and $\int_{E_{ij}} s \, d\mu + \int_{E_{ij}} t \, d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \leq i \leq n, 1 \leq j \leq m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon > 0$, then there exists a polynomial P such that $|f(z) = P(z)| < \varepsilon$ for all $z\varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f\varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2) - f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2 - z_1| \leq \delta$. Since f is uniformly continous, we have $\lim_{\delta \to 0} \omega(\delta) = 0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10,000 \, \omega(\delta) \quad (z\varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi\varepsilon C_c^{'}(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \le \omega(\delta), \qquad (3)$$

$$|(\partial\Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z) = -\frac{1}{\pi}\iint_X \frac{(\partial\Phi)(\zeta)}{\zeta - z}d\zeta d\eta \qquad (\zeta = \xi + i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi\delta^2}(1 - \frac{r^2}{\delta^2})^2 \qquad (0 \le r \le \delta), \qquad (6)$$

And define

$$A(z) = a(|z|) \qquad (7)$$

For all complex z. It is clear that $A\varepsilon C_c'(R^2)$. We claim that

$$\iint_{R^s} A = 1, \qquad (8)$$

$$\iint_{R^2} \partial A = 0, \qquad (9)$$

$$\iint_{R^3} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A / \partial\theta = 0$,

$$\partial A / \partial r = -a',$$

Now define

$$\Phi(z) = \iint_{R^2} f(z-\zeta)A d\xi d\eta = \iint_{R^2} A(z-\zeta)f(\zeta)d\xi d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z)$$
$$= \iint_{R^2} [f(z-\zeta) - f(z)]A(\xi)d\xi d\eta \qquad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since

$A\varepsilon C_c'(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial\Phi)(z) = \iint_{R^2} (\overline{\partial A})(z-\zeta)f(\zeta)d\xi d\eta$$
$$= \iint_{R^2} f(z-\zeta)(\partial A)(\zeta)d\xi d\eta$$
$$= \iint_{R^2} [f(z-\zeta) - f(z)](\partial A)(\zeta)d\xi d\eta \qquad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi = 0$ in G, where G is the set of all $z\varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z\varepsilon G); \qquad (14)$$

Note that $\partial f = 0$ in G, since f is holomorphic there. Now if $z\varepsilon G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z) = \int_0^\delta a(r)rdr\int_0^{2\pi} f(z - re^{i\theta})d\theta$$
$$= 2\pi f(z)\int_0^\delta a(r)rdr = f(z)\iint_{R^2} A = f(z) \qquad (15)$$

For all $z \varepsilon G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1, ..., D_n$, of radius 2δ, whose centers are not in K. Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that $S^2 - E_j$ is connected and so that $K \cap E_j = \phi$. with $r = 2\delta$. There are functions $g_j\varepsilon H(S^2 - E_j)$ and constants b_j so that the inequalities.

$$\left| Q_j(\zeta, z) \right| < \frac{50}{\delta}, \qquad (16)$$

$$\left| Q_j(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{4,000\delta^2}{|z - \zeta|^2} \qquad (17)$$

Hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j)g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1 \cup ... \cup E_n$. Then Ω is an open set which contains K. Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup ... \cup X_{j-1})$, for $2 \le j \le n$,

Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta \varepsilon X_j, z \varepsilon \Omega) \qquad (19)$$

And

$$F(z) = \frac{1}{\pi} \iint_X (\partial \Phi)(\zeta) R(\zeta, z) d\zeta d\eta \qquad (20)$$
$$(z \ \varepsilon \ \Omega)$$

Since,

$$F(z) = \sum_{j=1}^n \frac{1}{\pi} \iint_{X_i} (\partial \Phi)(\zeta) Q_j(\zeta, z) d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F \varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$\left| F(z) - \Phi(z) \right| < \frac{2\omega(\delta)}{\pi \delta} \iint_X | R(\zeta, z)$$

$$- \frac{1}{z - \zeta} | d\xi d\eta \quad (z \ \varepsilon \ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta \ \varepsilon \ X$ and $z \ \varepsilon \ \Omega$.

Now fix $z \ \varepsilon \ \Omega.$, put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \le \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi \int_0^{4\delta} \left(\frac{50}{\delta} + \frac{1}{\rho} \right) \rho d\rho = 808\pi\delta \qquad (23)$$

And

$$2\pi \int_{4\delta}^\infty \frac{4,000\delta^2}{\rho^2} \rho d\rho = 2,000\pi\delta. \qquad (24)$$

Hence (22) yields

$$\left| F(z) - \Phi(z) \right| < 6,000\omega(\delta) \qquad (z \ \varepsilon \ \Omega) \qquad (25)$$

Since $F \ \varepsilon \ H(\Omega), K \subset \Omega,$ and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f \varepsilon C_c'(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi} \iint_{R^2} \frac{(\partial f)(\zeta)}{\zeta - z} d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r, \theta) = f(z + re^{i\theta}), r > 0, \theta$ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2} e^{i\theta} \left[\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right] \varphi(r, \theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0$, of

$$-\frac{1}{2} \int_\varepsilon^\infty \int_0^{2\pi} \left(\frac{\partial \varphi}{\partial r} + \frac{i}{r} \frac{\partial \varphi}{\partial \theta} \right) d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial \varphi / \partial \theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\varepsilon^\infty \frac{\partial \varphi}{\partial r} dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\varepsilon, \theta) d\theta \qquad (5)$$

As $\varepsilon \to 0, \varphi(\varepsilon, \theta) \to f(z)$ uniformly. This gives (2)

If $X^\alpha \in a$ and $X^\beta \in k[X_1, ... X_n]$, then $X^\alpha X^\beta = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$\left(\sum_{\alpha \in A} c_\alpha X^\alpha\right)\left(\sum_{\beta \in \square^n} d_\beta X^\beta\right) = \sum_{\alpha, \beta} c_\alpha d_\beta X^{\alpha+\beta} \qquad (finite\ sums),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^\alpha, \alpha \in a$, is an

ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals (X^n), $n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^\alpha \mid \alpha \in A \rangle$ for the ideal corresponding to A (subspace generated by the $X^\alpha, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^\alpha, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{ \beta \in \square^n \mid \beta - \alpha \in \square^n, \quad some \ \alpha \in S \}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^\alpha, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^\beta \mid \beta \in A \rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^\beta = X^\alpha X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^\alpha \mid X^\beta \Leftrightarrow \beta - \alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that $A = \{\beta \in \square^n \mid \beta - \alpha_i \in \square^2, \ some \ \alpha_i \in S\}$ (The α_i's are the corners of A) Moreover, $a \overset{df}{=} \langle X^\alpha \mid \alpha \in A \rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by
$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find
$$f = a_1 g_1 + ... + a_s g_s + r, \qquad a_i, r \in k[X_1,...,X_n]$$
, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1),...,LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1,...,g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1,|...,g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1),...,LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1,...,X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1,...X_{n-1}][X_n] \to k[X_1,...X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1,...X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,...,X_n]$:

$$f(X_1,...X_n) = a_0(X_1,...X_{n-1})X_n^r + ... + a_r(X_1,...X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0 X^r + a_1 X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0.   Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1, ..., g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write

$$a = \sum b_i a_i, \qquad b_i \in A,$$

$a_i =$ *leading coefficient of* g_i

Now

$f - \sum b_i g_i X^{s-r_i}, \quad r_i = \deg(g_i),$ has degree $< \deg(f)$. By continuing in this way, we find that

$f \equiv f_t \qquad \mod(g_1, ... g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1}, ..., g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \qquad \mod(g_{d,1}, ... g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that

$f_t \in (g_{r-1,1}, ... g_{r-1,m_{r-1}}, ... g_{0,1}, ... g_{0,m_0})$ Hence

$$f_t \in (g_1, ... g_m g_{r-1,1}, ... g_{r-1,m_{r-1}}, ..., g_{0,1}, ..., g_{0,m_0})$$

and so the polynomials $g_1, ..., g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x. x$ is most conveniently defined

as a function from $D \, to \, D$, which must then be regarded as an element of D. Let $\psi : [D \rightarrow D] \rightarrow D$ be the function that picks out elements of D to represent elements of $[D \rightarrow D]$ and $\phi : D \rightarrow [D \rightarrow D]$ be the function that maps elements of D to functions of D. Since $\psi(f)$ is intended to represent the function f as an element of D, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D \rightarrow D]}$ Furthermore, we often want to view every element of D as representing some function from D to D and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that ϕ and ψ are inverses--- that is, D is isomorphic to the space of functions from D to D that can be the interpretations of functional abstractions: $D \cong [D \rightarrow D]$. Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \rightarrow D]$, where A is some predetermined domain containing interpretations for elements of C. Each element of D corresponds to either an element of A or an element of $[D \rightarrow D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \rightarrow X]$ from domains to domains --- that is, finding domains X such that $X \cong A + [X \rightarrow X]$, and such that for any domain Y also satisfying this equation, there is an embedding of X to Y --- a pair of maps

$$X \quad \overset{f}{\underset{f^R}{\square}} \quad Y$$

Such that

$$f^R o f = id_X$$

$$f o f^R \subseteq id_Y$$

Where $f \subseteq g$ means that f *approximates* g in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering F not as a function on domains, but as a

functor on a category of domains. Instead of a least fixed point of the function, *F*.

Definition 1.3: Let **K** be a category and $F : K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a **K-object** and $a : F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a **K-object** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category **K** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$$

Recall that a cocone μ of an $\omega - chain$ Δ is a *K*-object *X* and a collection of K –*arrows* $\{\mu_i : D_i \to X \mid i \ge 0\}$ such that $\mu_i = \mu_{i+1} o\, f_i$ for all $i \ge 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of $\mu's$ components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with the property that if $v : \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k : X \to X'$ such that for all $i \ge 0,, v_i = k\, o\, \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - co\lim its$. Dually, an $\omega^{op} - chain$ in **K** is a diagram of the following form:

$$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} .....$$

A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain$ Δ is a **K**-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \ge 0\}$ such that for all $i \ge 0$, $\mu_i = f_i\, o\, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op} - chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $v : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \ge 0, \mu_i\, o\, k = v_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of **K,** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each **K**-object A. It is also convenient to write $\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \ge 1\}$. For the images of Δ and μ under **F** we write

$$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} .....$$

and $F(\mu) = \{F(\mu_i) \mid i \ge 0\}$

We write F^i for the *i*-fold iterated composition of *F* – that is,

$$F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$$

,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let **K** be a category with initial object $\perp$ and let $F : K \to K$ be a functor. Define the $\omega - chain \Delta$ by

$$\Delta = \perp \xrightarrow{\perp \to F(\perp)} F(\perp) \xrightarrow{F(\perp \to F(\perp))} F^2(\perp) \xrightarrow{F^2(\perp \to F(\perp))} ....$$

If both $\mu : \Delta \to D$ and $F(\mu) : F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d : F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^-

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, ...X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, ...x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1})...$$
$$..P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \le P(x_1, x_2, ...x_n) \le 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \le k \le n$ that whenever

$$P(pa_k) \ne 0, if\ P(nd_k \mid pa_k) \ne 0$$
$$and\ P(x_k \mid pa_k) \ne 0$$
$$then\ P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a

given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, \ldots X_{k-1}\}$$

Let

$$D_k = \{X_{k+1}, X_{k+2}, \ldots X_n\}$$

follows $\sum_{d_k}$

We define the mth *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ *Where* $\Phi_m(x)$ *is the* m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ *has degree* $\varphi(m)$. *The roots of* $\Phi_m(x)$ *are just the primitive* m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \le k \prec m, (k,m)=1, \quad where$$

$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another, or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$. Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5   If m divides n, then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. Since $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6   If m and n are relatively prime, then

$$Q(\xi_m, \xi_n) = Q(\xi_{nm})$$

and

$$Q(\xi_m) \cap Q(\xi_n) = Q$$

(Recall the $Q(\xi_m, \xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m \xi_n$ is a primitive mn^{th} root of unity, so that

$$Q(\xi_{mn}) \subseteq Q(\xi_m, \xi_n)$$
$$[Q(\xi_m, \xi_n) : Q] \le [Q(\xi_m) : Q][Q(\xi_n : Q]$$
$$= \varphi(m)\varphi(n) = \varphi(mn);$$

Since $[Q(\xi_{mn}) : Q] = \varphi(mn);$ this implies that $Q(\xi_m, \xi_n) = Q(\xi_{nm})$ We know that $Q(\xi_m, \xi_n)$ has degree $\varphi(mn)$ over Q, so we must have

$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(n)$$

and

$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(m)$$

$$[Q(\xi_m) : Q(\xi_m) \cap Q(\xi_n)] \ge \varphi(m)$$

And thus that $Q(\xi_m) \cap Q(\xi_n) = Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m, \xi_n) = Q(\xi_{[m,n]})$$

And

$$Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)});$$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF.   Write $m = p_1^{e_1} \ldots \ldots p_k^{e_k}$ *and* $p_1^{f_1} \ldots p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)

$$Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$$

Thus

$$Q(\xi_m, \xi_n) = Q(\xi_{p_1^{e_1}})........Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}})$$

$$= Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}})$$

$$= Q(\xi_{p_1^{\max(e_1, f_1)}})........Q(\xi_{p_1^{\max(e_k, f_k)}})$$

$$= Q(\xi_{p_1^{\max(e_1, f_1)}........p_1^{\max(e_k, f_k)}})$$

$$= Q(\xi_{[m,n]});$$

An entirely similar computation shows that $Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i, y_i) = \log_2 \frac{P(x_i/y_i)}{P(x_i)} \; bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input –output pair (x_i, y_i) for which

$$P(x_i/y_j) = 1 \; and \; I(x_i, y_j) = \log_2 \frac{1}{P(x_i)} \quad bits;$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j) = P(x_i)$ and also $I(x_i, y_j) = 0;$ that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X, Y) = \sum_{i.j} P(x_i, y_j)I(x_i, y_j) = \sum_{i.j} P(x_i, y_j)\log_2 \left[\frac{P(x_i/y_j)}{P(x_i)} \right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i, y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X, Y) = \sum_{i.j} P(x_i, y_j)$$

$$= \sum_{i.j} P(x_i, y_j)\log_2 \left[\frac{1}{P(x_i)} \right]$$

$$-\sum_{i.j} P(x_i, y_j)\log_2 \left[\frac{1}{P(x_i/y_j)} \right]$$

$$\sum_{i.j} P(x_i, y_j)\log_2 \left[\frac{1}{P(x_i)} \right]$$

$$= \sum_i \left[P(x_i/y_j)P(y_j) \right] \log_2 \frac{1}{P(x_i)}$$

$$\sum_i P(x_i)\log_2 \frac{1}{P(x_i)} = H(X)$$

$$I(X, Y) = H(X) - H(X/Y)$$

Where $H(X/Y) = \sum_{i,j} P(x_i, y_j)\log_2 \frac{1}{P(x_i/y_j)}$

is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X, Y) = I(Y, X)$$

And by interchanging input and output it is also true that

$$I(X, Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j)\log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j) \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \leq 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i \log_2(\frac{Q_i}{P_i}) \leq 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \leq 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

$$+ \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$$P(x_1) = \alpha \ and \ P(x_2) = 1 - \alpha, \ and \ \text{transition probabilities}$$

$$P(y_3/x_2) = 1 - p \ and \ P(y_2/x_1) = 0,$$

$$and \ P(y_3/x_1) = 0$$

$$and \ P(y_1/x_2) = p$$

$$and \ P(y_3/x_2) = 1 - p$$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean n-space. $p(y|x)$ for the density $p_n(y_1,...,y_n | x_1,...x_n)$ and F for F_n. For any real number a, let

$$A = \left\{ (x,y) : \log \frac{p(y|x)}{p(y)} > a \right\} \quad (1)$$

Then for each positive integer u, there is a code (u, n, λ) such that

$$\lambda \leq ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \quad (2)$$

Where

$$P\{(X,Y) \in A\} = \int_A ... \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y|x)$$

and

$$P\{X \in F\} = \int_F ... \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$$P\{Y \in A_{x^1} | X = x^{(1)}\} \geq 1 - \varepsilon$$

where $A_x = \{y : (x,y)\varepsilon A\};$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)}, ..., x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\left\{Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i | X = x^{(k)}\right\} \geq 1 - \varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets $B_i, i = 1, 2, ..., u,$ form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \geq u$ by showing that

$$\varepsilon \leq te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \quad . \quad \text{We}$$

proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (\text{If } t = 0, \text{ take } B = \phi). \text{ Then}$$

$$P\{(X,Y) \in A\} = \int_{(x,y) \in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y\,|\,x)dy\,dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y\,|\,x)dy\,dx + \int_x p(x)$$

I. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1,\dots,s_m\}$, we shall write $(s_1,\dots,s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b \,|\, a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab \,|\, a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1,\dots,a_m)$ and $b = (b_1,\dots,b_n)$, then $ab = (a_1 b_1,\dots,a_i b_j,\dots,a_m b_n)$. Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a \mapsto a+a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p \neq A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \neq 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \neq| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and

$(a,b) \in c$, then $(a,0) = (a,b)(1,0) \in c$ and $(0,b) = (a,b)(0,1) \in c$. This shows that $c = a \times b$ with

$$a = \{a \,|\, (a,b) \in c \text{ some } b \in b\}$$

and

$$b = \{b \,|\, (a,b) \in c \text{ some } a \in a\}$$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A \to B$. A *homomorphism of A*-algebra $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1,\dots,x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1,\dots,X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite,* and B is finitely generated as an A-module. Let k be a field, and let A be a k-algebra. If $1 \neq 0$ in A, then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A. If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$. **Polynomial rings.** Let k be a field. A *monomial* in $X_1,\dots,X_n$ is an expression of the form $X_1^{a_1}\dots X_n^{a_n}, \quad a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by $X^\alpha, \alpha = (a_1,\dots,a_n) \in \square^n$. The elements of the polynomial ring $k[X_1,\dots,X_n]$ are finite sums

$$\sum c_{a_1\dots a_n} X_1^{a_1}\dots X_n^{a_n}, \quad c_{a_1\dots a_n} \in k, \quad a_j \in \square$$

With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1,\dots,X_n]$ as a k-vector space. The ring $k[X_1,\dots,X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1,\dots,X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in $k[X]$.** The division algorithm allows us to divide a nonzero polynomial into another: let f and g be

polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or $\deg r < \deg g$. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

*(Pure) **lexicographic** ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha = (a_1, \ldots a_n)$ and $\beta = (b_1, \ldots b_n)$ be two elements of $\square^n$; then $\alpha > \beta$ and $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha - \beta \in \square$, the left most nonzero entry is positive. For example,

$XY^2 > Y^3Z^4$; $X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha - \beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ *(total degree greater)*
$XY^5Z^2 > X^4YZ^3$, $X^5YZ > X^4YZ^2$.

Orderings on $k[X_1, \ldots X_n]$. Fix an ordering on the monomials in $k[X_1, \ldots X_n]$. Then we can write an element f of $k[X_1, \ldots X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$

as

$f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2$ *(lex)*

or

$f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2$ *(grevlex)*

Let $\sum a_\alpha X^\alpha \in k[X_1, \ldots, X_n]$, in decreasing order:

$f = a_{\alpha_0} X^{\alpha_0} + a_{\alpha_1} X^{\alpha_1} + \ldots, \qquad \alpha_0 > \alpha_1 > \ldots, \quad \alpha_0 \neq 0$

Then we define.

- The *multidegree* of f to be multdeg(f)$= \alpha_0$;
- The *leading coefficient of* f to be LC(f)$= a_{\alpha_0}$;
- The *leading monomial of* f to be LM(f) $= X^{\alpha_0}$;
- The *leading term of* f to be LT(f) $= a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + \ldots$, the multidegree is $(1,2,1)$, the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in** $k[X_1, \ldots X_n]$. Fix a monomial ordering in $\square^2$. Suppose given a polynomial f and an ordered set $(g_1, \ldots g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1, \ldots a_s$ and r such that $f = a_1 g_1 + \ldots + a_s g_s + r$ Where either $r = 0$ or no monomial in r is divisible by any of $LT(g_1), \ldots, LT(g_s)$ **Step 1:** If $LT(g_1) | LT(f)$, divide g_1 into f to get

$$f = a_1 g_1 + h, \qquad a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1, \ldots, X_n]$$

If $LT(g_1) | LT(h)$, repeat the process until $f = a_1 g_1 + f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f = a_1 g_1 + \ldots + a_s g_s + r_1$ With $LT(r_1)$ not divisible by any $LT(g_1), \ldots LT(g_s)$ **Step 2:** Rewrite $r_1 = LT(r_1) + r_2$, and repeat Step 1 with r_2 for f :

$f = a_1 g_1 + \ldots + a_s g_s + LT(r_1) + r_3$ (different $a_i 's$) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a = (Y^2 - X^3)$ contains $Y^2 - X^3$ but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if

$$\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$$

all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal,* and let $A = \{\alpha | X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A$, $\beta \in \square^n \Rightarrow \alpha + \beta \in$ (*)

And a is the k-subspace of $k[X_1, \ldots, X_n]$

generated by the $X^\alpha, \alpha \in A$. Conversely, of A is a subset of $\square^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,...,X_n]$ generated by $\{X^\alpha \mid \alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,...,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,...,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)},...,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!} N(n,c) = 1\left\{\sum_{j=1}^{n} jc_j = n\right\} \prod_{j=1}^{n} (\frac{1}{j})^{c_j} \frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \square_+^n$.

Lemma1.7 For nonnegative integers $m_1,...,m_n$,

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = \left(\prod_{j=1}^{n} \left(\frac{1}{j}\right)^{m_j}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]}/c_j! = 1/(c_j - m_j)!$ when $c_j \ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum jm_j$. Then, with the first sum indexed by $c = (c_1,...c_n) \in \square_+^n$ and the last sum indexed by $d = (d_1,...,d_n) \in \square_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = \sum_c P[C^{(n)} = c] \prod_{j=1}^{n} (c_j)^{[m_j]}$$

$$= \sum_{c:c_j \ge m_j \text{ for all } j} 1\left\{\sum_{j=1}^{n} jc_j = n\right\} \prod_{j=1}^{n} \frac{(c_j)^{[m_j]}}{j^{c_j} c_j!}$$

$$= \prod_{j=1}^{n} \frac{1}{j^{m_j}} \sum_d 1\left\{\sum_{j=1}^{n} jd_j = n - m\right\} \prod_{j=1}^{n} \frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m \le n)$, corresponding to the fact that if $n - m \ge 0$, then $d_j = 0$ for $j > n - m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1,...,d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]} = j^{-r} 1\{jr \le n\} \qquad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = E\left(\prod_{j=1}^{n} Z_j^{[m_j]}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\}, \qquad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1,2,...n\}$, so that $|I| = n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha| = (n - j)!$, since the elements of $\{1,2,...,n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n - rj)!1(rj \le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n-rj)! \, 1(rj \leq n)$$

$$\times \frac{n^{[rj]}}{j^r r!} \frac{1}{n!} = 1(rj \leq n) \frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \geq 0} (-1)^l \binom{k+l}{l} S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0, 1, ..., n$,

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1$. In particular, for $n \geq 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)}, ..., C_b^{(n)})$ for any $1 \leq b \leq n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1, ..., c_b) \in \square_+^b$ with $m = \sum i c_i,$

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c]$$

$$= \left\{ \prod_{i=1}^{b} \left(\frac{1}{i}\right)^{c_i} \frac{1}{c_i!} \right\} \sum_{\substack{l \geq 0 \text{ with} \\ \sum i l_i \leq n-m}} (-1)^{l_1 + ... + l_b} \prod_{i=1}^{b} \left(\frac{1}{i}\right)^{l_i} \frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)}, ..., C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = ... = m_n = 0$

The limit distribution of cycle counts
It follows immediately from Lemma 1.2 that for each fixed j, as $n \rightarrow \infty$,

$$P[C_j^{(n)} = k] \rightarrow \frac{j^{-k}}{k!} e^{-1/j}, \quad k = 0, 1, 2, ...,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)} \rightarrow_d Z_j$ where $Z_j \square P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6   The process of cycle counts converges in distribution to a Poisson process of $\square$ with intensity j^{-1}. That is, as $n \rightarrow \infty$,

$$(C_1^{(n)}, C_2^{(n)}, ...) \rightarrow_d (Z_1, Z_2, ...) \qquad (1.1)$$

Where the $Z_j, j = 1, 2, ...,$ are independent Poisson-distributed random variables with

$$E(Z_j) = \frac{1}{j}$$

Proof. To establish the converges in distribution one shows that for each fixed $b \geq 1$, as $n \rightarrow \infty$,

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c] \rightarrow P[(Z_1, ..., Z_b) = c]$$

Error rates
The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b = 1$. Using properties of alternating series with decreasing terms, for $k = 0, 1, ..., n$,

$$\frac{1}{k!} \left(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!} \right) \leq \left| P[C_1^{(n)} = k] - P[Z_1 = k] \right|$$

$$\leq \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!} \frac{n}{n+2} \leq \sum_{k=0}^{n} \left| P[C_1^{(n)} = k] - P[Z_1 = k] \right| \leq \frac{2^{n+1} - 1}{(n+1)!} \quad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ... \right) < \frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $P\left[A_n(C^{(n)}) \right]$ under conditions (A_0) and (B_{01}), where

$$A_n(C^{(n)}) = \bigcap_{1 \leq i \leq n} \bigcap_{r_i' + 1 \leq j \leq r_i} \left\{ C_{ij}^{(n)} = 0 \right\},$$

and $\zeta_i = (r_i' / r_{id}) - 1 = O(i^{-g'})$ as $i \rightarrow \infty$, for some $g' > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0m}(Z') = n]}{P[T_{0m}(Z) = n]}$$

$$\prod_{\substack{1 \leq i \leq n \\ r_i' + 1 \leq j \leq r_i}} \left\{ 1 - \frac{\theta}{i r_i} (1 + E_{i0}) \right\} \qquad (1.1)$$

$P[T_{0n}(Z') = n]$

$$= \frac{\theta d}{n} \exp\left\{ \sum_{i \geq 1} [\log(1 + i^{-1}\theta d) - i^{-1}\theta d] \right\}$$

$$\left\{ 1 + O(n^{-1}\varphi'_{\{1,2,7\}}(n)) \right\} \qquad (1.2)$$

and

$P[T_{0n}(Z') = n]$

$$= \frac{\theta d}{n} \exp\left\{ \sum_{i \geq 1} [\log(1 + i^{-1}\theta d) - i^{-1}\theta d] \right\}$$

$$\left\{ 1 + O(n^{-1}\varphi'_{\{1,2,7\}}(n)) \right\} \qquad (1.3)$$

Where $\varphi'_{\{1,2,7\}}(n)$ refers to the quantity derived from Z'. It thus follows that $P[A_n(C^{(n)})] \square Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the r'_i and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta^*_i = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\varphi'_{\{1,2,7\}}(n)$ and $n^{-1}\varphi'_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \leq b \leq n/8$ and $n \geq n_0$, with n_0

$d_{TV}(L(C[1,b]), L(Z[1,b]))$

$$\leq d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b]))$$

$$\leq \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0), (D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\overset{\square}{C}[1,b] \mid T_{0b}(C) = l) = L(\overset{\square}{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b]))$

$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$

$$= \max_A \sum_{r \in A} P[T_{0b}(Z) = r]$$

$$\left\{ 1 - \frac{P[T_{bn}(Z) = n - r]}{P[T_{0n}(Z) = n]} \right\} \qquad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b]))$$

$$= \sum_{r \geq 0} P[T_{0b} = r]\left\{ 1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]} \right\}_+$$

$$\leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]}$$

$$\times \left\{ \sum_{s=0}^{n} P[T_{0b} = s](P[T_{bn} = n - s] - P[T_{bn} = n - r] \right\}_+$$

$$\leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r]$$

$$\times \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{\left\{ P[T_{bn} = n - s] - P[T_{bn} = n - r] \right\}}{P[T_{0n} = n]}$$

$$+ \sum_{s=0}^{[n/2]} P[T_{0b} = r] \sum_{s=[n/2]+1}^{n} P[T = s]P[T_{bn} = n - s] / P[T_{0n} = n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$(\max_{n/2 < s \leq n} P[T_{0b} = s]) / P[T_{0n} = n]$

$$\leq \frac{2\varepsilon_{\{10.5(1)\}}(n/2, b)}{n} \frac{3n}{\theta P_\theta[0,1]},$$

$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n) \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{1}{2}|r - s|$$

$$\leq \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{ 1 + \frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \right\} P$$

$$+ \frac{6}{\theta P_\theta[0,1]} \varepsilon_{\{10.5(1)\}}(n/2, b) \qquad (1.5)$$

Required order under Conditions $(A_0), (D_1)$ and (B_{11}), if $S(\infty) < \infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty) < \infty$. Examining the Conditions $(A_0), (D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l \geq 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1 > 1$. A first observation is that a similar problem arises with

the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\overset{\square}{n}_1$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l \geq 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)} = s] - P[T_{bn}^{(m)} = s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from $\overset{\square}{n}_1$ of the form $\phi_1^{\theta}(n) + u_1^*(n)$, which is only small if $a_1 > 1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta > 0$, since $a_2 > 1$ is in any case assumed. For $s \geq n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn} = s] - P[T_{bn} = s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn} = s] - P[T_{bn} = s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta > 0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w) = 1_{\{w=s\}} - 1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn} = s] - P[T_{bn} = s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^{\theta}(n) + u_1^*(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form $|P[T_{bn} = s] - P[T_{bn} = s+t]| = O(n^{-2}t + n^{-1-a_1+\delta})$ for any $\delta > 0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1} + n^{-a_1+\delta})$ for any $\delta > 0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left| d_{TV}(L(C[1,b]), L(Z[1,b])) - \frac{1}{2}(n+1)^{-1}|1-\theta| E |T_{0b} - ET_{0b}| \right| \leq \varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b) = O(n^{-1}b[n^{-1}b + n^{-\beta_{12}+\delta}])$ for any $\delta > 0$ under Conditions $(A_0), (D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b]))$$
$$= \sum_{r \geq 0} P[T_{0b} = r]\left\{ 1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]} \right\}_+$$

Now we observe that

$$\left| \sum_{r \geq 0} P[T_{0b} = r]\left\{ 1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]} \right\}_+ - \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right|$$
$$\times \left| \sum_{s=[n/2]+1}^{n} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r]) \right|$$
$$\leq 4n^{-2}ET_{0b}^2 + (\max_{n/2 < s \leq n} P[T_{0b} = s])/P[T_{0n} = n]$$
$$+ P[T_{0b} > n/2]$$
$$\leq 8n^{-2}ET_{0b}^2 + \frac{3\varepsilon_{\{10.5(2)\}}(n/2,b)}{\theta P_\theta[0,1]}, \qquad (1.1)$$

We have

$$\left| \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right.$$
$$\times \left(\left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r] \right\}_+ \right.$$
$$\left. - \left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s]\frac{(s-r)(1-\theta)}{n+1} P[T_{0n} = n] \right\}_+ \right) \right|$$

$$\leq \frac{1}{n^2 P[T_{0n} = n]} \sum_{r \geq 0} P[T_{0b} = r]\sum_{s \geq 0} P[T_{0b} = s]|s-r|$$
$$\times \left\{ \varepsilon_{\{10.14\}}(n,b) + 2(r \vee s)|1-\theta| n^{-1}\left\{ K_0\theta + 4\phi_{\{10.8\}}^*(n) \right\} \right\}$$
$$\leq \frac{6}{\theta n P_\theta[0,1]} ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$
$$+ 4|1-\theta| n^{-2}ET_{0b}^2\left\{ K_0\theta + 4\phi_{\{10.8\}}^*(n) \right\}$$
$$(\frac{3}{\theta n P_\theta[0,1]}) \Big\}, \qquad (1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]} P[T_{0b} = r] \left| \left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} \right\}_+ \right.$$

$$\left. - \left\{ \sum_{s=0} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} \right\}_+ \right|$$

$$\le \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s>[n/2]} P[T_{0b} = s] \frac{(s-r)|1-\theta|}{n+1}$$

$$\le |1-\theta| n^{-1} E(T_{0b} 1\{T_{0b} > n/2\}) \le 2|1-\theta| n^{-2} ET_{0b}^2, \qquad (1.3)$$

and then by observing that

$$\sum_{r>[n/2]} P[T_{0b} = r] \left\{ \sum_{s\ge 0} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} \right\}$$

$$\le n^{-1} |1-\theta| (ET_{0b} P[T_{0b} > n/2] + E(T_{0b} 1\{T_{0b} > n/2\}))$$

$$\le 4|1-\theta| n^{-2} ET_{0b}^2 \qquad (1.4)$$

Combining the contributions of (1.2) –(1.3), we thus find that

$$\left| d_{TV}(L(\overset{\Box}{C}[1,b]), L(\overset{\Box}{Z}[1,b])) \right.$$

$$\left. -(n+1)^{-1} \sum_{r\ge 0} P[T_{0b} = r] \left\{ \sum_{s\ge 0} P[T_{0b} = s](s-r)(1-\theta) \right\}_+ \right|$$

$$\le \varepsilon_{\{7.8\}}(n,b)$$

$$= \frac{3}{\theta P_\theta[0,1]} \left\{ \varepsilon_{\{10.5(2)\}}(n/2,b) + 2n^{-1} ET_{0b} \varepsilon_{\{10.14\}}(n,b) \right\}$$

$$+ 2n^{-2} ET_{0b}^2 \left\{ 4 + 3|1-\theta| + \frac{24|1-\theta| \phi_{\{10.8\}}^*(n)}{\theta P_\theta[0,1]} \right\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0), (D_1)$ and (B_{12}), provided that $S(\infty) < \infty$; this supplementary condition can be removed if $\phi_{\{10.8\}}^*(n)$ is replaced by $\phi_{\{10.11\}}^*(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty) < \infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0} P[T_{0b} = r] \left\{ \sum_{s\ge 0} P[T_{0b} = s](s-r)(1-\theta) \right\}_+$$

$$= \frac{1}{2} |1-\theta| E |T_{0b} - ET_{0b}|$$

Example 1.0. Consider the point $O = (0,...,0) \in \Box^n$. For an arbitrary vector r, the coordinates of the point $x = O + r$ are equal to the respective coordinates of the vector $r : x = (x^1,...x^n)$ and $r = (x^1,...,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\Box^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\Box^n : \Box^n = \{points\}, \quad \Box^n = \{vectors\}$

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\Box^n$ treated in this way is called an *n-dimensional affine space.* (An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\Box^n$ considered as an affine space we can precede in two opposite directions: $\Box^n$ as an Euclidean space $\Leftarrow \Box^n$ as an affine space $\Rightarrow \Box^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\Box^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\Box^n$ a Euclidean space. Namely, we define the length of a vector $a = (a^1,...,a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + ... + (a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B) := \left| \overrightarrow{AB} \right| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B) \le d(A,C) + d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b) := a^1 b^1 + \dots + a^n b^n \qquad (3)$$

Thus $|a| = \sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b = (a,b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos \alpha := \frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2 \le |a|^2 |b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a + tb,$ where $t \in R$. As $(a + tb, a + tb) \ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x) = x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i. From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2, \dots$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2, \dots,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 +$$

$$\dots + \frac{\partial f}{\partial x^n} h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \square^n$ at $t = t_0$ and with the velocity vector $x(t_0) = \upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$f(x(t_0 + \Delta t) - f(x_0)$$
$$= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t)$$
$$+ \beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t|$$
$$= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \dots + \frac{\partial f}{\partial x^n} x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \to \square$, $U \subset \square^n$,

$$d(f + g) = df + dg \qquad (1)$$
$$d(fg) = df.g + f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $\dot{x}(t_0) = \upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t = t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\Box^m$ instead of $\Box$. The only difference is that now the differential of a map $F : U \to \Box^m$ at a point x will be a linear function taking vectors in $\Box^n$ to vectors in $\Box^m$ (instead of $\Box$). For an arbitrary vector $h \in |\Box^n$,

$$F(x+h) = F(x) + dF(x)(h)$$
$$+ \beta(h)|h| \qquad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. We have $dF = (dF^1, ..., dF^m)$ and

$$dF = \frac{\partial F}{\partial x^1} dx^1 + ... + \frac{\partial F}{\partial x^n} dx^n$$

$$= \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & .... & \frac{\partial F^1}{\partial x^n} \\ ... & ... & ... \\ \frac{\partial F^m}{\partial x^1} & ... & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ ... \\ dx^n \end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\Box^n$, the differential of a map $F : U \to \Box^m$ (where $U \subset \Box^n$) maps the velocity vector $\dot{x}(t)$ to the velocity vector of the curve $F(x(t))$ in $\Box^m$:

$$\frac{dF(x(t))}{dt} = dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t + \Delta t) = x(t) + \dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t) \to 0$ when $\Delta t \to 0$. By the definition of the differential,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h| \qquad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. we obtain

$$F(x(t+\Delta t)) = F(x + \underbrace{\dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t}_{h})$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t) +$$

$$\beta(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t\right|$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F : U \to V$ and $G : V \to W$, where $U \subset \Box^n, V \subset \Box^m, W \subset \Box^p$ (open domains). Let $F : x \mapsto y = F(x)$. Then the differential of the composite map $GoF : U \to W$ is the composition of the differentials of F and G:

$$d(GoF)(x) = dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\Box^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\Box^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t) = G(F(x(t)))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\Box^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence

$$d(GoF)(\dot{x}) = dG(dF(\dot{x})) \qquad \text{for an arbitrary}$$

vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\Box^n$ by $(x^1, ..., x^n)$ and in $\Box^m$ by $(y^1, ..., y^m)$, and write

$$dF = \frac{\partial F}{\partial x^1} dx^1 + ... + \frac{\partial F}{\partial x^n} dx^n \qquad (1)$$

$$dG = \frac{\partial G}{\partial y^1} dy^1 + ... + \frac{\partial G}{\partial y^n} dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF) = \frac{\partial G}{\partial y^1} dF^1 + ... + \frac{\partial G}{\partial y^m} dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i = dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF) = \begin{pmatrix} \frac{\partial G^1}{\partial y^1} & .... & \frac{\partial G^1}{\partial y^m} \\ ... & ... & ... \\ \frac{\partial G^p}{\partial y^1} & ... & \frac{\partial G^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & .... & \frac{\partial F^1}{\partial x^n} \\ ... & ... & ... \\ \frac{\partial F^m}{\partial x^1} & ... & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ ... \\ dx^n \end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} & .... & \frac{\partial z^1}{\partial x^n} \\ ... & ... & ... \\ \frac{\partial z^p}{\partial x^1} & ... & \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} & .... & \frac{\partial z^1}{\partial y^m} \\ ... & ... & ... \\ \frac{\partial z^p}{\partial y^1} & ... & \frac{\partial z^p}{\partial y^m} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & .... & \frac{\partial y^1}{\partial x^n} \\ ... & ... & ... \\ \frac{\partial y^m}{\partial x^1} & ... & \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^{\mu}}{\partial x^a} = \sum_{i=1}^{m} \frac{\partial z^{\mu}}{\partial y^i} \frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y \in \Box^m$ on $x \in \Box^n$ is given by the map F, the dependence of $z \in \Box^p$ on $y \in \Box^m$ is given by the map G, and the dependence of $z \in \Box^p$ on $x \in \Box^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U \subset \Box^n$. Consider also another copy of $\Box^n$, denoted for

distinction $\Box^n_y$, with the standard coordinates $(y^1...y^n)$. A system of coordinates in the open domain U is given by a map $F : V \to U$, where $V \subset \Box^n_y$ is an open domain of $\Box^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;
(2) F is invertible;
(3) $F^{-1} : U \to V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \Box^n_y$
In other words,

$$F : (y^1..., y^n) \mapsto x = x(y^1..., y^n) \qquad (1)$$

Here the variables $(y^1..., y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\Box^2$ specified in polar coordinates as
$$x(t) : r = r(t), \varphi = \varphi(t) \qquad (1)$$
We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x / \partial r, \partial x / \partial \varphi$ are vectors depending on point in $\Box^2$. We can compare this with the formula in the "standard" coordinates:

$$\dot{x} = e_1 \dot{x} + e_2 \dot{y} \qquad . \qquad \text{Consider the vectors}$$
$\partial x / \partial r, \partial x / \partial \varphi$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos \varphi, \sin \varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r \sin \varphi, r \cos \varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x / \partial r, \partial x / \partial \varphi$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$

$(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r, \varphi) \ (r = r_0 \ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \partial x / \partial r, e_\varphi := \partial x / \partial \varphi$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i. In particular, the elements of the basis $e_i = \partial x / \partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1, ..., x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F : \Box^n \to \Box^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0) : \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \Box^n$ to vectors attached to the point $F(x) \in \Box^m$

$$dF = \frac{\partial F}{\partial x^1} dx^1 + ... + \frac{\partial F}{\partial x^n} dx^n$$

$$(e_1, ..., e_m) \begin{pmatrix} \dfrac{\partial F^1}{\partial x^1} \cdots \dfrac{\partial F^1}{\partial x^n} \\ \cdots \ \cdots \ \cdots \\ \dfrac{\partial F^m}{\partial x^1} \cdots \dfrac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1} dx^1 + ... + \frac{\partial f}{\partial x^n} dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\Box^2$ given in the standard coordinates:

$A = -y dx + x dy$ In the polar coordinates we will have $x = r \cos \varphi, y = r \sin \varphi$, hence

$$dx = \cos \varphi dr - r \sin \varphi d\varphi$$
$$dy = \sin \varphi dr + r \cos \varphi d\varphi$$

Substituting into A, we get
$$A = -r \sin \varphi (\cos \varphi dr - r \sin \varphi d\varphi)$$
$$+ r \cos \varphi (\sin \varphi dr + r \cos \varphi d\varphi)$$
$$= r^2 (\sin^2 \varphi + \cos^2 \varphi) d\varphi = r^2 d\varphi$$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U :

$$\omega(\upsilon) = \omega_1 \upsilon^1 + ... + \omega_n \upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i \upsilon^i$, where $e_i = \partial x / \partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta_j^i \qquad (2) \qquad \text{at}$$

every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \dfrac{dx}{dt'} \right\rangle$ and
$\left\langle \omega(x(t(t'))), \dfrac{dx}{dt'} \right\rangle$ As

$$\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle = \left| \left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle \cdot \frac{dt}{dt'} \right.,$$

Let p be a rational prime and let $K = \Box(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p - 1$ over $\Box$. We wish to show that $O_K = \Box[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\Box[\zeta] \subseteq O_K$. We give a proof without

assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, ..., \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + ... + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p - 1$ By linearity of the trace, we find that

$Tr_{K/\mathbb{Q}}(1 - \zeta) = Tr_{K/\mathbb{Q}}(1 - \zeta^2) = ...$
$= Tr_{K/\mathbb{Q}}(1 - \zeta^{p-1}) = p$

We also need to compute the norm of $1 - \zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + ... + 1 = \Phi_p(x)$$
$$= (x - \zeta)(x - \zeta^2)...(x - \zeta^{p-1});$$

Plugging in $x = 1$ shows that

$$p = (1 - \zeta)(1 - \zeta^2)...(1 - \zeta^{p-1})$$

Since the $(1 - \zeta^j)$ are the conjugates of $(1 - \zeta)$, this shows that $N_{K/\mathbb{Q}}(1 - \zeta) = p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1 - \zeta)O_K \cap \mathbb{Z} = p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1 - \zeta)$ in O_K, so the inclusion $(1 - \zeta)O_K \cap \mathbb{Z} \supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1 - \zeta)O_K \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1 - \zeta)O_K \cap \mathbb{Z} = \mathbb{Z}$ Thus we can write
$$1 = \alpha(1 - \zeta)$$
For some $\alpha \in O_K$. That is, $1 - \zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha \in O_K$, $Tr_{K/\mathbb{Q}}((1 - \zeta)\alpha) \in p.\mathbb{Z}$
PROOF. We have

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) = \sigma_1((1-\zeta)\alpha) + ... + \sigma_{p-1}((1-\zeta)\alpha)$$
$$= \sigma_1(1-\zeta)\sigma_1(\alpha) + ... + \sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha)$$
$$= (1-\zeta)\sigma_1(\alpha) + ... + (1-\zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1 - \zeta^j$ is a multiple of $1 - \zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = \mathbb{Q}(\zeta_p)$ be the pth cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, ..., \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write
$$\alpha = a_0 + a_1\zeta + ... + a_{p-2}\zeta^{p-2} \quad \text{With } a_i \in \mathbb{Q}.$$
Then
$$\alpha(1-\zeta) = a_0(1-\zeta) + a_1(\zeta - \zeta^2) + ...$$
$$+ a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0$ We also have $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in p\mathbb{Z}$, so $a_0 \in \mathbb{Z}$ Next consider the algebraic integer

$$(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + ... + a_{p-2}\zeta^{p-3};$$ This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \mathbb{Z}$, and continuing in this way we find that all of the a_i are in $\mathbb{Z}$. This completes the proof.

Example 1.4 Let $K = \mathbb{Q}$, then the local ring $\mathbb{Z}_{(p)}$ is simply the subring of $\mathbb{Q}$ of rational numbers with denominator relatively prime to p. Note that this ring $\mathbb{Z}_{(p)}$ is not the ring $\mathbb{Z}_p$ of p-adic integers; to get $\mathbb{Z}_p$ one must complete $\mathbb{Z}_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is

clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta / \gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta / \gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta / \gamma \in (a \cap O_K)O_{K,p}$, as claimed. We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p} / pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha / \beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as

$$x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + ... + \frac{\alpha_0}{\beta_0} \text{ With } \alpha_i \in O_K \text{ and}$$

$\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1...\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma / \beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/\square}(\alpha O_K) = \left| N_{K/\square}(\alpha) \right|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\square$ is Galois. Let σ be an element of $Gal(K/\square)$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\square}(\sigma(\alpha)O_K) = N'_{K/\square}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\square)$, we have $N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N'_{K/\square}(\alpha O_K)^n$ Since $N_{K/\square}(\alpha)$ is a rational integer and O_K is a free $\square$ - module of rank n,

$O_K / N_{K/\square}(\alpha)O_K$ Will have order $N_{K/\square}(\alpha)^n$; therefore

$$N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N_{K/\square}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m$.

J. Modeling

Herewith, we propose a hybrid approach for modeling the load profile in order to support decisions of policy makers and engineers, especially for long term planning. This approach consists of a combination of deterministic (devices' operation), probabilistic (user groups, user behavior) and stochastic models (weather and external parameters). The main advantages in this approach are that the planner can incorporate deterministic models of both power generating and consuming devices, and define (in the preferred degree of detail) user groups, users' behaviors, market influences and other interactions (depending on the study objectives). A modeling methodology should be defined as a function of the study objectives (goals), data availability (statistical, survey, measurement data or a combination of the aforementioned), user groups' definition (customer segmentation, target user groups) and the study boundaries (degree of detail required, limitations and hypotheses). The definition of user groups is crucial for an efficient allocation of the costs related to the processes of generation, transmission, and distribution of electrical energy. Identification of similar consumption patterns among different groups of consumers (i.e. residential, office, commercial, industrial) can support an efficient cost allocation and prevent the problem of double counting [28]. The scope of this section is to point out the main aspects involved within the interaction of small size prosumers with the electrical grid. During this effort, emphasis is given in both technical and social

aspects. The authors acknowledge that there is a gap between the practices of engineers and policy makers while pursuing societal change. Even though they often share the same objectives, in many cases the linkage between the consumer and aspects related to technology, regulation and markets, is not addressed adequately. Figure 1, illustrates the components involved in the proposed methodology, and which represent the interactions between the user (prosumer) and the electrical grid (power system physical layer). These components are explicitly described in the following sub-sections.

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