Smart Grid Dynamic Pricing

Akash K Singh, PhD
IBM Corporation Sacramento, USA

Abstract

Dynamic energy pricing is a promising development that addresses the concern of finding an environmentally friendly solution to meeting energy needs of customers while minimizing their electrical energy bill. In this paper, we mathematically formulate the electrical energy bill minimization problem for cooperative networked consumers who have a single energy bill, such as those working in a commercial/industrial building. The idea is to schedule user requests for appliance use at different times during a fixed interval based on dynamic energy prices during that interval. Two different methods are presented to minimize the energy cost of such users under non-interruptible or interruptible jobs. The methods relay on a quasi-dynamic pricing function for unit of energy consumed, which comprises of a base price and a penalty term. The methods minimize the energy cost of the users while meeting all the scheduling constraints and heeding the pricing function. The proposed methods result in significant savings in the energy bill under different usage pricing, and scheduling constraints.

Keywords- Advanced metering infrastructure (AMI), communication technologies, quality-of-service (QoS), smart grid, standards

I. INTRODUCTION

Demand response and dynamic retail pricing in electricity transmission and distribution (T&D) networks have been long advocated for improving system efficiency, mitigating wholesale price volatility, reducing system peak load, and the need to hold excessive reserve capacity. Reducing the annual system peak load lowers the required system-wide maximum capacity and lessens the need for expensive mega power plants (peakers) that are brought online for only a couple of hours per year. The impact on the required level of near real-time reserve capacity margins could be even greater. As the contribution of renewable resources to the supply of electricity grows in magnitude and extent, increased supply-side stochasticity challenges the system operators. System reliability constraints then compel the system operators to maintain additional reserve capacity to deal with the increased uncertainty. Dynamic pricing is one of the mechanisms that could be used for mitigating the effect of these uncertainties by allowing the consumers to react—in their own monetary or environmental interest—to the wholesale market prices, which reflect the real-time fluctuations in the system's capacity. While this real-time price-based coupling between supply and demand, should, at least in principle, mitigate the effects of stochastic fluctuations on the supply side, it challenges system operators in new ways. The first challenge concerns the endogenous uncertainty introduced by the demand side due to uncertainty in consumer behavior, preferences, and reactions to real-time price variations. The second challenge concerns system stability (both price and supply/demand stability) as we will explore in the sequel. Investigating the trade-offs between the endogenous uncertainties induced by consumer behavior and mitigation of exogenous system uncertainties is outside the scope of this paper. Herein, we will focus entirely on system stability and efficiency questions that arise when the retail prices are tied to the spot prices of the wholesale markets. The literature on dynamic pricing in communication or transportation networks—so that certain system objectives are met—is extensive, see for instance [8], [7], [4], [3] and the references therein. However, the specific characteristics of power grids arising from the distinctively close interplay between physics, market operation, and economics, along with the safety-critical nature of the system pose new and unique challenges to be addressed. The implications and consequences of various forms of dynamic pricing such as Critical Peak Pricing, Time-of-Use Pricing, and Real-Time Pricing, have been investigated by many economists and regulatory agencies. They mostly argue in favor of real-time pricing, characterized by passing on a price, that best reflects the wholesale market prices, to the end consumers. See for instance the papers by Borenstein et. al. [2], Hogan [6], technical report by the IEADSMP [5] and the report on California's 2011 deadline for mandatory real-time pricing [14]. In [10], the authors argued that simply relaying the wholesale market spot prices to the end consumers may create system instabilities. In particular, it appears that real-time pricing as defined above, in the absence of well-designed financial instruments for hedging, could potentially aggravate price volatility in wholesale markets. Whether real-time pricing will mitigate, or aggravate wholesale price volatility depends on many factors including implementation details, contract types, and most importantly, the mathematical relations between the cost functions of the producers and the value functions of consumers.

The framework developed in this paper considers the consumer as an autonomous agent myopically adjusting her usage in response to price signals, based on maximization of a quasi-linear smooth concave utility function. It is assumed that supply follows demand precisely, in the following sense: at each instant of time, any amount of electricity demanded by the consumers must be matched by the producers and the per unit price associated with this exchange is the exant´e price corresponding to the marginal cost of supplying the predicted demand. The consumers then (myopically) adjust their usage by maximizing their utility functions for the next time period.

DEMAND RESPONSE, response of distributed generation and operations of electricity storage (e.g. using vehicle to grid) will be a crucial part of power systems management in the future. There are two main reasons why these Distributed Energy Resources (DER)1 will be playing a major role in electricity systems operation in the future:

• The rising share of both renewable energy sources and distributed generation in the energy mix decreases the controllability of the supply side. Utilizing the flexibility potential in demand and distributed generation will mitigate the associated balancing problems.
• The electrification of everything will drive our ageing electricity networks to their limits. As an alternative to grid reinforcements, the flexibility potential in local demand and generation can be used to cope with grid overload situations.

Currently, approaches to integrate DER in system operations are predominantly based on one-way signalling from the utility to the end-customer's devices. In this paper, we argue that this leads to a suboptimal solution for two reasons. Firstly, the approach does not unlock the full DER flexibility potential.

Secondly, In case of mass application, it leads to market inefficiencies and uncertainty in the price response of DER clusters. We argue that, these important drawbacks will not occur when the responsive end-customer's systems are fully integrated in the electricity markets. Further, we describe a smart grid technology that integrates demand and supply flexibility in the operation of the electricity system, rather than just letting it respond. Over the last few years, this technology has been researched and developed into a market ready system, and has been deployed in a number of successful field trials. With this approach demand response moves from influencing, with an uncertain overall response, into control with the dynamic price as a control signal.

A. Prices to devices vs market integration

As discussed in the previous chapter, demand response and response of distributed generation will be crucial for power systems management in the future smart electricity grid. Throughout the world, there are a number of automatic response programs aimed at retail customers, i.e. households small enterprises, etc. These programs are predominantly based on one-way signalling from the utility to consuming devices at the end-customer. This is either direct switching of device groups or dynamic pricing based on a 'prices to devices' approach. There is a number of reasons to choose for such an approach. Among these are: (i) simple one-way communications leading to low system complexity, (ii) no privacy issues, and (iii) the availability of a dynamic price profile from the wholesale market. Of these two approaches, direct switching is the most simple one. This approach comes down to either switching devices 'on' or 'off'. Switching 'on' gives an external start of the operation cycle of devices such as water boilers in homes and ice machines in hotels. This allows the utility to schedule these loads during off-peak hours. However, it ignores user preferences. The switching 'off' category comprehends interrupting the operational cycle of household appliances such as freezers and air conditioning systems, or industrial installations such as freezing houses and circulation pumps. Again, this ignores user preferences. On top of that, this approach totally ignores the state the device is in. In the case of a freezer, for instance, this state is the inner temperature and whether the compressor is running or not. Ignoring the device state has two consequences:

- The exact reaction of the response pool is uncertain as devices may be off already. In these cases, switching 'off' comes down to 'preventing a device to switch on.' In the freezer example, an 'off' signal is received while the compressor is not running at the time.
- The approach is suboptimal as the response potential is used only partially. Due to the unknown device state, the minimal off time needs to be based on a worst case scenario. In the freezer example, the worst case is switching at the moment when the freezer's controller would switch on the compressor because the upper control temperature has been reached. In that case, the freezer can be kept from switching its compressor on until the freezer temperature reaches some safe value below the freezing point. Freezers having a lower temperature at the time of switching could be switched off for a longer time, however, the central system is agnostic of this. The suboptimality in the available response potential can be solved by moving to the price reaction approach. Then, the responsive DER units are not externally switched, but rather respond to an external price signal. Using this price signal, the

operation of the DER device is optimized economically by a local intelligent controller. This controller tries to operate demand during low-priced periods, and supply at high-priced periods, taking the device state and the user preferences into account. However, when using the 'prices-to-devices' approach, the reaction of a demand response pool to a certain price signal is still not known beforehand. Again, the ability of a device to react to a price signal highly depends on the particular state the device is in at that moment. As this information is not known centrally, the reaction of the pool as a whole is unknown. Further, in the long run, when this approach is used for a substantial group of customers, the approach will develop a market inefficiency. In western economies, more than 50% of the electricity is used by retail customers and smaller businessto- business customers located in distribution networks. Here, the (currently unused) potential for automatic response is high, and so will be the influence of this response on the market price of electricity. Consequently, when end-user systems are going to react to market prices on a bigger scale, without participating in these markets, the prices of these markets will not be right anymore. This leads to market inefficiencies. An ad-hoc solution to these two problems is a forecasting model for the demand pool's reaction to a certain price signal. This model is then used for determining the right price to get a desired pool reaction. Note that this type of forecasting of demand response is harder than forecasting of demand patterns for big customer groups. The latter task is done in the current electricity system with an accuracy up to 98%. However, the responsiveness to a price signal of a demand pool depends on the available operational flexibility within the pool. This is highly dependent on the response history of the pool. Consequently, for this forecasting problem it is much harder to reach an acceptable accuracy. So, expectedly, a certain level of uncertainty will remain. Figure 1 positions these approaches relative to two axes: central and local decision making, and one-way versus two way communications. When going from top-down switching to a price response, the decision taking moves from the central to the local level. However, the approach stays on the left side of the axes, where the system reaction is uncertain and the market inefficient. When a way is created to integrate the DER in the electricity markets, one moves to the right side of the figure, where the reaction of the response pool is known beforehand. On top of that, the market prices will be automatically right without the need of forecasting. Currently, only the bigger (production or consumption) capacities are having access to the wholesale electricity markets. As will be shown later on, it is possible to integrate smaller DER capacities in these markets as well. Although our approach is based on 2-way communications, the system is highly scalable and protects privacy sensitive information. As the communication loop starts with the local devices communicating their momentary flexibility and related preferences, the pool reaction to a certain price signal is known beforehand. With this approach demand response moves from influencing, with an uncertain overall response, into market-based control with the dynamic price as a control signal triggering a certain system reaction. Table I gives a comparison of the characteristics of the three approaches.

B. Dynamic energy pricing

With the current emphasis on environmentally friendly solutions alongside the concerns of a recovering economy, dynamic energy pricing may be exploited as an effective means of utilizing green energies while reducing the electricity costs by a significant amount (i.e., by an average of 20%) [1]. Consumers need access to dynamic electricity pricing to reduce greenhouse gas (GHG) emissions and save money on their bills [2]. The Association of Home Appliance Manufacturers (AHAM) released a white paper strongly advocating that "residential electricity prices must be based on time of use" to fully enable smart grid technology [2]. Energy pricing may be classified as two major types: realtime/ dynamic pricing and time-of-use (TOU) pricing. An economic view of real-time and TOU energy pricing has been presented in [3] where it is shown that dynamic pricing is the ideal method to capture the true cost of producing energy [3]. Also, dynamic changes in energy prices provide an incentive for the customer to reduce their energy consumption during "peak" energy-use hours [3]. Since dynamic energy pricing results in a time shift of consumption from peak time to offpeak time, the grid power capacity requirement reduces, which can result in around 10% gain for the whole energy economy [3]. By transitioning to dynamic energy pricing and by providing relevant information to the consumers (e.g., energy consumption comparison with similar households/facilities or the mix of green vs. brown energy sources that are being used by the utility companies to produce electricity at a given time), there will be strong incentives to reduce the overall energy use to reduce cost or to change the energy usage profile to make it more environmentally friendly. It is clear that a time-based pricing is useful when there is a significant difference between usage of peak and off-peak times. This is often the case as indicated, for example, by studies published by the Demand Response Research Center on Automated Critical Peak Pricing [4], which emphasizes the difference in peak, off peak, and "needle peak" energy demands. The time-based pricing model that we use in this paper is a quasi-dynamic pricing model as detailed next. The price of one unit of energy consumption comprises of two parts: (i) A TOU-dependent base price, which is specified in advance,

and captures the slow dynamics of energy usage; an example is the hourly price of a unit of energy consumption in the current day provided the day before, and (ii) A penalty term, which penalizes the users when their peak power consumption over some recent window of time goes above a predetermined TOU-dependent threshold. A smart grid, which delivers electricity from suppliers to consumers, controls appliances at consumers' homes, and provides the required data and applications to optimize energy costs, is a very large and complex system with a lot of sensing/metering, data fusion and information processing as well as intelligent control and decision making built into it. It comprises of at least a physical power generation facility, power distribution, and power delivery infrastructure. To achieve end user energy savings, it is necessary to install power meters, sensors/actuators for appliances and other power consuming resources in a facility (e.g., air conditioning, heating and ventilation, lighting, and machine rooms) to track and control energy consumption. This by itself requires significant upfront investment, planning, and installation work. Hence, only a select number of areas in the country have 978-

II. THE WOLF PRICING CURVE

Figure 3 contains two arrows, one depicting ACE and the other the price pressure associated with the imbalance between supply and demand as measured by ACE. The relation between these arrows can be used to create a WOLF pricing curve, as is shown in Figure 6. When ACE is zero, the WOLF price is equal to the base price, the result of the bid based auction run by the ISO. When ACE is negative, which is to the left of the graph, the WOLF price is above the base price. When ACE is positive, which is to the right of the graph, the WOLF price is below the base price. The WOLF pricing curve produces an estimate of short run marginal costs for those participating in the market place. Generators with a short run marginal cost below the expected WOLF price have an incentive to increase production. As shown on Figure 1, the increase in production will also increase the short run marginal cost of the generator. The incentive to increase production exists until the generator's short run marginal cost approaches the expected WOLF price. The action of increasing generation will also increase ACE. The increase in ACE will, as shown in Figure 6, decrease the WOLF price. This completes a negative feedback loop, as that concept is used in control theory. Thus, the generator's response to the anticipated WOLF price will have a tendency to move the WOLF price toward the generator's short run marginal cost at the same time that the generator's short run marginal cost moves toward the WOLF price. Thus, WOLF pricing for imbalances will consolidate the short run marginal cost of the various generators at the WOLF price while moving the WOLF price toward that consolidation of the short run marginal costs of the various generators.

The WOLF pricing curve plotted as a straight line in Figure 6 is plotted on a dimensionless scale. The slope of the WOLF pricing curve should be steep enough to encourage participants to assist the system operators in moving ACE toward zero. One such shape is an exponential curve, where the equal changes in ACE results in the WOLF price changing by equal multiples, as is demonstrated in (1).

-ACE/Ç 1

PRICEWOLF = PRICEBASE * B

Increasingly, utility operators are also having trouble with power surpluses due to various externalities.

• Government programs

- Tax incentives—Many governments offer producers investment tax credits based on the production from renewable resources. Subtracting these credits from the gross marginal cost of operating a unit can make the net marginal cost negative.
- Renewable energy mandates and credits—Some governments require utilities and other participants in the market to own such generation or buy credits from owners of such generation. Again, subtracting these credits from the gross marginal cost of operating a unit can make the net marginal cost negative.

• Physics

- Generating system inertia—Base loaded generators are notoriously unresponsive to calls for rapid incremental or decremental production changes. Such production changes reduce the efficiency with which the generators operate, much like a jack rabbit starting and stopping reduces the fuel efficiency of a car and increases maintenance costs.
- Cogeneration—Many facilities also produce thermal energy for chemical processes or heating. In some cases the electrical generation is necessary for continued thermal production.

As a result, some bid based auctions have actually resulted in negative bids and settlement prices. These negative bids reflect the actual conditions that the effective short run marginal costs for some generators can be negative, whether permanently in the case of government programs or for short periods of the day in the case of the physics of some generating systems. Certainly the short run marginal cost can be negative when there is a sudden decline in the need for power and generators struggle to reduce their output. Thus, the WOLF pricing curve can be designed to be linear once ACE is slightly positive, resulting in a continuing downward slope and negative prices. ACE will seldom average out to zero over an ISO bidding period. Accordingly,

the average WOLF price during an ISO bidding period will be different from the ISO base price. The size of that difference will depend on the slope chosen for the WOLF pricing curve and the distribution of the ISO ACE. Table 1 shows the ratio of the average WOLF price versus the ISO base price for a variety of WOLF pricing curves using the distribution of ACE on AEMO for the first eight (8) hours of 2009 September 16. In all cases, the B term in (1) is 2.0. Thus, the WOLF price doubles for each "C" decrease in ACE. The middle row is for "C" equal to 0.015 Hertz. Thus, the WOLF price at 49.985 Hertz is twice the base price at 50.000 Hertz. The three columns are for various price decrements starting when ACE is above 50.02 Hertz.

The ratios presented in Table 1 are before any recognition of elasticity effects. Participants in the dispatch of the system will respond to the extreme prices produced by WOLF for unscheduled flows of electricity. This was demonstrated in 2002/2003 while India was implementing a formulary auction for Unscheduled Interchange, as is illustrated in Figure 7. The choice of the "C" term will depend on how tightly the operators wish to control ACE. Fig. 5 was meant to illustrate the situation when a generator produces the assigned amount of energy but not following the standard pattern of a flat top and a flat top. Some generators will not produce the assigned amount of energy, such as generators that do not participate in the bid based auction or do not accept their energy allocation. That may require a different

A. Multi-Stakeholder Field Test

In the previous sections, we argued there is necessity to introduce distributed control in the electricity infrastructure in order to cope with the interrelated trends of increasing sustainable electricity sources and distributed generation. We have shown how a specific implementation of distributed control can be used for commercial portfolio balancing as well as for DSO congestion management. An important remaining question is: how to combine the two? In real-life situations, large networks with many stakeholders involved and having multiple optimization objectives for different scenarios should be expected. Each stakeholder has its own interest and these interests will conflict at certain periods in time. For example, a low price of electricity may stimulate consumption of electricity, but the immediate resulting increase in consumption may overload the grid locally. The key stakeholders are:

- The Prosumer: an end-customer that may be capable of generating electricity by his own means. Such a Prosumer primarily wants to maximise the economic value of their investment in such devices as well as minimize the costs for their consumption of energy;
- The Distribution System Operator (DSO), who operates the grid, wants to limit load fluctuations as much as possible by optimizing the usage of their assets in this way; and
- The Energy Supplier, who trades energy on the wholesale market, delivers electricity to the Prosumers, and buys surplus electricity back from the Prosumers. In the PowerMatching City field test, Multi-stakeholder optimization is being tested under real-life conditions. The field test consists of a cluster of about thirty real life households and two supplemented laboratory sites, resulting in a total of approximately one hundred DER devices. The devices range from micro-CHP, heat pumps, photovoltaic, (urban) wind, household appliances (laundry) as well as plug-in electric cars. Implementation of the coordination is done by creating a Virtual Power Plant (VPP) based on the Power Matcher concept This VPP varies the overall price for the electricity commodity. Congestion management in the electricity grid is handled by network agents that vary the transport tariff locally in order to mitigate network overloading. The latter introduces local differences in the dynamic price. At time of writing, the first phase of testing the VPP functionality is being finalized with promising preliminary results. The network agents are currently under construction and will be thoroughly tested in a later stage.

B. Market Introduction

The first two field tests were mainly focusing on technology aspects. Now that the technology has been thoroughly proven, our aim will switch towards the development of large scale commercial demonstration projects targeting user aspects and product development. These next step demonstrations need to involve all relevant stakeholders and technology providers. One of these large-scale demonstrations will be located at the Danish island of Bornholm as part of the EcoGrid.eu project. For market introduction we are establishing a partnership with a software vendor and system integrator for software product commercialization, dissemination and standardization. Further, we are creating a broad partnership to develop off-theshelf PowerMatcher-Ready products.

Responsiveness of demand, distributed generation and storage will be vitally important for the operation of the future electricity infrastructure. The rising share of renewable energy sources and distributed generation, ageing of electricity networks and the electrification of everything create this urgent need. We argued that current approaches to DER response, which are predominantly based on one-way price signaling from the utility to the end-customer's devices will lead to a suboptimal solution. The "prices-to-devices" approach does not unlock the full DER flexibility potential. Secondly,

it leads to market inefficiencies and uncertainties in the price response of DER clusters when applied on a mass scale. A true market integration of DER will overcome these problems. Then, demand response moves from influencing, with an uncertain overall response, into control with the dynamic price as a control signal. We have shown that full market integration of DER is technically feasible and we have shown how a specific implementation of DER market integration, the PowerMatcher, can be used for commercial portfolio balancing as well as for DSO congestion management.

C. Quadratic Energy Pricing Model

Without of loss of generality, a smooth differentiable quadratic cost function from [27] and [28] is used to model the utility charge in terms of the cumulative usage. The model is expressed as (14) where represents the cumulative electricity consumed by the customer, while and are model parameters. This model shows that the total cost, , is proportional to the squared cumulative electricity. Thus it reflects the dynamic charges on the actual usage. If we want to incorporate the pricing elasticity, is simply changed into where represents the different price function at different time interval .

D. Reward Function for Mean and Variance

Customer incentives should be devised based on how close the demand input is relatively close to the actual usage. The closeness can be measured by three quality indexes: mean, variance, and correlation coefficient. The following model is formulated in such a way that a customer will obtain a higher

E. Bayesian Load Forecasting

Many load prediction models [22]–[25] have been developed based on probabilistic theories, simulation techniques, and neural networks. These methods are able to obtain an accurate load prediction based on historical data combined with some external factors (e.g., environment or weather conditions). The OPEN system makes it possible that the load prediction can be made by synthesizing customer orders with historical consumptions as well as environmental conditions. Using the Bayesian model, it is anticipated that a more robust prediction can be made if customers' demand orders are incorporated into the forecasting process. For Bayesian prediction models, readers are referred to the Appendix.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r) = \sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\ \qquad\qquad + I_i^{ext}(r,t), \qquad t \geq 0, 1 \leq i \leq p, \\ V_i(t,r) = \phi_i(t,r) \qquad\qquad t \in [-T,0] \end{cases} \tag{1}$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d . The vector r and $\bar{r}$ represent points in Ω . The function $S: R \to (0,1)$ is the normalized sigmoid function:

$$S(z) = \frac{1}{1+e^{-z}} \tag{2}$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i = 1,..., p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i = 1,..., p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p \times p$ matrix of functions $J = \{J_{ij}\}_{i,j=1,...,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i = 1,..., p,$ determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i = 1,..., p,$ determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i = 1,..., p,$ determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S: R^p \to R^p$, defined by $S(x) = [S(\sigma_1(x_1 - h_1)),..., S(\sigma_p - h_p))]$, and the diagonal $p \times p$ matrix $L_0 = diag(l_1,...,l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt}+l_i)$ is replaced by $(\frac{d}{dt}+l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt}+l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of

generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\overline{\Omega^2}, R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r}) \neq \tau_{ij}(\bar{r},r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,r\in\Omega\times\Omega)} \tau_{i,j}(r,\bar{r}) \qquad (3)$$

Hence we choose $T = \tau_m$

F. Mathematical Framework

A convenient functional setting for the non-delayed packet field equations is to use the space $F = L^2(\Omega, R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U\rangle_F = \sum_{i=1}^{p}\int_\Omega V_i(r)U_i(r)dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m,0],F)$ with $\|\phi\| = \sup_{t\in[-\tau_m,0]}\|\phi(t)\|F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t+\theta), \theta\in[-\tau_m,0]$, we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0V(t) + L_1S(V_t) + I^{ext}(t), \\ V_0 = \phi \in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1 : C \to F, \\ \phi \to \int_\Omega J(.,\bar{r})\phi(\bar{r},-\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \le \|J\|_{L^2(\Omega^2,R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $J \in L^2(\Omega^2, R^{p\times p})$,
2. The external current $I^{ext} \in C^0(R,F)$,
3. $\tau \in C^0(\overline{\Omega^2}, R_+^{p\times p}), \sup_{\overline{\Omega^2}} \tau \le \tau_m$.

Then for any $\phi \in C$, there exists a unique solution $V \in C^1([0,\infty),F)\cap C^0([-\tau_m,\infty,F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

G. *Boundedness of Solutions*

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t\in R^+}\|I^{ext}(t)\|_F < \infty$.

Proof :Let us defined $f : R\times C \to R^+$ as

$$f(t,V_t) \overset{def}{=} \langle -L_0V_t(0) + L_1S(V_t) + I^{ext}(t), V(t)\rangle_F = \frac{1}{2}\frac{d\|V\|_F^2}{dt}$$

We note $l = \min_{i=1,\dots p} l_i$

$$f(t,V_t) \le -l\|V(t)\|_F^2 + (\sqrt{p|\Omega|}\,\|J\|_F + I)\|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F \ge 2\frac{\sqrt{p|\Omega|}.\|J\|_F + I}{l} \overset{def}{=} R,\ f(t,V_t) \le -\frac{lR^2}{2} \overset{def}{=} -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \ge 0s$ and that $f < 0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C < R$ and set $T = \sup\{t \mid \forall s\in[0,t], V(s) \in \overline{B_R}\}$. Suppose that $T \in R$, then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have $\frac{d}{dt}\|V\|_F^2 |_{t=T} = f(T,V_T) \le -\delta < 0$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and

small enough, $V(T+\varepsilon)\in\overline{B_R}$ which contradicts the definition of T. Thus $T\notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0)\in\partial B_R$ implies that $\forall t>0, V(t)\in B_R$. Finally we consider the case $V(0)\in C\overline{B_R}$. Suppose that $\forall t>0, V(t)\notin\overline{B_R},$ then $\forall t>0, \frac{d}{dt}\|V\|_F^2\le -2\delta,$ thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T>0\,|\,V(T)\in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on X. for $E\varepsilon M$, define

$$\phi(E)=\int_E s\,d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X (s+t)d\mu=\int_X s\,d\mu+\int_X t d\mu \qquad (2)$$

Proof : If s and if $E_1, E_2, \ldots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E)=\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E)=\sum_{i=1}^{n}\alpha_i\sum_{r=1}^{\infty}\mu(A_i\cap E_r)$$

$$=\sum_{r=1}^{\infty}\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E_r)=\sum_{r=1}^{\infty}\phi(E_r)$$

Also, $\varphi(\phi)=0,$ so that φ is not identically ∞.

Next, let s be as before, let $\beta_1,\ldots,\beta_m$ be the distinct values of t,and let $B_j=\{x: t(x)=\beta_j\}$ If $E_{ij}=A_i\cap B_j,$ the $\int_{E_{ij}}(s+t)d\mu=(\alpha_i+\beta_j)\mu(E_{ij})$

and $\int_{E_{ij}} s d\mu+\int_{E_{ij}} t d\mu=\alpha_i\mu(E_{ij})+\beta_j\mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1\le i\le n, 1\le j\le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon>0,$ then there exists a polynomial P such that $|f(z)=P(z)|<\varepsilon$ for all $z\varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f\varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta>0,$ let $\omega(\delta)$ be the supremum of the numbers $|f(z_2)-f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2-z_1|\le\delta$. Since f is uniformly continous, we have $\lim_{\delta\to 0}\omega(\delta)=0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z)-P(z)|<10{,}000\ \omega(\delta)\quad (z\varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi\varepsilon C_c^{'}(R^2),$ such that for all z

$$|f(z)-\Phi(z)|\le\omega(\delta), \qquad (3)$$

$$|(\partial\Phi)(z)|<\frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z)=-\frac{1}{\pi}\iint_X\frac{(\partial\Phi)(\zeta)}{\zeta-z}d\zeta d\eta \qquad (\zeta=\xi+i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r)=0$ if $r>\delta,$ put

$$a(r)=\frac{3}{\pi\delta^2}\left(1-\frac{r^2}{\delta^2}\right)^2 \qquad (0\le r\le\delta), \qquad (6)$$

And define

$$A(z)=a(|z|) \qquad (7)$$

For all complex z. It is clear that $A\varepsilon C_c^{'}(R^2)$. We claim that

$$\iint_{R^s} A = 1, \qquad (8)$$

$$\iint_{R^2} \partial A = 0, \qquad (9)$$

$$\iint_{R^3} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A / \partial\theta = 0$,

$$\partial A / \partial r = -a',$$

Now define

$$\Phi(z) = \iint_{R^2} f(z-\zeta) A d\xi d\eta = \iint_{R^2} A(z-\zeta) f(\zeta) d\xi d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z) = \iint_{R^2} [f(z-\zeta) - f(z)] A(\xi) d\xi d\eta \quad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A \varepsilon C_c'(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial\Phi)(z) = \iint_{R^2} (\overline{\partial A})(z-\zeta) f(\zeta) d\xi d\eta$$

$$= \iint_{R^2} f(z-\zeta)(\partial A)(\zeta) d\xi d\eta$$

$$= \iint_{R^2} [f(z-\zeta) - f(z)](\partial A)(\zeta) d\xi d\eta \qquad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi = 0$ in G, where G is the set of all $z \varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z \varepsilon G); \qquad (14)$$

Note that $\partial f = 0$ in G, since f is holomorphic there. Now if $z \varepsilon G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z) = \int_0^{\delta} a(r) r dr \int_0^{2\pi} f(z - re^{i\theta}) d\theta$$

$$= 2\pi f(z) \int_0^{\delta} a(r) r dr = f(z) \iint_{R^2} A = f(z) \qquad (15)$$

For all $z \varepsilon G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1, \ldots, D_n$, of radius 2δ, whose centers are not in K. Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that $S^2 - E_j$ is connected and so that $K \cap E_j = \phi$. with $r = 2\delta$. There are functions $g_j \varepsilon H(S^2 - E_j)$ and constants b_j so that the inequalities.

$$|Q_j(\zeta, z)| < \frac{50}{\delta}, \qquad (16)$$

$$\left| Q_j(\zeta, z) - \frac{1}{z-\zeta} \right| < \frac{4{,}000\delta^2}{|z-\zeta|^2} \qquad (17)$$

Hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j) g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1 \cup \ldots \cup E_n$. Then Ω is an open set which contains K. Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup \ldots \cup X_{j-1})$, for $2 \le j \le n$,

Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta \varepsilon X_j, z \varepsilon \Omega) \qquad (19)$$

And

$$F(z) = \frac{1}{\pi} \iint_X (\partial\Phi)(\zeta) R(\zeta, z) d\zeta d\eta \qquad (20)$$

$$(z \varepsilon \Omega)$$

Since,

$$F(z) = \sum_{j=1} \frac{1}{\pi} \iint_{X_i} (\partial\Phi)(\zeta) Q_j(\zeta, z) d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F \varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$|F(z)-\Phi(z)| < \frac{2\omega(\delta)}{\pi\delta}\iint_X |R(\zeta,z)| \left|-\frac{1}{z-\zeta}\right| d\xi d\eta \quad (z\ \varepsilon\ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta\ \varepsilon\ X$ and $z\ \varepsilon\ \Omega$. Now fix $z\ \varepsilon\ \Omega.$, put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \le \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta}+\frac{1}{\rho}\right)\rho d\rho = 808\pi\delta \qquad (23)$$

And

$$2\pi\int_{4\delta}^{\infty}\frac{4,000\delta^2}{\rho^2}\rho d\rho = 2,000\pi\delta. \qquad (24)$$

Hence (22) yields

$$|F(z)-\Phi(z)| < 6,000\omega(\delta) \qquad (z\ \varepsilon\ \Omega) \quad (25)$$

Since $F\ \varepsilon\ H(\Omega),\ K \subset \Omega,$ and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f \varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2}\frac{(\partial f)(\zeta)}{\zeta - z}d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta) = f(z + re^{i\theta}),\ r > 0,\ \theta$ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0,$ of

$$-\frac{1}{2}\int_{\varepsilon}^{\infty}\int_0^{2\pi}\left(\frac{\partial\varphi}{\partial r} + \frac{i}{r}\frac{\partial\varphi}{\partial\theta}\right)d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi / \partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{\varepsilon}^{\infty}\frac{\partial\varphi}{\partial r}dr = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon,\theta)d\theta \qquad (5)$$

As $\varepsilon \to 0,\ \varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^{\alpha} \in a$ and $X^{\beta} \in k[X_1,\ldots X_n]$, then $X^{\alpha}X^{\beta} = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$(\sum_{\alpha\in A} c_{\alpha}X^{\alpha})(\sum_{\beta\in \square^n} d_{\beta}X^{\beta}) = \sum_{\alpha,\beta} c_{\alpha}d_{\beta}X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^{\alpha}, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,\ldots X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals $(X^n), n \ge 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^{\alpha} \mid \alpha \in A\rangle$ for the ideal corresponding to A **(subspace generated by the** $X^{\alpha}, \alpha \in a$**).**

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^{\alpha}, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{\beta \in \square^n \mid \beta - \alpha \in \square^n, \quad some\ \alpha \in S\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^{\alpha}, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^{\beta} \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^{\beta} = X^{\alpha}X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^{\alpha} \mid X^{\beta} \Leftrightarrow \beta - \alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,\ldots\alpha_s\}$ of A such that

$A=\{\beta\in\mathbb{Z}^n \mid \beta-\alpha_i\in\mathbb{N}^2,\ some\ \alpha_i\in S\}$

(The α_i's are the corners of A) Moreover, $a\overset{df}{=}\langle X^\alpha \mid \alpha\in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i\in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by $\{LT(f)\mid f\in a\}$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n\in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a=(g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f\in a$. On applying the division algorithm, we find $f=a_1g_1+...+a_sg_s+r, \quad a_i,r\in k[X_1,...,X_n]$, where either $r=0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r=f-\sum a_ig_i\in a$, and therefore $LT(r)\in LT(a)=(LT(g_1),...,LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r=0$, and $g\in(g_1,...,g_s)$.

DEFINITION 1.1. A finite subset $S=\{g_1,|...,g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1),...,LT(g_s))=LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1,...,X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n=1,\ k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1,...X_{n-1}][X_n]\to k[X_1,...X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1,...X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,...,X_n]$:

$$f(X_1,...X_n)=a_0(X_1,...X_{n-1})X_n^r+...+a_r(X_1,...X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X)=a_0X^r+a_1X^{r-1}+...+a_r, \qquad a_i\in A, \qquad a_0\neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1,...,g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f\in a,$ and suppose f has degree $s>r$, say, $f=aX^s+...$ Then $a\in a'$, and so we can write

$$a=\sum b_ia_i, \qquad b_i\in A,$$

$a_i=$ *leading coefficient of* g_i

Now

$f-\sum b_ig_iX^{s-r_i}, \quad r_i=\deg(g_i),$ has degree $<\deg(f)$. By continuing in this way, we find that $f\equiv f_t \mod(g_1,...g_m)$ With f_t a polynomial of degree $t<r$. For each $d<r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1},...,g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same

argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \mod(g_{d,1},...g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1},...g_{r-1,m_{r-1}},...g_{0,1},...g_{0,m_0})$ Hence

$$f_t \in (g_1,...g_m g_{r-1,1},...g_{r-1,m_{r-1}},...,g_{0,1},...,g_{0,m_0})$$

and so the polynomials $g_1,...,g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x.x$ is most conveniently defined as a function from $D to D$, which must then be regarded as an element of *D*. Let $\psi:[D \to D] \to D$ be the function that picks out elements of *D* to represent elements of $[D \to D]$ and $\phi: D \to [D \to D]$ be the function that maps elements of *D* to functions of *D*. Since $\psi(f)$ is intended to represent the function f as an element of *D*, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D \to D]}$ Furthermore, we often want to view every element of *D* as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that $\phi and \psi$ are inverses--- that is, *D* is isomorphic to the space of functions from *D to D* that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of *C*. Each element of *D* corresponds to either an element of *A* or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains *X* such that $X \cong A + [X \to X]$, and such that for any domain *Y* also satisfying this equation, there is an embedding of *X* to *Y* --- a pair of maps

$$X \underset{f^R}{\overset{f}{\rightleftarrows}} Y$$

Such that

$$f^R o f = id_X$$

$$f o f^R \subseteq id_Y$$

Where $f \subseteq g$ means that $f\ approximates\ g$ in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering *F* not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, *F*.

Definition 1.3: Let ***K*** be a category and $F: K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a ***K-object*** and $a: F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a ***K-object*** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category ***K*** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$$

Recall that a cocone μ of an $\omega - chain$ Δ is a *K*-object *X* and a collection of K –*arrows* $\{\mu_i : D_i \to X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} o f_i$ for all $i \geq 0$. We sometimes write $\mu: \Delta \to X$ as a reminder of the arrangement of $\mu's$ components Similarly, a colimit $\mu: \Delta \to X$ is a cocone with the property that if $\nu: \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k: X \to X'$ such that for all $i \geq 0,, \nu_i = k o \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - colimits$. Dually, an $\omega^{op} - chain$ in ***K*** is a diagram of the following form:

$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} \ldots\ldots$ A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain$ Δ is a ***K***-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \geq 0\}$ such that for all $i \geq 0,\ \mu_i = f_i \, o \, \mu_{i+1}$. An ω^{op}-limit of an $\omega^{op} - chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \geq 0, \mu_i \, o \, k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of ***K,*** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each ***K***-object A. It is also convenient to write $\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots\ldots$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under ***F*** we write

$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} \ldots\ldots$

and $F(\mu) = \{F(\mu_i) \mid i \geq 0\}$

We write F^i for the ***i***-fold iterated composition of ***F*** – that is, $F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let ***K*** be a category with initial object $\perp$ and let $F : K \to K$ be a functor. Define the $\omega - chain \, \Delta$ by

$\Delta = \perp \xrightarrow{!\perp \to F(\perp)} F(\perp) \xrightarrow{F(!\perp \to F(\perp))} F^2(\perp) \xrightarrow{F^2(!\perp \to F(\perp))} \ldots\ldots\ldots$

If both $\mu : \Delta \to D$ and $F(\mu) : F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d : F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^-

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, \ldots\ldots\ldots X_n$ be the resultant ordering. Next define.

$P(x_1, x_2, \ldots x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1}) \ldots$

$\ldots P(x_2 \mid pa_2) P(x_1 \mid pa_1),$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \leq P(x_1, x_2, \ldots x_n) \leq 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \leq k \leq n$ that whenever

$P(pa_k) \neq 0, if \;\; P(nd_k \mid pa_k) \neq 0$

$and \;\; P(x_k \mid pa_k) \neq 0$

$then \; P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$ND_k = \{X_1, X_2, \ldots . X_{k-1}\}$

Let

$D_k = \{X_{k+1}, X_{k+2}, \ldots . X_n\}$

follows $\sum_{d_k}$

We define the m^{th} *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ has degree $\varphi(m)$. *The roots of* $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$
$1 \le k \prec m, (k,m) = 1, \quad where$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another,or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n, then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. Since $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6 If m and n are relatively prime, then
$$Q(\xi_m, \xi_n) = Q(\xi_{nm})$$
and
$$Q(\xi_m) \cap Q(\xi_n) = Q$$
(Recall the $Q(\xi_m, \xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m \xi_n$ is a primitive mn^{th} root of unity, so that
$Q(\xi_{mn}) \subseteq Q(\xi_m, \xi_n)$
$[Q(\xi_m, \xi_n) : Q] \le [Q(\xi_m) : Q][Q(\xi_n : Q]$
$= \varphi(m)\varphi(n) = \varphi(mn);$
Since $[Q(\xi_{mn}) : Q] = \varphi(mn);$ this implies that $Q(\xi_m, \xi_n) = Q(\xi_{nm})$ We know that $Q(\xi_m, \xi_n)$ has degree $\varphi(mn)$ over Q, so we must have
$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(n)$$
and
$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(m)$$

$$[Q(\xi_m) : Q(\xi_m) \cap Q(\xi_n)] \ge \varphi(m)$$
And thus that $Q(\xi_m) \cap Q(\xi_n) = Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m, \xi_n) = Q(\xi_{[m,n]})$$
And
$$Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)});$$
here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m = p_1^{e_1} \ldots\ldots p_k^{e_k}$ *and* $p_1^{f_1} \ldots . p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)
$$Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}}) \ldots Q(\xi_{p_k^{e_k}})$$
and
$$Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}}) \ldots Q(\xi_{p_k^{f_k}})$$
Thus
$$\begin{aligned} Q(\xi_m, \xi_n) &= Q(\xi_{p_1^{e_1}}) \ldots\ldots\ldots Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}}) \ldots Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}}) \ldots Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{\max(e_1, f_1)}}) \ldots\ldots\ldots Q(\xi_{p_1^{\max(e_k, f_k)}}) \\ &= Q(\xi_{p_1^{\max(e_1, f_1)} \ldots\ldots\ldots p_1^{\max(e_k, f_k)}}) \\ &= Q(\xi_{[m,n]}); \end{aligned}$$

An entirely similar computation shows that $Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as
$$I(x_i, y_i) = \log_2 \frac{P(x_i / y_i)}{P(x_i)} \; bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input –output pair (x_i, y_i) for which $P(x_i/y_j)=1$ *and* $I(x_i, y_j)=\log_2 \frac{1}{P(x_i)}$ bits; that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j)=P(x_i)$ and also $I(x_i, y_j)=0$; that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y)=\sum_{i.j} P(x_i, y_j) I(x_i, y_j)=\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i, y_j)=P(x_i/y_j)P(y_j)=P(y_j/x_i)P(x_i)$$

$$P(y_j)=\sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i)=\sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y)=\sum_{i.j} P(x_i, y_j)$$

$$=\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$-\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i/y_j)}\right]$$

$$\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$=\sum_i\left[P(x_i/y_j)P(y_j)\right]\log_2\frac{1}{P(x_i)}$$

$$\sum_i P(x_i)\log_2\frac{1}{P(x_i)}=H(X)$$

$$I(X,Y)=H(X)-H(X/Y)$$

Where $H(X/Y)=\sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i/y_j)}$ is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X)-H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j)=P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X,Y)=I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y)=H(Y)-H(Y/X)$$

Where

$$H(Y)=\sum_j P(y_j)\log_2\frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol

after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i,y_j)\log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i,y_j)\log_2 \frac{P(x_i,y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i,y_j)\frac{P(x_i)P(y_j)}{P(x_i,y_j)} \le 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i \log_2\left(\frac{Q_i}{P_i}\right) \le 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \le 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i,y_j)\log_2 \frac{1}{P(x_i,y_j)}$$

$$= \sum_{i,j} P(x_i,y_j)\log_2 \frac{P(x_i)P(y_j)}{P(x_i,y_j)}$$

$$+ \sum_{i,j} P(x_i,y_j)\log_2 \frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ *and* $P(x_2) = 1-\alpha,$ and transition probabilities

$P(y_3/x_2) = 1-p$ *and* $P(y_2/x_1) = 0,$

and $P(y_3/x_1) = 0$

and $P(y_1/x_2) = p$

and $P(y_3/x_2) = 1-p$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean *n*-space. $p(y \mid x)$ for the density $p_n(y_1,...,y_n \mid x_1,...x_n)$ and F *for* F_n. For any real number a, let

$$A = \left\{(x,y) : \log\frac{p(y \mid x)}{p(y)} > a\right\} \qquad (1)$$

Then for each positive integer u, there is a code (u,n,λ) such that

$$\lambda \le ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \qquad (2)$$

Where

$P\{(X,Y) \in A\} = \int_A ... \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y \mid x)$

and

$$P\{X \in F\} = \int_F ... \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$$P\{Y \in A_{x^1} \mid X = x^{(1)}\} \ge 1-\varepsilon$$

where $A_x = \{y : (x,y) \varepsilon A\};$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)},.....,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\left\{Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i \mid X = x^{(k)}\right\} \ge 1-\varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets $B_i, i = 1,2,...,u,$ form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \ge u$ by showing that $\varepsilon \le te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$B = \bigcup_{j=1}^{t} B_j$. *(If* $t = 0,$ *take* $B = \phi$*). Then*

$$P\{(X,Y) \in A\} = \int_{(x,y)\in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y \mid x)dy\,dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y \mid x)dy\,dx + \int_x p(x)$$

H. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$a \in a, r \in A \Rightarrow ra \in A$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1, ..., s_m\}$, we shall write $(s_1, ..., s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b \mid a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab \mid a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_n)$, then $ab = (a_1 b_1, ..., a_i b_j, ..., a_m b_n)$ .Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a \mapsto a + a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p \neq A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \neq 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \neq| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and $(a,b) \in c$, then $(a,0) = (a,b)(1,0) \in c$ and $(0,b) = (a,b)(0,1) \in c$. This shows that $c = a \times b$ with

$a = \{a \mid (a,b) \in c \ \ some \ \ b \in b\}$

and

$b = \{b \mid (a,b) \in c \ \ some \ \ a \in a\}$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A \to B$. A *homomorphism of* A -algebra $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1, ..., x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1, ..., X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k -algebra. If $1 \neq 0$ in A, then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$. **Polynomial rings.** Let k be a field. A *monomial* in $X_1, ..., X_n$ is an expression of the form $X_1^{a_1} ... X_n^{a_n}, \quad a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^{α}, $\alpha = (a_1, ..., a_n) \in \square^n$. The elements of the polynomial ring $k[X_1, ..., X_n]$ are finite sums $\sum c_{a_1 ....a_n} X_1^{a_1} ... X_n^{a_n}, \quad c_{a_1 ....a_n} \in k, \quad a_j \in \square$ With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1, ..., X_n]$ as a k -vector space. The ring $k[X_1, ..., X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1, ..., X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or $\deg r < \deg g$. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) ***lexicographic*** *ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha=(a_1,...a_n)$ and $\beta=(b_1,...b_n)$ be two elements of $\mathbb{N}^n$; then $\alpha>\beta$ *and* $X^\alpha>X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha-\beta\in\mathbb{N}$, the left most nonzero entry is positive. For example, $XY^2>Y^3Z^4;\quad X^3Y^2Z^4>X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha>\beta$ if $\sum a_i>\sum b_i$, or $\sum a_i=\sum b_i$ and in $\alpha-\beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7>X^5Y^5Z^4$ *(total degree greater)*

$XY^5Z^2>X^4YZ^3,\quad X^5YZ>X^4YZ^2$.

Orderings on $k[X_1,...X_n]$. Fix an ordering on the monomials in $k[X_1,...X_n]$. Then we can write an element f of $k[X_1,...X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$f=4XY^2Z+4Z^2-5X^3+7X^2Z^2$

as

$f=-5X^3+7X^2Z^2+4XY^2Z+4Z^2\quad(lex)$

or

$f=4XY^2Z+7X^2Z^2-5X^3+4Z^2\quad(grevlex)$

Let $\sum a_\alpha X^\alpha\in k[X_1,...,X_n]$, in decreasing order:

$f=a_{\alpha_0}X^{\alpha_0}+_{\alpha_1}X^{\alpha_1}+...,\qquad\alpha_0>\alpha_1>...,\quad\alpha_0\neq0$

Then we define.

- The *multidegree* of f to be multdeg(f)= α_0;
- The *leading coefficient of* f to be *LC*(f)= a_{α_0};
- The *leading monomial of* f to be *LM*(f) = X^{α_0};
- The *leading term of* f to be *LT*(f) = $a_{\alpha_0}X^{\alpha_0}$

For the polynomial $f=4XY^2Z+...,$ the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in $k[X_1,...X_n]$.** Fix a monomial ordering in $\mathbb{N}^2$. Suppose given a polynomial f and an ordered set $(g_1,...g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,...a_s$ and r such that $f=a_1g_1+...+a_sg_s+r$ Where either $r=0$ or no monomial in r is divisible by any of $LT(g_1),...,LT(g_s)$ **Step 1:** If $LT(g_1)\,|\,LT(f)$, divide g_1 into f to get

$$f=a_1g_1+h,\qquad a_1=\frac{LT(f)}{LT(g_1)}\in k[X_1,...,X_n]$$

If $LT(g_1)\,|\,LT(h)$, repeat the process until $f=a_1g_1+f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f=a_1g_1+...+a_sg_s+r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),...LT(g_s)$

Step 2: Rewrite $r_1=LT(r_1)+r_2$, and repeat Step 1 with r_2 for f : $f=a_1g_1+...+a_sg_s+LT(r_1)+r_3$ (different a_i's) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a=(Y^2-X^3)$ contains Y^2-X^3 but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if $\sum c_\alpha X^\alpha\in a\Rightarrow X^\alpha\in a$ all α with $c_\alpha\neq0$.

PROPOSITION 1.3. Let a be a *monomial ideal*, and let $A=\{\alpha\,|\,X^\alpha\in a\}$. Then A satisfies the condition $\alpha\in A,\ \beta\in\mathbb{N}^n\Rightarrow\alpha+\beta\in$ (*) And a is the k-subspace of $k[X_1,...,X_n]$ generated by the $X^\alpha,\alpha\in A$. Conversely, of A is a subset of $\mathbb{N}^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,...,X_n]$ generated by $\{X^\alpha\,|\,\alpha\in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,...,X_n]$ generated by the set of monomials it contains. If $X^\alpha\in a$ and $X^\beta\in k[X_1,...,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)}, ..., C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!} N(n,c) = 1\left\{\sum_{j=1}^{n} jc_j = n\right\} \prod_{j=1}^{n} \left(\frac{1}{j}\right)^{c_j} \frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \mathbb{Z}_+^n$.

Lemma1.7 For nonnegative integers $m_{1,...,} m_n$,

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = \left(\prod_{j=1}^{n} \left(\frac{1}{j}\right)^{m_j}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]} / c_j^{!} = 1/(c_j - m_j)!$ when $c_j \ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum jm_j$. Then, with the first sum indexed by $c = (c_1, ...c_n) \in \mathbb{Z}_+^n$ and the last sum indexed by $d = (d_1, ..., d_n) \in \mathbb{Z}_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = \sum_{c} P[C^{(n)} = c] \prod_{j=1}^{n} (c_j)^{[m_j]}$$

$$= \sum_{c: c_j \ge m_j \text{ for all } j} 1\left\{\sum_{j=1}^{n} jc_j = n\right\} \prod_{j=1}^{n} \frac{(c_j)^{[m_j]}}{j^{c_j} c_j!}$$

$$= \prod_{j=1}^{n} \frac{1}{j^{m_j}} \sum_{d} 1\left\{\sum_{j=1}^{n} jd_j = n - m\right\} \prod_{j=1}^{n} \frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m \le n)$, corresponding to the fact that if $n - m \ge 0$, then $d_j = 0$ for $j > n - m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1, ..., d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]} = j^{-r} 1\{jr \le n\} \qquad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = E\left(\prod_{j=1}^{n} Z_j^{[m_j]}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\}, \qquad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1, 2, ...n\}$, so that $|I| = n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha| = (n - j)!$, since the elements of $\{1, 2, ..., n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n - rj)!1(rj \le n)$ permutations in the intersection. There are $n^{[rj]} / (j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n - rj)!1(rj \le n)$$

$$\times \frac{n^{[rj]}}{j^r r!} \frac{1}{n!} = 1(rj \le n) \frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \ge 0} (-1)^l \binom{k+l}{l} S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0, 1, ..., n$,

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j=1.$ In particular, for $n\geq 2,$ the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)},...,C_b^{(n)})$ for any $1\leq b\leq n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c=(c_1,...,c_b)\in \square_+^b$ with $m=\sum ic_i,$

$$P[(C_1^{(n)},...,C_b^{(n)})=c]$$

$$=\left\{\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{c_i}\frac{1}{c_i!}\right\}\sum_{\substack{l\geq 0 \text{ with}\\ \sum il_i\leq n-m}}(-1)^{l_1+...+l_b}\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{l_i}\frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)},...,C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1}=...=m_n=0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed $j,$ as $n\to\infty,$

$$P[C_j^{(n)}=k]\to\frac{j^{-k}}{k!}e^{-1/j},\quad k=0,1,2,...,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j;$ we use the notation $C_j^{(n)}\to_d Z_j$ where $Z_j\square P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\square$ with intensity j^{-1}. That is, as $n\to\infty,$

$$(C_1^{(n)},C_2^{(n)},...)\to_d(Z_1,Z_2,...) \qquad (1.1)$$

Where the $Z_j, j=1,2,...,$ are independent Poisson-distributed random variables with $E(Z_j)=\frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b\geq 1,$ as $n\to\infty,$

$$P[(C_1^{(n)},...,C_b^{(n)})=c]\to P[(Z_1,...,Z_b)=c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b=1$. Using properties of alternating series with decreasing terms, for $k=0,1,...,n,$

$$\frac{1}{k!}\left(\frac{1}{(n-k+1)!}-\frac{1}{(n-k+2)!}\right)\leq\left|P[C_1^{(n)}=k]-P[Z_1=k]\right|$$

$$\leq\frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!}\frac{n}{n+2}\leq\sum_{k=0}^{n}\left|P[C_1^{(n)}=k]-P[Z_1=k]\right|\leq\frac{2^{n+1}-1}{(n+1)!} \qquad (1.11)$$

Since

$$P[Z_1>n]=\frac{e^{-1}}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+...\right)<\frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $\mathrm{P}\left[A_n(C^{(n)})\right]$ under conditions (A_0) and $(B_{01}),$ where

$$A_n(C^{(n)})=\bigcap_{1\leq i\leq n}\ \bigcap_{r_i'+1\leq j\leq r_i}\left\{C_{ij}^{(n)}=0\right\},$$

and $\zeta_i=(r_i'/r_{id})-1=O(i^{-g'})$ as $i\to\infty,$ for some $g'>0.$ We start with the expression

$$P[A_n(C^{(n)})]=\frac{P[T_{0m}(Z')=n]}{P[T_{0m}(Z)=n]}$$

$$\prod_{\substack{1\leq i\leq n\\ r_i'+1\leq j\leq r_i}}\left\{1-\frac{\theta}{ir_i}(1+E_{i0})\right\} \qquad (1.1)$$

$$P[T_{0n}(Z')=n]$$

$$=\frac{\theta d}{n}\exp\left\{\sum_{i\geq 1}[\log(1+i^{-1}\theta d)-i^{-1}\theta d]\right\}$$

$$\left\{1+O(n^{-1}\varphi'_{\{1,2,7\}}(n))\right\} \qquad (1.2)$$

and

$$P[T_{0n}(Z')=n]$$

$$=\frac{\theta d}{n}\exp\left\{\sum_{i\geq 1}[\log(1+i^{-1}\theta d)-i^{-1}\theta d]\right\}$$

$$\left\{1+O(n^{-1}\varphi_{\{1,2,7\}}(n))\right\} \qquad (1.3)$$

Where $\varphi'_{\{1,2,7\}}(n)$ refers to the quantity derived from Z'. It thus follows that $P[A_n(C^{(n)})]\square Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the r_i' and computable

explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\varphi'_{\{1,2,7\}}(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \le b \le n/8$ and $n \ge n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$
$$\le d_{TV}(L(\bar{C}[1,b]), L(\bar{Z}[1,b]))$$
$$\le \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0),(D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\bar{C}[1,b] \mid T_{0b}(C) = l) = L(\bar{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$$d_{TV}(L(\bar{C}[1,b]), L(\bar{Z}[1,b]))$$
$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$
$$= \max_A \sum_{r \in A} P[T_{0b}(Z) = r]$$
$$\left\{1 - \frac{P[T_{bn}(Z) = n-r]}{P[T_{0n}(Z) = n]}\right\} \qquad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\bar{C}[1,b]), L(\bar{Z}[1,b]))$$
$$= \sum_{r \ge 0} P[T_{0b} = r]\left\{1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]}\right\}_+$$
$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]}$$
$$\times \left\{\sum_{s=0}^{n} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r]\right\}_+$$
$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r]$$
$$\times \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{\{P[T_{bn} = n-s] - P[T_{bn} = n-r]\}}{P[T_{0n} = n]}$$
$$+ \sum_{s=0}^{[n/2]} P[T_{0b} = r] \sum_{s=[n/2]+1}^{n} P[T = s] P[T_{bn} = n-s] / P[T_{0n} = n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2 < s \le n} P[T_{0b} = s]) / P[T_{0n} = n]$$
$$\le \frac{2\varepsilon_{\{10.5(1)\}}(n/2, b)}{n} \frac{3n}{\theta P_\theta[0,1]},$$
$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n) \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{1}{2}|r-s|$$
$$\le \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{1 + \frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\} P$$
$$+ \frac{6}{\theta P_\theta[0,1]} \varepsilon_{\{10.5(1)\}}(n/2, b) \qquad (1.5)$$

Required order under Conditions $(A_0),(D_1)$ and (B_{11}), if $S(\infty) < \infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty) < \infty$. Examining the Conditions $(A_0),(D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l \ge 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1 > 1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\bar{n}_1$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l \ge 2$, than are made in (B_{11}). The

critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)} = s] - P[T_{bn}^{(m)} = s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from n_1 of the form $\phi_1^{\theta}(n) + u_1^*(n)$, which is only small if $a_1 > 1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta > 0$, since $a_2 > 1$ is in any case assumed. For $s \geq n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn} = s] - P[T_{bn} = s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn} = s] - P[T_{bn} = s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta > 0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w) = 1_{\{w=s\}} - 1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn} = s] - P[T_{bn} = s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^{\theta}(n) + u_1^*(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form $|P[T_{bn} = s] - P[T_{bn} = s+t]| = O(n^{-2}t + n^{-1-a_1+\delta})$ for any $\delta > 0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1} + n^{-a_1+\delta})$ for any $\delta > 0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left| d_{TV}(L(C[1,b]), L(Z[1,b])) - \frac{1}{2}(n+1)^{-1}|1-\theta| E|T_{0b} - ET_{0b}| \right| \leq \varepsilon_{\{7.8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b) = O(n^{-1}b[n^{-1}b + n^{-\beta_{12}+\delta}])$ for any $\delta > 0$ under Conditions $(A_0), (D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$

$$= \sum_{r \geq 0} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]} \right\}_+$$

Now we observe that

$$\left| \sum_{r \geq 0} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]} \right\}_+ - \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right|$$

$$\times \left| \sum_{s=[n/2]+1}^{n} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r]) \right|$$

$$\leq 4n^{-2}ET_{0b}^2 + (\max_{n/2 < s \leq n} P[T_{0b} = s]) / P[T_{0n} = n]$$

$$+ P[T_{0b} > n/2]$$

$$\leq 8n^{-2}ET_{0b}^2 + \frac{3\varepsilon_{\{10.5(2)\}}(n/2, b)}{\theta P_\theta[0,1]}, \qquad (1.1)$$

We have

$$\left| \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right.$$

$$\times \left(\left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r] \right\}_+ \right.$$

$$\left. \left. - \left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} P[T_{0n} = n] \right\}_+ \right) \right|$$

$$\leq \frac{1}{n^2 P[T_{0n} = n]} \sum_{r \geq 0} P[T_{0b} = r] \sum_{s \geq 0} P[T_{0b} = s] |s-r|$$

$$\times \left\{ \varepsilon_{\{10.14\}}(n,b) + 2(r \vee s)|1-\theta| n^{-1} \left\{ K_0\theta + 4\phi_{\{10.8\}}^*(n) \right\} \right\}$$

$$\leq \frac{6}{\theta n P_\theta[0,1]} ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$

$$+ 4|1-\theta| n^{-2} ET_{0b}^2 \left\{ K_0\theta + 4\phi_{\{10.8\}}^*(n) \right\}$$

$$\left(\frac{3}{\theta n P_\theta[0,1]} \right) \Big\}, \qquad (1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]} P[T_{0b} = r] \left| \left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} \right\}_+ \right.$$

$$\left. - \left\{ \sum_{s=0} P[T_{0b} = s] \frac{(s-r)(1-\theta)}{n+1} \right\}_+ \right|$$

$$\le \sum_{r=0}^{[n/2]} P[T_{0b}=r] \sum_{s>[n/2]} P[T_{0b}=s] \frac{(s-r)|1-\theta|}{n+1}$$
$$\le |1-\theta| n^{-1} E(T_{0b} 1\{T_{0b} > n/2\}) \le 2|1-\theta| n^{-2} E T_{0b}^2, \qquad (1.3)$$

and then by observing that

$$\sum_{r>[n/2]} P[T_{0b}=r] \left\{ \sum_{s\ge 0} P[T_{0b}=s] \frac{(s-r)(1-\theta)}{n+1} \right\}$$
$$\le n^{-1} |1-\theta| (E T_{0b} P[T_{0b} > n/2] + E(T_{0b} 1\{T_{0b} > n/2\}))$$
$$\le 4|1-\theta| n^{-2} E T_{0b}^2 \qquad (1.4)$$

Combining the contributions of (1.2) –(1.3), we thus find tha

$$\Big| \; d_{TV}(L(\bar{C}[1,b]), L(\bar{Z}[1,b]))$$
$$-(n+1)^{-1} \sum_{r\ge 0} P[T_{0b}=r] \left\{ \sum_{s\ge 0} P[T_{0b}=s](s-r)(1-\theta) \right\}_+ \Big|$$
$$\le \varepsilon_{\{7.8\}}(n,b)$$
$$= \frac{3}{\theta P_\theta[0,1]} \left\{ \varepsilon_{\{10.5(2)\}}(n/2,b) + 2n^{-1} E T_{0b} \varepsilon_{\{10.14\}}(n,b) \right\}$$
$$+ 2n^{-2} E T_{0b}^2 \left\{ 4 + 3|1-\theta| + \frac{24|1-\theta| \phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \right\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0), (D_1)$ and (B_{12}), provided that $S(\infty) < \infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty) < \infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0} P[T_{0b}=r] \left\{ \sum_{s\ge 0} P[T_{0b}=s](s-r)(1-\theta) \right\}_+$$
$$= \frac{1}{2} |1-\theta| \, E \, |T_{0b} - E T_{0b}|$$

Example 1.0. Consider the point $O = (0,...,0) \in \mathbb{R}^n$. For an arbitrary vector r, the coordinates of the point $x = O + r$ are equal to the respective coordinates of the vector $r : x = (x^1, ...x^n)$ and $r = (x^1, ..., x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\mathbb{R}^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\mathbb{R}^n$: $\mathbb{R}^n$ = {points}, $\mathbb{R}^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\mathbb{R}^n$ treated in this way is called an *n-dimensional affine space*. (An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\mathbb{R}^n$ considered as an affine space we can precede in two opposite directions: $\mathbb{R}^n$ as an Euclidean space $\Leftarrow$ $\mathbb{R}^n$ as an affine space $\Rightarrow$ $\mathbb{R}^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\mathbb{R}^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\mathbb{R}^n$ a Euclidean space. Namely, we define the length of a vector $a = (a^1, ..., a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + ... + (a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B) := |\overrightarrow{AB}| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B) \le d(A,C) + d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b) := a^1 b^1 + ... + a^n b^n \qquad (3)$$

Thus $|a| = \sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b = (a,b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos\alpha := \frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2 \le |a|^2 |b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a + tb,$ where $t \in R$. As $(a+tb, a+tb) \ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x) = x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i .From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + ... + \frac{\partial f}{\partial x^n} dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2, ...$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2, ...,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 + ... + \frac{\partial f}{\partial x^n} h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \mathbb{R}^n$ at $t = t_0$ and with the velocity vector $x(t_0) = \upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$\begin{aligned} &f(x(t_0 + \Delta t) - f(x_0) \\ &= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t) \\ &+ \beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t| \\ &= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t \end{aligned}$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + ... + \frac{\partial f}{\partial x^n} x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$,

$$d(f+g) = df + dg \qquad (1)$$

$$d(fg) = df.g + f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $x(t_0) = \upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t = t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the

usual derivative Now, almost without change the theory generalizes to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$. The only difference is that now the differential of a map $F:U\to\mathbb{R}^m$ at a point x will be a linear function taking vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (instead of $\mathbb{R}$). For an arbitrary vector $h\in|\mathbb{R}^n$,

$$F(x+h)=F(x)+dF(x)(h)$$
$$+\beta(h)|h| \qquad (3)$$

Where $\beta(h)\to 0$ when $h\to 0$. We have $dF=(dF^1,...,dF^m)$ and

$$dF=\frac{\partial F}{\partial x^1}dx^1+...+\frac{\partial F}{\partial x^n}dx^n$$
$$=\begin{pmatrix}\frac{\partial F^1}{\partial x^1}&\cdots&\frac{\partial F^1}{\partial x^n}\\ \cdots&\cdots&\cdots\\ \frac{\partial F^m}{\partial x^1}&\cdots&\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \cdots\\ dx^n\end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\mathbb{R}^n$, the differential of a map $F:U\to\mathbb{R}^m$ (where $U\subset\mathbb{R}^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\mathbb{R}^m$:

$$\frac{dF(x(t))}{dt}=dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t+\Delta t)=x(t)+\dot{x}(t).\Delta t+\alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t)\to 0$ when $\Delta t\to 0$. By the definition of the differential,

$$F(x+h)=F(x)+dF(x)(h)+\beta(h)|h \qquad (3)|$$

Where $\beta(h)\to 0$ when $h\to 0$. we obtain

$$F(x(t+\Delta t))=F(x+\underbrace{\dot{x}(t).\Delta t+\dot{\alpha}(\Delta t)\Delta t}_{h})$$
$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t)+$$
$$\beta(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t+\dot{\alpha}(\Delta t)\Delta t\right|$$
$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t)\to 0$ when $\Delta t\to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F:U\to V$ and $G:V\to W$, where $U\subset\mathbb{R}^n, V\subset\mathbb{R}^m, W\subset\mathbb{R}^p$ (open domains). Let $F:x\mapsto y=F(x)$. Then the differential of the composite map $GoF:U\to W$ is the composition of the differentials of F and G:

$$d(GoF)(x)=dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\mathbb{R}^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\mathbb{R}^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t)=G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\mathbb{R}^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence $d(GoF)(\dot{x})=dG(dF(\dot{x}))$ for an arbitrary vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\mathbb{R}^n$ by $(x^1,...,x^n)$ and in $\mathbb{R}^m$ by $(y^1,...,y^m)$, and write

$$dF=\frac{\partial F}{\partial x^1}dx^1+...+\frac{\partial F}{\partial x^n}dx^n \qquad (1)$$

$$dG=\frac{\partial G}{\partial y^1}dy^1+...+\frac{\partial G}{\partial y^n}dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF)=\frac{\partial G}{\partial y^1}dF^1+...+\frac{\partial G}{\partial y^m}dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i=dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF)=\begin{pmatrix}\frac{\partial G^1}{\partial y^1}&\dots&\frac{\partial G^1}{\partial y^m}\\ \dots&\dots&\dots\\ \frac{\partial G^p}{\partial y^1}&\dots&\frac{\partial G^p}{\partial y^m}\end{pmatrix}\begin{pmatrix}\frac{\partial F^1}{\partial x^1}&\dots&\frac{\partial F^1}{\partial x^n}\\ \dots&\dots&\dots\\ \frac{\partial F^m}{\partial x^1}&\dots&\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \dots\\ dx^n\end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix}\frac{\partial z^1}{\partial x^1}&\dots&\frac{\partial z^1}{\partial x^n}\\ \dots&\dots&\dots\\ \frac{\partial z^p}{\partial x^1}&\dots&\frac{\partial z^p}{\partial x^n}\end{pmatrix}=\begin{pmatrix}\frac{\partial z^1}{\partial y^1}&\dots&\frac{\partial z^1}{\partial y^m}\\ \dots&\dots&\dots\\ \frac{\partial z^p}{\partial y^1}&\dots&\frac{\partial z^p}{\partial y^m}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial y^1}{\partial x^1}&\dots&\frac{\partial y^1}{\partial x^n}\\ \dots&\dots&\dots\\ \frac{\partial y^m}{\partial x^1}&\dots&\frac{\partial y^m}{\partial x^n}\end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a}=\sum_{i=1}^{m}\frac{\partial z^\mu}{\partial y^i}\frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y\in\mathbb{R}^m$ on $x\in\mathbb{R}^n$ is given by the map F, the dependence of $z\in\mathbb{R}^p$ on $y\in\mathbb{R}^m$ is given by the map G, and the dependence of $z\in\mathbb{R}^p$ on $x\in\mathbb{R}^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U\subset\mathbb{R}^n$. Consider also another copy of $\mathbb{R}^n$, denoted for distinction $\mathbb{R}^n_y$, with the standard coordinates $(y^1...y^n)$. A system of coordinates in the open domain U is given by a map $F:V\to U$, where $V\subset\mathbb{R}^n_y$ is an open domain of $\mathbb{R}^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;
(2) F is invertible;
(3) $F^{-1}:U\to V$ is also smooth

The coordinates of a point $x\in U$ in this system are the standard coordinates of $F^{-1}(x)\in\mathbb{R}^n_y$

In other words,

$$F:(y^1...,y^n)\mapsto x=x(y^1...,y^n) \qquad (1)$$

Here the variables $(y^1...,y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\mathbb{R}^2$ specified in polar coordinates as

$$x(t):r=r(t),\varphi=\varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t\mapsto x(t)$ can be considered as the composition of the maps $t\mapsto(r(t),\varphi(t)),(r,\varphi)\mapsto x(r,\varphi)$. Then, by the chain rule, we have

$$\dot{x}=\frac{dx}{dt}=\frac{\partial x}{\partial r}\frac{dr}{dt}+\frac{\partial x}{\partial\varphi}\frac{d\varphi}{dt}=\frac{\partial x}{\partial r}\dot{r}+\frac{\partial x}{\partial\varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x/\partial r, \partial x/\partial\varphi$ are vectors depending on point in $\mathbb{R}^2$. We can compare this with the formula in the "standard" coordinates: $\dot{x}=e_1\dot{x}+e_2\dot{y}$. Consider the vectors $\partial x/\partial r, \partial x/\partial\varphi$. Explicitly we have

$$\frac{\partial x}{\partial r}=(\cos\varphi,\sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial\varphi}=(-r\sin\varphi,r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r=0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x/\partial r, \partial x/\partial\varphi$ are, respectively, the velocity vectors for the curves $r\mapsto x(r,\varphi)$ $(\varphi=\varphi_0\ fixed)$ and $\varphi\mapsto x(r,\varphi)$ $(r=r_0\ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates

the velocity vector will have components $(\dot{r},\dot{\varphi})$ if as a basis we take $e_r := \partial x/\partial r, e_\varphi := \partial x/\partial \varphi$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i .In particular, the elements of the basis $e_i = \partial x/\partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1,...,x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F: \mathbb{R}^n \to \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0): \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + ... + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1,...,e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1}dx^1 + ... + \frac{\partial f}{\partial x^n}dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\mathbb{R}^2$ given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r\cos\varphi, y = r\sin\varphi$, hence

$$dx = \cos\varphi dr - r\sin\varphi d\varphi$$

$$dy = \sin\varphi dr + r\cos\varphi d\varphi$$

Substituting into A, we get

$$A = -r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi)$$

$$+r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi)$$

$$= r^2(\sin^2\varphi + \cos^2\varphi)d\varphi = r^2 d\varphi$$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U :

$$\omega(\upsilon) = \omega_1\upsilon^1 + ... + \omega_n\upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i\upsilon^i$, where $e_i = \partial x/\partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta^i_j \qquad (2)$$ at every point x .

Theorem 1.9. For arbitrary 1-form ω and path γ , the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \frac{dx}{dt'} \right\rangle$ and $\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle$ As

$$\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle = \left| \left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle . \frac{dt}{dt'} \right|,$$

Let p be a rational prime and let $K = \mathbb{Q}(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p-1$ over $\mathbb{Q}$. We wish to show that $O_K = \mathbb{Q}[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Q}[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not

divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, ..., \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + ... + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p - 1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta) = Tr_{K/\mathbb{Q}}(1-\zeta^2) = ...$$
$$= Tr_{K/\mathbb{Q}}(1-\zeta^{p-1}) = p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + ... + 1 = \Phi_p(x)$$
$$= (x-\zeta)(x-\zeta^2)...(x-\zeta^{p-1});$$

Plugging in $x = 1$ shows that

$$p = (1-\zeta)(1-\zeta^2)...(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta) = p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K \cap \mathbb{Z} = p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K, so the inclusion $(1-\zeta)O_K \cap \mathbb{Z} \supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1-\zeta)O_K \cap \mathbb{Z} = \mathbb{Z}$ Thus we can write

$$1 = \alpha(1-\zeta)$$

For some $\alpha \in O_K$. That is, $1-\zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha \in O_K$, $Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) \in p\mathbb{Z}$

PROOF. We have

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) = \sigma_1((1-\zeta)\alpha) + ... + \sigma_{p-1}((1-\zeta)\alpha)$$
$$= \sigma_1(1-\zeta)\sigma_1(\alpha) + ... + \sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha)$$
$$= (1-\zeta)\sigma_1(\alpha) + ... + (1-\zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = \mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, ..., \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write

$\alpha = a_0 + a_1\zeta + ... + a_{p-2}\zeta^{p-2}$ With $a_i \in \mathbb{Q}$. Then

$$\alpha(1-\zeta) = a_0(1-\zeta) + a_1(\zeta - \zeta^2) + ...$$
$$+ a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0$ We also have

$Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in p\mathbb{Z}$, so $a_0 \in \mathbb{Z}$ Next consider the algebraic integer

$(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + ... + a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \mathbb{Z}$, and continuing in this way we find that all of the a_i are in $\mathbb{Z}$. This completes the proof.

Example 1.4 Let $K = \mathbb{Q}$, then the local ring $\mathbb{Z}_{(p)}$ is simply the subring of $\mathbb{Q}$ of rational numbers with denominator relatively prime to p. Note that this ring $\mathbb{Z}_{(p)}$ is not the ring $\mathbb{Z}_p$ of p-adic integers; to get $\mathbb{Z}_p$ one must complete $\mathbb{Z}_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta / \gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta / \gamma \in a$ and a is an

ideal), so $\beta \in O_K$ and $\gamma \notin p.$ so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed.We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to $p.$ we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p} / pO_{K,p}$ Since $pO_{K,p} \cap O_K = p,$ this map is also surjection, since the residue class of $\alpha/\beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as $x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + \ldots + \frac{\alpha_0}{\beta_0}$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1\ldots\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p,$ we have $\beta\gamma/\beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in $K.$

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then $N'_{K/\square}(\alpha O_K) = \left|N_{K/\square}(\alpha)\right|$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\square$ is Galois. Let σ be an element of $Gal(K/\square)$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\square}(\sigma(\alpha)O_K) = N'_{K/\square}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\square)$, we have $N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N'_{K/\square}(\alpha O_K)^n$ Since $N_{K/\square}(\alpha)$ is a rational integer and O_K is a free $\square$ -module of rank n, $O_K / N_{K/\square}(\alpha)O_K$ Will have order $N_{K/\square}(\alpha)^n$; therefore

$$N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N_{K/\square}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m.$

III. OPTIMAL POLICY FOR PRICE INQUIRY

In this section, we study the optimal policy for price inquiry over the communication channel. We can model the decisions of price inquiry as a Markov decision process (MDP) [3]. We will discuss the fundamental elements in the MDP. Then, we apply the approach of value iteration to obtain the optimal policy. Note that we assume that the statistical laws of the price, like the mapping between the load and the price and the uncertainty on the load, are all known to the home appliance.

A. MDP Modeling

There are three elements in a MDP problem, namely action space, system state and reward. We discuss them in the context of price inquiry separately.

1) Action: Obviously, the action of the home appliance is the inquiry of the power price (denoted by 1) or not (denoted by 0). The decision of action is determined by the current system state and the policy of the home appliance. Actually, there is an implicit action for the home appliance, i.e. the power consumption level. In the most complicated case, the power consumption level should also be a function of the current system state and policy. However, to simplify the analysis, we assume that the home appliance assumes the power price of the latest inquiry, thus uniquely determines the power consumption level. Therefore, we do not consider the power consumption level as an action and do not incorporate it into the decision policy. We put an upper bound, denoted by T, for the number of time slots between two price inquiries. Then, the home appliance must send out a price inquiry within T time slots since the previous price inquiry. The

upper bound can address the jeopardy brought by modeling imperfection, e.g. some imprecise parameters may significantly lengthen the interval between two inquiries to a harmful level, and thus improves the robustness.

2) System State: The system state contains two parts, namely the power price in the previous inquiry and the elapsed time since the previous inquiry. For time slot t, the system state is given by (pτt, t − τt). Obviously, if the home appliance inquires the power price at time slot t, the corresponding system state is (pt, 0). We notice that, whenever the price substate is changed, the substate of elapsed time is reset to 0. Another important issue in the system state is the state transition probability respect to the action. Obviously, if the action is no inquiry, i.e. 0, the state is changed to (qi,Δ+1) if the previous state is (qi, Δ). When the action is inquiry, i.e. 1, the state is changed to (qj , 0) from the previous state (qi, Δ), where j is a random variable. We denote by Kij(Δ) the probability of transiting from price qi to price qj after Δ time slots. The transition probability is then given by

B. Value Iteration

Once defining the elements of MDP, we can apply Dynamic Programming (DP) [3] to obtain the optimal strategy. We denote by R(s) the optimal expected total reward when the initial state is s. Then, R(s) satisfies the following Bellman's equation [2], which is given by

$$R(s) = \max_a(r(s, a) + \beta E_{s,a} [R(s\_)]) , (11)$$

where r(s, a) is the expected reward due to action a and system state s, Es,a is the expectation conditioned on a and
s, s_ is the system state in the next time slot. The expectation can be computed using the state transition probability in (7). The instantaneous reward r(s, a) can be computed using (9) and (10). The Bellman's equation can be solved by using the following value iteration [3], which is given by

$$R(t)(s) = \max_a r(s, a) + \beta E_{s,a} R(t-1)(s\_), (12)$$

where the superscript t is the index of iteration. The iteration converges to the solution of the Bellman's equation as $t \to \infty$. The optimal action is obtained from

$$a^*(s) = \arg \max_a(r(s, a) + \beta E_{s,a} [R(s\_)]) . (13)$$

A. Workload Data and Electricity Spot Price Data Analysis
We consider the statistical properties (e.g. Mean and Variance) of both the workload data and the electricity spot price data. Statistical properties are usually extracted from populations formed by many single individuals. Both workload time series and spot electricity price time series have only one realization. For example, there is only one historical series of electricity price for each location and we have only one electricity price at each instant of time. In this paper, we discuss the properties of workload and price at a specific time given that there is only one available workload and price observation. As each workload time series or electricity price time series is unique, the solution is to look at the single element of the time series as the individuals of the population. For example, because there is only one realization of each location's electricity price time series, we have to look at the price of each location at different time. The initial time series must be transformed; that is, a unique time series must be transformed into populations of individuals to which statistical methods can be applied.

C. Performance Evaluation Based on Real-Life Data for Google IDCs

To demonstrate the effectiveness of the proposed optimal hedging algorithm, we simulate a two-IDC-location optimization assuming forward prices equal to (40, 45). The two IDC locations are California and Texas where the spot electricity price data are obtained. Without ambiguity, we use location 1 to represent California and location 2 to represent Texas in the following paper. Due to the availability of the data, we use the data from 4 January 2010 to 12 April 2010. In this paper, the unit for electricity price is $/MWh, the unit for unit electricity cost is $/Request, the unit for workload is MRequest/h, the unit for delay is s/Request and the unit for power is watts. The detailed parameters are shown in Table III. Fig. 3(a) illustrate the workload data from Google and spot electricity price data for California and Texas that we use for performance evaluation. We compare the simulation results for optimal hedging algorithm and optimal load distribution algorithm presented in [19]. The optimal hedging algorithm is calculated using the data and we obtain the optimal quantities and as shown in Table IV. With the optimal quantities of forward contracts, we obtain the unit cost time series incorporating optimal hedging as shown in Fig. 3(b). Further we also calculate the variances of unit cost for the cases with and without optimal hedging. We see the the variance of optimal hedging has been more than 59% less compared with the variance without optimal hedging . In this case, the operation risk under workload and electricity price uncertainties cost has been greatly reduced with optimal hedging at the same time the mean of unit cost does not increase. This indicates that the proposed method helps IDC providers not only save a lot in their electricity bill, but also reduce the operation risk.

D. Effect of Workload Variance and Electricity Price Variance Dynamics

It is important to evaluate whether our scheme scales when both workload uncertainty and electricity price uncertainty increases. With the rapid growth of diverse online applications, the workload uncertainty might further increase. We conduct simulation experiment to show how operations risk changes with the dynamic of workload uncertainty and electricity price uncertainty respectively.

E. Effect of Electricity Market Price Uncertainties

We now conduct simulation experiments to show how different forward electricity prices affect the optimal hedging results. The numerical results for hedging over different portfolio of forward electricity prices at two IDC locations are illustrated in Fig. 5. According to Table IV, we see that our proposed method is efficient for any portfolio of electricity prices. We also see that the optimal hedging results are different with different forward contract prices. Hence the optimal hedging decisions of areal so different. In this case, for different value of for IDC location attime0 with maturity time, we would buy different amount of electricity (i.e.,) at all IDC locations. This fact can be easily observed from the formulation of our bi level optimization problem.

IV. NEIGHBORHOOD LEVEL SCHEDULING

We next consider a decentralized approach to support neighborhood-level load scheduling. To do so, we assume that there exists a communication network between the EMCs in each neighborhood home. To simplify discussion here, we assume that all EMCs in the neighborhood are "one hop" away and can transmit/receive each other's signal sent over a common control channel. This channel may be supported by an underlying smart metering infrastructure or some other local communication network. Access to the control channel is granted to the EMCs by following the protocols of the supporting network. We assume here that the control channel has high capacity and supports the exchange of EMC scheduling packets with high priority and high reliability.

We describe below a scheduling-level exchange of packets over this channel that enables neighborhood EMCs to compete for power while collectively maintaining a relatively low and level peak demand. In contrast to purely pricing-based resource allocation, our proposed scheme does not require local EMCs to compete in a retail-level capacity market (where power is auctioned and bought). Instead, it extends access control methods typically used in communication networks to randomize an EMC's access to the local power capacity for a neighborhood. Specifically, we assume that time is divided into scheduling slots and a peak total power demand for the neighborhood is set (by the utility) as Pmax,t for slot t.2 The scheme described here aims to meet Pmax,t, which the utility can fix to be below the typical peak demand for time t. To describe our scheme, we will first assume (in Section III-A) that homes must compete for all of the available power. Presumably, such a scheme would be unacceptable to consumers, who would want a guarantee that at least some of their power needs can be met at all times. Therefore, in Section III-B, we modify the scheme to guarantee each home a certain level of power at all times; homes may then compete for additional power.

A. No Guaranteed Minimum

We assume that whenever a new load request is generated within a home, the EMC will seek to meet the load requirement by coordinating with other neighborhood homes over the common control channel. During scheduling slot t, all loads that are currently being supported are termed as active. All EMCs that have an active load will continuously monitor the common control channel. When a new load is requested within a home, its EMC will first require information about current active loads for the neighborhood. To do so, it will transmit a probe packet over the control channel in the next scheduling slot. To support transmissions from multiple EMCs in a given scheduling slot, we assume the slot is itself divided into M "mini slots." We assume each EMC selects a random mini slot (uniformly distributed in the range [1,M]) to send out a probe message. With sufficiently large M and slot durations, an EMC's probe transmission can be transmitted with negligible probability of collision with another EMC's transmission.3 Once the first EMC's (say EMC i's) probe transmission goes through successfully, all other contending EMCs cancel their probe transmission and wait another random number of mini slots. This time the number of mini slots chosen for the backoff is uniformly distributed in the range from m to M, where m minislots is a duration long enough to obtain responses from the EMCs currently supporting active loads and one additional transmission from EMC i. This is because the successful probe message will require all EMCs supporting active loads to respond with a short response packet containing the power levels of their supported loads. We assume that in a neighborhood of K homes, the m mini slots are long enough to support K such packets. EMCs from the K homes are assigned a transmission order during initial network formation and EMCs with active loads respond to the probe packet in this order over the m mini slots. Based on the response from the other EMCs, the EMC requesting the new load then computes the current

total power usage and determines if its desired power demand can be supported within the total allowable neighborhood load of Pmax,t. If so, the load joins the active set and sends a short admission packet containing its power consumption over the control channel; otherwise, the EMC enters random backoff at the scheduling layer (for this load request) and re-attempts the inquiry procedure after its backoff timer expires. The scheduling layer backoff mechanism recognizes that this EMC's load cannot be supported in this current scheduling slot and waits for a random number of scheduling slots before reattempting a probe message and admission to the active set. We assume that at the scheduling layer, an EMC entering backoff for a given load will select a random number of scheduling slots uniformly distributed in the range [1, w] for some integer w. Other EMCs attempting a new load admission in the current time slot (that had entered a second backoff after the initial probe message) continue to monitor the control channel over these m minislots. Based on the probe response packets and the presence/absence of an admission packet, these EMCs know the current total power usage for the neighborhood at the end of the m minislots. If their requested power can be supported, they continue to monitor the channel until their backoff timer (for the minislots) expires and then simply join the active set (if the total usage permits their admission within the constraint of Pmax,t) and send out an admission packet indicating the power level of this new active load. The steps of this distributed scheduling algorithm are presented in Fig. 2. The figure shows the procedure that must be followed by each EMC in the neighborhood. At the same time, those EMCs that have active loads must monitor the channel to respond to any probe messages sent out in each scheduling slot. As noted above, the transmission times for these response packets are prescribed when the neighborhood EMCs are initially connected on the control channel.

V. TRANSACTIVE MARKET OPERATIONS

As defined by [11], an active market is one where a price signal is sent from the central controller to end-use load, allowing the end-use load to react to price changes. An interactive market allows the end-use load to return information back to the central controller, resulting in an adjustment of the price signal. A transactive market, however, uses interactive controllers that react to price changes, returning information back to the central controller, and automatically acting on behalf of the end-use load. The following section will describe the behavior of the various components of the transactive market, including the doubleauction market, the transactive controller, and the capacity management device.

A. Double-Auction Market

A double-auction market is a traditional market strategy that can be described as a two-way market, where both suppliers and end-use loads submit bids for price and quantity into a single energy market simultaneously. The auction resolves the supply and demand bids into a common cleared market price and quantity, and delivers this information back to the participants. This approach is highly scalable, and allows all parties to participate in determining the price of electricity, including residential loads and distributed generation, in addition to the more traditional players, such as bulk power, transmission, distribution and ancillary service costs. In addition to the cleared price signal, the average price and standard deviation of the cleared price over a rolling 24-hour window are also sent to the transactive controllers. These values are calculated at five-minute intervals, which for this example, is also the market clearing interval. The exchange of this amount of information between the end-use customer and the central control agent requires a robust communication system, which can handle two-way message traffic. While in some applications, this may be cost or hardware prohibitive, that restriction will be ignored for this analysis.

B. Transactive Controller

For an end-use load to participate in a transactive market, it must be able to (1) change its energy consumption based on the cleared market price, (2) determine the price it is willing to pay for electricity, and (3) bid its desired demand. This section will describe a variant of the transactive control designed and deployed in [12-13] to control HVAC loads. This section will also examine how this variant of transactive control is currently implemented in the open source simulation environment GridLAB-D™ [14]. The HVAC transactive controller within GridLAB-D operates by responding to a combination of external signals from the market, and user specified internal setpoints. In this section, only the cooling mode of operation will be described. However, the operation of the heating mode is very similar. The cooling operation is described graphically in Fig. 1, where: The controller responds to the current price signal by moving the internal temperature setpoint, Tadjusted, as a function of the cleared market price; if the current price of electricity is greater than average, the cooling setpoint is moved higher, temporarily decreasing consumption, while lowering the cooling setpoint to take advantage of relatively low prices, temporarily increasing consumption. The range that the thermostat can move Tadjusted is bounded by the user-defined values of Tmin and Tmax, while the rate is defined by calculated values σ, khigh, and klow. The customer need only specify Tmin and

Tmax, setting their maximum tolerance for temperature deviations, and a comfort level, which defines σ, khigh, and klow, specifying the customer's willingness to participate in the market. The algorithm the controller uses to set the adjusted temperature setpoint can be described by:

By adjusting the internal setpoint as a function of price, the controller defines the energy consumption of the HVAC unit. By itself, this would be considered an active controller, only responding to market prices. A more detailed description of the active thermostat response is available in [15]. To be considered a transactive controller, it must also bid into the market, communicating desired buying price and quantity to the central auction. For each market clearing interval, the current bid price of the controller is determined by the current internal air.

If the current air temperature is greater than the desired set point, then the controller will bid a price that is greater than the average market price. If the current air temperature is below the desired set point, the bid price will be less than market average. Note, that while the prices received from and bid into the market are in terms of $/kWh, all of the controller responses and bids are determined by deviations from the mean in terms of standard deviation of the market prices over a 24-hour window. Using standard deviations of price, rather than absolute prices, makes the market and controller more adaptable. It automatically compensates for price differences between different regions and variances in long-term or seasonal market prices. Additionally, as the customer adjusts their desired setpoint, the various settings do not need to be recalibrated to the new set point. The transactive controller is also required to bid the demand of the HVAC system. Because the actual demand of an HVAC system varies with voltage and outdoor air temperature [3], among other parameters, the best indication of future demand comes from historical demand, rather than nameplate capacity. The transactive controller is required to store the previous actual power demand of the HVAC system, and bid this quantity into the auction. Differences in outdoor air temperature and voltage may cause discrepancies between the desired demand and the actual demand; however, the errors should be minor over short periods of time, hence the reason for five-minute market clearing times. Finally, in the case of the capacity management market, to be described in the following section, the transactive controller is also required to bid its current operational state, whether the HVAC system is currently ON or OFF. The need for this will be explained in the following section.

C. Capacity Management Market

The capacity management market is a special case of the transactive market that attempts to manage congestion at the distribution feeder level. The congestion limit may be caused by local conditions, such as thermal conductor ratings, or for higher level reasons, such as reduction of localized congestion on sub-transmission networks, or any combination of the above. This combination of constraints delves into the realm of multi-variable control, introducing a whole line of optimization problems. For the purpose of this discussion, however, the congestion limit will only be applied at the feeder level to reduce peak demand. The transactive controller deals with bidding the price and quantity of the HVAC load into the auction, called the responsive load. However, in this strategy, the controller does not account for the supplier bids or the non-bidding loads on the system, such as lights, refrigerators, or losses. The non bidding loads are called unresponsive loads, because the demand does not change as a function of price. The capacity management market fills the role of supply bidder, while also accounting for the unresponsive loads in this particular market strategy. To bid the supplier curve, the capacity management market simply bids the congestion limit at the uncongested market price, then forms a vertical line at the congestion quantity to the price cap of the market, as seen in Fig. 3. In general, the uncongested market price is the bulk cost of electricity plus mark-up, such as the Locational Marginal Price (LMP), or any other appropriate price signal, such as direct purchase of power from distributed generators.

VI. DEMAND RESPONSE MODEL

A. Congestion pricing background

In this paper we propose to apply the principle of congestion pricing in IP networks to demand response in the electricity grid. In their seminal paper [6], Kelly et al. have proposed the proportionally fair pricing (PFP) scheme in which each user declares a price per unit time that he is willing to pay for his flow. In that sense the network capacity is shared among the flows of all users in proportion to the prices paid by the users. It has been shown in [6] that in a weighted proportionally fair system where the weights are the prices the users pay per unit time, when each user chooses the price that maximizes the utility she gets from the network, the system converges to a state where the total utility of the network is maximized. In other words, in an ideal environment, the PFP proposal is able to decentralize the global optimal allocation of congestible resources. Another important result of [6] is that rate control (such as TCP) based on additive increase and multiplicative decrease achieves proportional fairness. It has also been proved that the decentralized congestion control mechanism is stable even under arbitrary network

topologies and heterogeneous round trip times (feedback delays). In Kelly's approach, the philosophy is that users who are willing to pay more should get more. As the network makes no explicit promises to the user, there is no need for over provisioning in the core of the network. One implementation of PFP is to give control to end systems (users). In this scheme, the TCP algorithm is modified to incorporate congestion prices by means of protocols like explicit congestion notification (ECN) [7]. Upon receiving feedback signals, f (t), which are related to shadow prices (in terms of packet marks), the users are free to react as they choose, but will incur charges when resources are congested. An end system can adjust its rate x(t) using a willingness to pay (WTP) parameter w:

$$X(t + 1) = x(t) + a(w - f(t)), \quad (1)$$

where a affects the rate of convergence of the algorithm.

B. Demand Response and Dynamic Pricing Services

The Energy Residential Gateway offers a set of Demand Response services, focused on peak clipping and valley filling actions, according to the algorithm shown in (Fig. 2). ERG reacts to emergency DR messages from the Utility, decreasing power demand at critical periods (Service no.1). This message contains the percentage of power reduction required. HAN devices will be re-configured in order to adjust household consumption to this new target.

Regarding Valley Filling, the gateway is ready to switch on or dimmer some HAN devices when a low price RTP message is received from the utility (Service no.2). In both scenarios, the ERG discovers automatically those HAN devices ready to be switched or dimmered.

C. Demand Management Game

According to the definition in Section III, it is easily seen that the demand management problem can be formulated by means of a congestion game. In such a setup,

- the players are the M users demanding resources during the day;
- the resources E are the N available edges ej0,1, representing the N time slots, through which the power demanded by user i flows;
- the strategy of player i is the demand vector di;
- the cost of each resource in E is ce = cej0,1(xej0,1).

Players define a strategy and update it based on the other players' strategies, until an equilibrium is reached. In order to take into account the preferences in demand scheduling over the day, each user weights the cost of each time slot
according to its preferences, so that the weighted cost vector for user i becomes ci(si, s−i) = wiej0,1cej0,1(xej0,1), and the game described becomes a weighted congestion game. The Nash equilibrium may be reached if all users update their strategy in order to minimize their cost given a fixed strategy of the other players. Given the network model, such problem consists in finding the minimum cost flow from the generation point (i.e. V0) to the user (i.e. V1). However, link cost is dependent on the total link load, and each user wishes to allocate more than one unit of load. A simple transformation may be used to convert the network graph with load-dependent link costs to a graph with fixed link costs, consisting in splitting each link of capacity K into K unit capacity links with increasing cost

ce(1), ⋯ ce(K).

Once this transformation has been made, the min cost flow problem may be solved using a linear program [13].

D. Grid Load Management Game

The demand management game may be extended to represent and manage load in each link of the electric distribution grid. Under such model (see Figure 3), the congestion game looses the symmetric property, since different users will in general be connected to different points of the distribution grid. Nevertheless, a non-symmetric congestion game is also guaranteed to have at least one Nash equilibrium, which may be achieved when each user seeks to minimize its cost function, given fixed strategies of the other players. As in the previous section, the cost function minimization problem is the classical min-cost flow problem from the generation point to the user.

E. Dynamic Pricing

Introducing into the basic operation model the dynamic pricing DSM mechanism leads to another modelling approach. Given that in dynamic pricing, such as real-time pricing, consumers face varying prices throughout the day, their reaction can be modelled with demand functions using elasticities to express consumers' sensitivities to price changes. The price demand elasticity expresses how much percent of demand would change when a change in price of 1% occurs (see equation (8) and subsection III-B for more explications). The demand function is approximated assuming a reference point when no DSM mechanism is applied. This point corresponds to a reference demand Dp and a reference price Prp (coincident with the marginal cost). The slope of the demand function is represented with elasticities εup

and εdown, according to equations (5) and (6). We assume the demand function to be linear which may differ slightly from reality. Due to the absence of real-time pricing schemes in Spain we could not construct a demand function based on real data. Other models in the literature use the same assumption of linearity, see [4]. Thus, we think a linear demand function may be a valid

VII. ELECTRICITY PRICING STRATEGIES

Various prices considered in this paper and their implementations are described below.

A. Fixed-existing rate

This is the fixed rate paid by consumers per kWh energy consumed. A rate of $0.095/kWh is applied to all the days throughout the year [5].

B. Critical Peak Pricing

This is a time-based pricing scheme with a peak-rate of $0.895/kWh and an off-peak rate of $0.056/kWh. The peak rate is applied for the five hours between 1pm and 6pm of the weekdays during the four summer months from June to September. Up to 15 weekdays can be declared as CPP days, which is done based on load forecasting. There is a 15:1 ratio between the peak-rate and off-peak rate [5]. For days other than critical days including the weekends an off-peak rate of $0.056/kWh is applied throughout the day.

C. Time of Use Pricing

This is a time-based pricing scheme with a peak rate of $0.234/kWh and an off-peak rate of $0.061/kWh. The peak rate is applied for the five hours between 1pm and 6pm of the weekdays during the four summer months from June to September. There is a 4:1 ratio between the peak-rate and offpeak rate [5]. For the weekends an off-peak rate of $0.061/kWh is applied throughout the day.

D. Declaring a CPP day

Utilities typically would do this based on load forecast for the next day. The consumers are informed a day ahead about a critical day through a signal sent to the home, to allow enough time to plan load-shifting. Although utilities would use a detailed forecasting method to declare a specific day as a CPP day, we have used peak temperature of the day as the deciding factor since we are analyzing historical data. The summer days were classified into six temperature ranges and a cut-off temperature of 96F was set. However, for June a cutoff temperature of 99F was used because the data showed that the high temperature days in June were few and spread out without much heat build-up. This allows utilities to include the extremely hot days as CPP days, which were mostly found to be in the months of July and August. The analysis of weather data for Manhattan from 1999 to 2009 showed that on an average around 11 to 12 CPP days could be declared per year.

E. Price Driven Real-time Demand Response:

Demand response (DR) is the ability of users to dynamically change their electricity loads. The change can be according to price signal, which may reflect the total demand of user side. DR is one of the most important capability to enable smart grid. To enable DR, several techniques need to be developed. Users need smart meter to report their current or future energy consumption and appliances can adjust their behaviors according to price signal. Sensing of renewable energy production and changing price are also important issues. Protocols of sending price signal and algorithms to control appliances behavior have to be developed.

F. Load and Price Forecasting

Load forecasting is widely studied in the electricity grid. Load forecasts are essential for dispatchers, who are the commercial or governmental bodies responsible for dispatching electricity to the grid. Load forecasting provides tools for operation and planning of a dispatcher where decisions such as purchasing or generating power, bringing peaker plants online, load switching and infrastructure development can be made [12]. Electricity market regulators and dispatchers rely on forecasting tools that provide short, medium and long-term forecasts. Short-term load forecasts cover hourly or daily forecasts where medium-term forecasts span a time interval from a week to a year and long-term forecasts cover several years. Forecasting techniques may differ according to this range. In this paper, we use prediction for a short time period, therefore we limit the scope of this section to short-term forecasting techniques. For short-term forecasting, the similar day approach searches the historical database of days to find a similar day with properties such as weather, day of the week, etc [13]. Regression is another widely used statistical technique for load forecasting. Regression methods aim to model the relationship of load and environmental factors, e.g. temperature [14]. Time series methods try to fit a model to data. Previous studies have employed a wide variety of time series methods such as Autoregressive Moving Average (ARMA), Autoregressive Integrated Moving Average (ARIMA), Autoregressive Moving Average with eXogenous variables (ARMAX) and Autoregressive Integrated Moving Average with eXogenous variables (ARIMAX) methods. ARIMAX has been shown to have successful load forecasting performance since it does not require stationarity and it can incorporate weather and time of use as exogenous variables [15]. On the other hand, using neural networks [16], expert systems [17], support vector machines [18] and fuzzy logic [19] are among the recent forecasting studies. The

accuracy of forecasting techniques may differ depending on the markets, utilities and regions because the geographical conditions and the usage patterns in one region are not identical with another [12]. A detailed survey of load forecasting techniques can be found in [12] and [20]. The techniques proposed for load forecasting can be used for price forecasting, as well. Nevertheless, several recent studies focus on price forecasting only. [21] proposes a technique based on weighted nearest neighbors approach for the prediction of the next-day electricity market prices and [22] employs game theory for price forecasting. Utilities and the system operators have large databases of previous loads and environmental conditions where they can employ the above techniques and achieve accurate forecasts. However, these data may not be available for the consumers. In this case, consumers can employ simpler techniques on limited data sets and achieve less accurate predictions, which are still adequate for making decisions such as determining the TOC. In this paper, we use a limited amount of data to avoid storage cost at the charging station.

VIII. PREDICTION BASED CHARGING

In this paper, we consider a smart grid employing dynamic pricing. The market prices of electricity, in addition to extra fees related with regulation and taxes, etc. are used in billing the consumers. To make dynamic pricing practical, consumers need to be equipped with smart appliances or smart outlets that can respond to the instant variations in price or the day-head prices need to be announced. We assume all the consumers use smart outlets that can communicate with the grid and learn the dynamic price of electricity. We consider the use of prediction for charging the batteries of PHEVs, therefore we assume the charging station is able to store limited amount of past price signals of the utility and run a light-weight prediction algorithm. Since the charging process can span several hours, the current price, in addition to several hours-ahead prices are needed to determine the best TOC. For instance, a PHEV may be plugged in at 6:59am which is an off-peak hour, just before the morning peak, i.e., 7am. At this time instant, the price of electricity is low and the PHEV may start charging. However, the charging process will continue for several hours which coincide with the peak hours. To avoid this, we propose a prediction-based charging scheme and consider the future price of the electricity. Our prediction-based charging scheme adopts a simple, light-weight classification technique, which is the k-nearest neighbors algorithm (k-NN). In k-NN, an unclassified sample point is assigned to the class of the nearest set of the previously classified points. k-NN does not require complex model fitting operations as the time series models do. k-NN considers the training set as the model.

In our scheme, the training set is the price signals recorded in the several previous days. The number of days used in the training set is denoted by □□. We employ a sliding window over the training set. The size of the sliding window is □□. The training set is split into time slots of one hour and □□ is the hourly average price for time slot □. Each two consecutive time slots (□□□□+1) are used by the classification algorithm to determine the corresponding class, □ for the time slot □□+2. We define maximum number of classes □□□□ based on the market prices, before running the prediction algorithm. First the training set is grouped under □□□□ classes, then the class of the □□+2 price signal is predicted. If the predicted price is greater than the price threshold, □□, i.e. □□□ > □□, charging is delayed for one hour. At the beginning of the next hour the prediction algorithm predicts the new price and if this exceeds the threshold, charging is again delayed. Otherwise, PHEV starts charging. At the beginning of each hour, until the PHEV is unplugged or fully charged, the charging station predicts □□, and decides whether to start charging or to delay it. We assume that PHEVs can be charged during daytime at any time. They are assumed to be plugged-in wherever there is a charging station available however, being plugged in does not necessarily mean that the PHEV starts charging. Charging starts only when the predicted prices are below □□ and continues at least for a minimum amount of time □□□ and until the battery is fully charged or PHEV is unplugged by the driver. Our scheme is a simple hourly price prediction technique. Since PHEV charging algorithms will be either on-board or they will be embedded on a charging station, their complexity needs to be low. However, this technique may not be convenient for price or load forecasting in electricity markets.

A. Authors and Affiliations

Dr Akash Singh is working with IBM Corporation as an IT Architect and has been designing Mission Critical System and Service Solutions; He has published papers in IEEE and other International Conferences and Journals.

He joined IBM in Jul 2003 as a IT Architect which conducts research and design of High Performance Smart Grid Services and Systems and design mission critical architecture for High Performance Computing Platform and Computational Intelligence and High Speed Communication systems. He is a member of IEEE (Institute for Electrical and Electronics Engineers), the AAAI (Association for the Advancement of Artificial Intelligence) and the AACR (American Association for Cancer Research). He is the recipient of numerous awards from World Congress in Computer Science, Computer Engineering and Applied Computing 2010, 2011, and IP Multimedia System 2008 and Billing and Roaming 2008. He is active

research in the field of Artificial Intelligence and advancement in Medical Systems. He is in Industry for 18 Years where he performed various role to provide the Leadership in Information Technology and Cutting edge Technology.

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