Smart Grid Cloud

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Abstract

This paper presents a model for smart grid data management based on specific characteristics of cloud computing, such as distributed data management for real-time data gathering, parallel processing for real-time information retrieval, and ubiquitous access. The appliance of the cloud computing model meets the requirements of data and computing intensive smart grid applications. We gathered these requirements by analyzing the set of well-known smart grid use cases, most of which demand flexible collaboration across organizational boundaries of network operators and energy service providers as well as the active participation of the end user. Hence, preserving confidentiality and privacy, whilst processing the massive amounts of smart grid data, is of paramount importance in the design of the proposed Smart Grid Data Cloud.

Keywords- **Advanced metering infrastructure (AMI), communication technologies, quality-of-service (QoS), smart grid, standards**

I. INTRODUCTION- THE DATA CLOUD MODEL FOR THE SMART GRID

The majority of smart grid use cases are characterized by the exponential growth of data from the many intelligent, communicating devices and the need for fast information retrieval. Ubiquitous access to actionable information about electricity consumption, generation, transmission and delivery across organizational boundaries of generators, network operators, retailers and consumers will form the backbone of a new electricity era. Typical for this new electricity era is also the increased dynamics through distributed and renewable generation both at the distribution and the transmission level. Liberalization and the creation of new market roles result in requirements for a more flexible platform-centric approach to data delivery and usage, in which all data are accessible to all actors whilst preserving confidentiality of competitors' data and privacy of end users. A major challenge is that ICT and data management costs prevent initiatives from becoming widely adopted after pilot projects. Sensors and actuators make devices along transmission and distribution communicative and controllable. However, capital and operational costs of data management for energy market actors to make informed decisions about how to control these resources reach prohibitive levels if

each actor must replicate infrastructures for managing data from the same sources in order to retrieve information according to their different needs. Figure 2 shows data and information flow separated from the control flow. The rationale behind this separation is that actuators will be triggered by the legitimate parties on events. Whilst sensors and other data sources external to the power system, such as weather sensors and electronic markets, will generate a continuous flow of data, which is subject to the on demand information retrieval needs of each party. In such a scenario a data service provider would provision their data management and processing resources to any legitimate interested party. The main contribution of the data service provider is to offer reusable services such as data collection, data validation and cleaning, data analysis, information retrieval, and archival. The value added and reduced costs are remarkable: Each energy market actor needs the data delivered by a set of sensors: These sets, however, are overlapping. Using traditional IT, each party would be replicating the needed infrastructure and software for data management for the majority of the needed data sources. For the provisioning of such smart grid data as a service, we propose a platform-centric cloud computing approach, which has proven to accommodate internet-scale data and computing intensive applications in a multi-tenant setting [13, 14, 15]. In the research community there are current efforts to classify and deliver appropriate definition of cloud computing [16]. In this paper, we are providing a conceptual view on cloud computing as a platform for smart grid data management. The emphasis is on the specific characteristics of cloud computing, which result in an internet-scale platform and can facilitate the data intensive needs of the smart grid use cases detailed in Section II. Figure 3 depicts the model of ubiquitous data storage and data access for the Smart Grid Data Cloud. The specific internet-scale characteristics, such as interoperability, ease-of-use and extensibility, distributed data management and parallel processing techniques, as well as concepts for managing confidentiality and privacy in such a "hyper-tenant" environment are explained in the following subsections.

A. Interoperability and Extensibility

The Smart Grid Data Cloud consists of interconnected data centers accessible from IP-based networks via simple web- based APIs, such as REST

[17]. Sensors as data sources put their data in a continuous flow via the put API of the data cloud. Legitimate energy market actors can query for information via simple get API. It is necessary to use load balancing mechanisms depending on the type of data, the locality of the data source as well as patterns of data access for information retrieval by the energy market actors. For example, metered consumption data is sensed in intervals via smart meters at any location, and queried by retailers, end users and distribution system operators as well as other service providers (e.g. energy management service provider for end users) in different aggregation levels and times. Other sensor data may be delivered by sensors throughout the different transmission and regional networks, whilst information in the form of data analysis results are retrieved by any of the transmission system operators focusing on their networks and adjacent transmission and distribution hubs. The simple web-based APIs can also enhance the required extensibility for the smart grid domain. Any cloud-connected party with access rights can deploy their own applications and services, such as a portfolio management application or an algorithm for data analysis, in the cloud computing platform. These services can be shared or provisioned for use by other cloud-connected parties on the data that those parties have access rights to. Such flexible and open access design can remarkably increase business innovation around the available information, as recently demonstrated in other IT-driven domains, e.g. [18, 14]. Additionally, costs for the infrastructure and data management are shared proportionally by the utilizing parties. This characteristic is often referred to as "pay-per-use" in the cloud computing domain. Whilst easy to interface web-based APIs enable interoperability and extensibility around the Smart Grid Data Cloud allowing any internet-enabled device to put data and get information, the networking protocols utilized within and between the data centers allow for scalability and availability of the cloud computing platform. The specific networking protocols are tightly coupled with the data management and processing scheme used by the cloud computing platform, which will be explained in the next subsection.

Distributed Data Management and Parallel Processing
Existing cloud computing platforms, which have proven to enable application development on internet-scale and extensibility by 3rd parties, share one common trait. They utilize distributed data structures and appropriate data management algorithms for large scale data-intensive applications. The resulting fault tolerance allows the data centers to consist of large numbers of commodity hardware [19, 20, 21]. For highly parallelizable applications, the cost advantages of using inexpensive, PC-based clusters over highend multiprocessor servers can be substantial: the multiprocessor server is about three times more expensive but has 22 times fewer CPUs, three times less RAM, and slightly more disk space [22].

II. SMART GRID CLOUD COMPUTING OPERATING SYSTEM

Cloud computing operating system is the overall management system of data center behind the smart grid cloud platform, it is constructed on software resources such as stand-alone operating systems and hardware infrastructures like servers, storages and networks. Simply put, our cloud operating system has the following functions:

• It manages and drives large mass of basic hardware like servers and storages. It integrates physical resources of the data center into a logical server.
• It provides common and standard APIs to cloud applications.
• It is responsible for managing massive computing tasks and dispatching resources.

Cloud computing operating system is the key step to realize cloud computing. It simplifies the computing and provides a more efficient computational model. As shown in figure2, our cloud computing operating system includes 4 parts as follows: virtualization management, DFS, safety management control, cloud computing services interface. By the way, cloud computing services interface provides interface to the development and application which will manage the underlying data center visibly.

A. Virtualization Management

Virtualization management simplifies the management of underlying data center and helps enterprises to reduce the burden and expenses in management, visualization and maintenance of heterogeneous storage infrastructures. It monitors each application status visibly all times. Once found abnormal actions, it will warn the administrator to repair, avoiding affecting the normal operation of the entire system. According to user's access situation and server's CPU loading condition, virtualization management will call dynamic load-balancing algorithm. And it will evaluate, configure and supply storage resources automatically, to realize the purpose of user shared resources and effective improvement of resources utilization.

B. DFS(Distributed File System)

Due to block based DFS, customer data is deployed redundantly in masses of cheap storages. Parallel and distributed computing provides excellent data redundancy. Distributed concurrent data processing technology not only makes high-

performance services provided simultaneously to users by storage nodes possible, but also ensures higher efficiency of data transportation. Therefore, block based DFS is more suited to mass throughput cloud computing platform. There are several current popular DFS: HDFS by Hadoop, FastDFS by Google, Iustre by Oracle.etc. FastDFS is customized for Internet Applications. It is aware of redundancy backup, load balance and linear expansion, emphasizes targets like high availability and performance. It is mainly designed for large capacity and high page view of small files. Compared to other Google FS, characteristics of FastDFS architecture and design concept mainly reflects in three aspects: lightweight, packet mode and equivalence structure. So FastDFS is much suitable for our cloud computing operating system as it can provide better cloud services to inner enterprise.

C.   Safety Management Control

Private cloud computing separated computing from storage and realized multiple users' sharing for the same basic resources, but at the same time, numerous users sharing the same resources, it will create higher challenges to data security. Specifically, private cloud computing security involves several main aspects as follows: data access risk, data storage risk, information management risk, data isolation risk, legal investigation support risk, sustainable development and transfer risk, etc. Security control in our operating system needs a comprehensive prevention and control of infrastructure security design, cloud computing center operating system architecture, authentication and encryption strategy to ensure information security of the entire system.

III. CLOUD COMPUTING OF ELECTRIC POWER SYSTEM

A.   The proposed intelligent cloud of electric power system

Electric power system is a super-system with distributed parameters. Due to electric power system's own characteristics, it cannot store energy in a large scale, and electric generation, transmission, distribution and usage must also be completed simultaneously. Production control of the electric power must be strong real time, high reliability, with the characteristics of natural distribution. Electric power production and management is naturally formed as a set of architecture of "hierarchical management, hierarchical control, distributed processing ". Many years of practice shows that this is the reflection of the intrinsic nature characteristics of power system. "Intelligent cloud" can automatically split large calculation into small pieces through the power system intranet and deliver to a huge system which is constituted by many servers to compute and analyze, and then return the results to the user.

Through the intelligent cloud, huge information can be handled in a very short period of time, which can get to the supercomputer's level of service. Through distributed computing, electric power system is running similarly to the Internet, and electric power system intelligent cloud can switch resources according to application, get access to the computer and storage resources on demand. The purpose of Intelligent Cloud is to move the running grid nodes or computation on a single computer system to a huge number of "intelligent cloud" in system, and the cloud processes the request of the point or the computer. Using electric power system intelligent cloud, we needn't enhance computing power of the node or the computer any more, and we can get computing power and resources from intelligent cloud, improving greatly computing power of every point of overall system. Currently, power grid at all levels has a certain processor and storage resources, the advantage of the realization of intelligent cloud is that we can keep the existing distribution of our computers and make the most use of the physical structure of information networks of current electric power system, allocating calculation and storage resources for the current task.

B.   The components of electric power system intelligent cloud

China's existing electric power system is characterized by the geographical features where power grid is distributed, different network topology, electrical characteristics of power grids, which divides power grid into multiple subnets. The scheduling, operation, monitoring, protection, distribution and marketing of each subnet is managed by centre of the subnet which is maintaining the detailed parameters of the power grid which is managed by the subnet. The networks have established more detailed electric power system model for the power grid in the networks while the networks have made the simplification and equivalence to some extent for the adjacent grid model, and do computer simulation based on this system model, providing an important basis for the power scheduling, operation, monitoring, protection, transmission and distribution and marketing. This simplification and equivalent method overcomes the difficulty in collecting data for China's vast territory, reducing the complexity of system simulation. But the simplification and equivalent model has limited application scope while Intelligent Cloud can solve this problem. Given the sensitivity of electric power system data and the integrity of the inner network of China power system, power system in China can use the existing physical network devices of system inner network to establish a private cloud of electric power system. With this cloud computing model, the power system can fully control cloud computing, this cloud storage and access to computing resources can be completely controlled by the power system

itself, rather than the public cloud services provider, which equivalents to using the system to establish our own inner cloud.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r) = \sum_{j=1}^{p} \int_{\Omega} J_{ij}(r,\bar{r}) S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{1j})]d\bar{r} \\ \qquad\qquad\qquad + I_i^{ext}(r,t), \qquad t \geq 0, 1 \leq i \leq p, \\ V_i(t,r) = \phi_i(t,r) \qquad\qquad t \in [-T,0] \end{cases} \quad (1)$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d . The vector r and $\bar{r}$ represent points in Ω . The function $S : R \rightarrow (0,1)$ is the normalized sigmoid function:

$$S(z) = \frac{1}{1+e^{-z}} \qquad (2)$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i = 1,...,p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i = 1,...,p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p \times p$ matrix of functions $J = \{J_{ij}\}_{i,j=1,...,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i = 1,...,p,$ determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i = 1,...,p,$ determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i = 1,...,p,$ determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S : R^p \rightarrow R^p$, defined by $S(x) = [S(\sigma_1(x_1 - h_1)),...,S(\sigma_p - h_p))]$, and the diagonal $p \times p$ matrix

$L_0 = diag(l_1,...,l_p).$ Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt}+l_i)$ is replaced by $(\frac{d}{dt}+l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt}+l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\overline{\Omega}^2, R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r}) \neq \tau_{ij}(\bar{r},r),$ thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,r\in\Omega\times\overline{\Omega})} \tau_{i,j}(r,\bar{r}) \qquad (3)$$

Hence we choose $T = \tau_m$

C. *Mathematical Framework*
A convenient functional setting for the non-delayed packet field equations is to use the space $F = L^2(\Omega, R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U \rangle_F = \sum_{i=1}^{p} \int_{\Omega} V_i(r)U_i(r)dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m,0],F)$ with $\|\phi\| = \sup_{t\in[-\tau_m,0]} \|\phi(t)\| F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t+\theta), \theta \in [-\tau_m,0]$, we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0V(t) + L_1S(V_t) + I^{ext}(t), \\ \qquad\qquad V_0 = \phi \in C, \end{cases} \quad (2)$$

Where

$$\begin{cases} L_1 : C \to F, \\ \phi \to \int_{\Omega} J(.,\bar{r})\phi(\bar{r}, -\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \le \|J\|_{L^2(\Omega^2, R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty$.

Proposition 1.0   If the following assumptions are satisfied.

1. $J \in L^2(\Omega^2, R^{p\times p})$,

2.   The external current $I^{ext} \in C^0(R, F)$,

3. $\tau \in C^0(\overline{\Omega^2}, R_+^{p\times p}), \sup_{\overline{\Omega^2}} \tau \le \tau_m$.

Then for any $\phi \in C$, there exists a unique solution $V \in C^1([0,\infty), F) \cap C^0([-\tau_m, \infty, F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

D.   Boundedness of Solutions
A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t \in R^+} \|I^{ext}(t)\|_F < \infty.$

Proof :Let us defined $f : R \times C \to R^+$ as

$$f(t, V_t) \overset{def}{=} \left\langle -L_0 V_t(0) + L_1 S(V_t) + I^{ext}(t), V(t) \right\rangle_F = \frac{1}{2}\frac{d\|V\|_F^2}{dt}$$

We note $l = \min_{i=1,\dots p} l_i$

$$f(t, V_t) \le -l\|V(t)\|_F^2 + (\sqrt{p|\Omega|}\|J\|_F + I)\|V(t)\|_F$$
Thus, if

$$\|V(t)\|_F \ge 2\frac{\sqrt{p|\Omega|}.\|J\|_F + I}{l} \overset{def}{=} R, f(t, V_t) \le -\frac{lR^2}{2} \overset{def}{=} -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \ge 0s$ and that $f < 0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C < R$ and set

$T = \sup\{t \mid \forall s \in [0,t], V(s) \in \overline{B_R}\}$.   Suppose that $T \in R$, then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R,   we   also   have $\frac{d}{dt}\|V\|_F^2 \mid_{t=T} = f(T, V_T) \le -\delta < 0$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and small enough, $V(T + \varepsilon) \in \overline{B_R}$ which contradicts the definition of T. Thus $T \notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0) \in \partial B_R$ implies that $\forall t > 0, V(t) \in B_R$. Finally we consider the case $V(0) \in C\overline{B_R}$.   Suppose   that $\forall t > 0, V(t) \notin \overline{B_R}$,   then $\forall t > 0, \frac{d}{dt}\|V\|_F^2 \le -2\delta,$ thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts   our   assumption.   Thus $\exists T > 0 \mid V(T) \in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on $X.$ for $E\varepsilon M$, define

$$\phi(E) = \int_E s\, d\mu \qquad (1)$$

Then ϕ is a measure on M.
$$\int_X (s+t)d\mu = \int_X s\, d\mu + \int_X t d\mu \qquad (2)$$

Proof : If s and if $E_1, E_2, \dots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E) = \sum_{i=1}^n \alpha_i \mu(A_i \cap E) = \sum_{i=1}^n \alpha_i \sum_{r=1}^\infty \mu(A_i \cap E_r)$$
$$= \sum_{r=1}^\infty \sum_{i=1}^n \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^\infty \phi(E_r)$$

Also, $\varphi(\phi) = 0,$ so that φ is not identically ∞.
Next, let s be as before, let $\beta_1, \dots, \beta_m$ be the distinct values of t,and let $B_j = \{x : t(x) = \beta_j\}$ If $E_{ij} = A_i \cap B_j$,   the $\int_{E_{ij}} (s+t)d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$

and $\int_{E_{ij}} s \, d\mu + \int_{E_{ij}} t \, d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \le i \le n, 1 \le j \le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon > 0$, then there exists a polynomial P such that $|f(z) = P(z)| < \varepsilon$ for all $z \varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f \varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2) - f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2 - z_1| \le \delta$. Since f is uniformly continous, we have $\lim_{\delta \to 0} \omega(\delta) = 0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10,000 \ \omega(\delta) \quad (z \varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi \varepsilon C_c'(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \le \omega(\delta), \qquad (3)$$

$$|(\partial \Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z) = -\frac{1}{\pi} \iint_X \frac{(\partial \Phi)(\zeta)}{\zeta - z} d\zeta \, d\eta \qquad (\zeta = \xi + i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi \delta^2} \left(1 - \frac{r^2}{\delta^2}\right)^2 \qquad (0 \le r \le \delta), \qquad (6)$$

And define

$$A(z) = a(|z|) \qquad (7)$$

For all complex z. It is clear that $A \varepsilon C_c'(R^2)$. We claim that

$$\iint_{R^s} A = 1, \qquad (8)$$

$$\iint_{R^2} \partial A = 0, \qquad (9)$$

$$\iint_{R^3} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\frac{\partial A}{\partial \theta} = 0$,

$$\frac{\partial A}{\partial r} = -a',$$

Now define

$$\Phi(z) = \iint_{R^2} f(z - \zeta) A \, d\xi \, d\eta = \iint_{R^2} A(z - \zeta) f(\zeta) d\xi \, d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z)$$
$$= \iint_{R^2} [f(z - \zeta) - f(z)] A(\xi) d\xi \, d\eta \qquad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A \varepsilon C_c'(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial \Phi)(z) = \iint_{R^2} (\overline{\partial A})(z - \zeta) f(\zeta) d\xi \, d\eta$$
$$= \iint_{R^2} f(z - \zeta)(\partial A)(\zeta) d\xi \, d\eta$$
$$= \iint_{R^2} [f(z - \zeta) - f(z)](\partial A)(\zeta) d\xi \, d\eta \qquad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial \Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial \Phi = 0$ in

G, where G is the set of all $z\varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z\varepsilon G); \qquad (14)$$

Note that $\partial f = 0$ in G, since f is holomorphic there. Now if $z\varepsilon G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z) = \int_0^\delta a(r)rdr\int_0^{2\pi} f(z - re^{i\theta})d\theta$$

$$= 2\pi f(z)\int_0^\delta a(r)rdr = f(z)\iint_{R^2} A = f(z) \qquad (15)$$

For all $z \, \varepsilon \, G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1, ..., D_n$, of radius 2δ, whose centers are not in K. Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that $S^2 - E_j$ is connected and so that $K \cap E_j = \phi$. with $r = 2\delta$. There are functions $g_j\varepsilon H(S^2 - E_j)$ and constants b_j so that the inequalities.

$$\left|Q_j(\zeta, z)\right| < \frac{50}{\delta}, \qquad (16)$$

$$\left|Q_j(\zeta, z) - \frac{1}{z - \zeta}\right| < \frac{4,000\delta^2}{|z - \zeta|^2} \qquad (17)$$

Hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j)g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1 \cup ... \cup E_n$. Then Ω is an open set which contains K. Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup ... \cup X_{j-1})$, for $2 \le j \le n$,

Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta\varepsilon X_j, z \, \varepsilon \, \Omega) \qquad (19)$$

And

$$F(z) = \frac{1}{\pi}\iint_X (\partial\Phi)(\zeta)R(\zeta, z)d\zeta d\eta \qquad (20)$$

$$(z \, \varepsilon \, \Omega)$$

Since,

$$F(z) = \sum_{j=1}^{} \frac{1}{\pi}\iint_{X_i} (\partial\Phi)(\zeta)Q_j(\zeta, z)d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F\varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$\left|F(z) - \Phi(z)\right| < \frac{2\omega(\delta)}{\pi\delta}\iint_X | R(\zeta, z)$$

$$- \frac{1}{z - \zeta}|d\xi d\eta \quad (z \, \varepsilon \, \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta \, \varepsilon \, X$ and $z \, \varepsilon \, \Omega$. Now fix $z \, \varepsilon \, \Omega$., put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \le \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta} + \frac{1}{\rho}\right)\rho d\rho = 808\pi\delta \qquad (23)$$

And

$$2\pi\int_{4\delta}^\infty \frac{4,000\delta^2}{\rho^2}\rho d\rho = 2,000\pi\delta. \qquad (24)$$

Hence (22) yields

$$\left|F(z) - \Phi(z)\right| < 6,000\omega(\delta) \qquad (z \, \varepsilon \, \Omega) \qquad (25)$$

Since $F \, \varepsilon \, H(\Omega), K \subset \Omega,$ and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f\varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2} \frac{(\partial f)(\zeta)}{\zeta - z}d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta) = f(z + re^{i\theta})$, $r > 0$, θ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0$, of

$$-\frac{1}{2}\int_{\varepsilon}^{\infty}\int_0^{2\pi}\left(\frac{\partial\varphi}{\partial r} + \frac{i}{r}\frac{\partial\varphi}{\partial\theta}\right)d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi/\partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{\varepsilon}^{\infty}\frac{\partial\varphi}{\partial r}dr = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon,\theta)d\theta \qquad (5)$$

As $\varepsilon \to 0$, $\varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^\alpha \in a$ and $X^\beta \in k[X_1,...X_n]$, then $X^\alpha X^\beta = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$(\sum_{\alpha\in A}c_\alpha X^\alpha)(\sum_{\beta\in\square^n}d_\beta X^\beta) = \sum_{\alpha,\beta}c_\alpha d_\beta X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^\alpha, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals (X^n), $n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^\alpha \mid \alpha \in A\rangle$ for the ideal corresponding to A (subspace generated by the $X^\alpha, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^\alpha, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{\beta \in \square^n \mid \beta-\alpha \in \square^n, \quad some \ \alpha \in S\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^\alpha, \alpha \in \mid S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^\beta \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^\beta = X^\alpha X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^\alpha \mid X^\beta \Leftrightarrow \beta-\alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that $A = \{\beta \in \square^n \mid \beta-\alpha_i \in \square^2, \ some \ \alpha_i \in S\}$ (The α_i's are the corners of A) Moreover, $a \overset{df}{=} \langle X^\alpha \mid \alpha \in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find $f = a_1g_1 + ... + a_sg_s + r$, $a_i, r \in k[X_1,...,X_n]$, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_ig_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1),...,LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1,...,g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1, | ..., g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1),...,LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1,...,X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1,...X_{n-1}][X_n] \to k[X_1,...X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1,...X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,...,X_n]$:

$$f(X_1,...X_n) = a_0(X_1,...X_{n-1})X_n^r + ... + a_r(X_1,...X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0X^r + a_1X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1,...,g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write

$$a = \sum b_i a_i, \quad b_i \in A,$$

a_i =leading coefficient of g_i

Now

$f - \sum b_i g_i X^{s-r_i}$, $r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \quad \mod(g_1,...g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1},...,g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \quad \mod(g_{d,1},...g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1},...g_{r-1,m_{r-1}},...g_{0,1},...g_{0,m_0})$ Hence

$$f_t \in (g_1,...g_m g_{r-1,1},...g_{r-1,m_{r-1}},...,g_{0,1},...,g_{0,m_0})$$

and so the polynomials $g_1,...,g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x . x$ is most conveniently defined as a function from $D\ to\ D$, which must then be regarded as an element of D. Let $\psi : [D \to D] \to D$ be the function that picks out elements of D to represent elements of $[D \to D]$ and $\phi : D \to [D \to D]$ be the function that maps elements of D to functions of D. Since $\psi(f)$ is intended to represent the function f as an element of D, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D \to D]}$ Furthermore, we often want to view every element of D as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that ϕ and ψ are inverses--- that is, D is isomorphic to the space of functions from D to D that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda-calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of C. Each element of D corresponds to either an element of A or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains X such that $X \cong A + [X \to X]$, and such that for any domain Y also satisfying this equation, there is an embedding of X to Y --- a pair of maps

$$X \quad \overset{f}{\underset{f^R}{\square}} \quad Y$$

Such that

$$f^R \, o \, f = id_X$$
$$f \, o \, f^R \subseteq id_Y$$

Where $f \subseteq g$ means that f approximates g in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering F not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, F.

Definition 1.3: Let K be a category and $F : K \to K$ as a functor. A fixed point of F is a pair (A,a), where A is a **K-object** and $a : F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a **K-object** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega-chain$ in a category K is a diagram of the following form:

$$\Delta = D_o \overset{f_o}{\longrightarrow} D_1 \overset{f_1}{\longrightarrow} D_2 \overset{f_2}{\longrightarrow} .....$$

Recall that a cocone μ of an $\omega-chain$ Δ is a K-object X and a collection of K –*arrows* $\{\mu_i : D_i \to X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} o \, f_i$ for all $i \geq 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of $\mu's$ components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with

the property that if $\nu : \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k : X \to X'$ such that for all $i \geq 0,, v_i = k \, o \, \mu_i$. Colimits of $\omega-chains$ are sometimes referred to as $\omega-co\lim its$. Dually, an $\omega^{op}-chain$ in K is a diagram of the following form:

$$\Delta = D_o \overset{f_o}{\longleftarrow} D_1 \overset{f_1}{\longleftarrow} D_2 \overset{f_2}{\longleftarrow} .....$$

A cone $\mu : X \to \Delta$ of an $\omega^{op}-chain$ Δ is a **K**-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \geq 0\}$ such that for all $i \geq 0$, $\mu_i = f_i \, o \, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op}-chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \geq 0, \mu_i \, o \, k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of **K,** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each **K**-object A. It is also convenient to write $\Delta^- = D_1 \overset{f_1}{\longrightarrow} D_2 \overset{f_2}{\longrightarrow} .....$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under **F** we write

$$F(\Delta) = F(D_o) \overset{F(f_o)}{\longrightarrow} F(D_1) \overset{F(f_1)}{\longrightarrow} F(D_2) \overset{F(f_2)}{\longrightarrow} .....$$

and $F(\mu) = \{F(\mu_i) \mid i \geq 0\}$

We write F^i for the **i-**fold iterated composition of F that is, $F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let **K** be a category with initial object $\perp$ and let $F : K \to K$ be a functor. Define the $\omega-chain \Delta$ by

$$\Delta = \perp \overset{!\perp \to F(\perp)}{\longrightarrow} F(\perp) \overset{F(!\perp \to F(\perp))}{\longrightarrow} F^2(\perp) \overset{F^2(!\perp \to F(\perp))}{\longrightarrow} ....$$

If both $\mu : \Delta \to D$ and $F(\mu) : F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d : F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^-

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of

these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, ...X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, ...x_n) = P(x_n | pa_n) P(x_{n-1} | Pa_{n-1})...$$
$$..P(x_2 | pa_2) P(x_1 | pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i | pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \le P(x_1, x_2, ...x_n) \le 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \le k \le n$ that whenever

$$P(pa_k) \ne 0, if \ P(nd_k | pa_k) \ne 0$$
$$and \ \ P(x_k | pa_k) \ne 0$$
$$then \ P(x_k | nd_k, pa_k) = P(x_k | pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k | nd_k) = P(x_k | pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, ...X_{k-1}\}$$
$$Let$$
$$D_k = \{X_{k+1}, X_{k+2}, ...X_n\}$$

follows $\sum_{d_k}$

We define the mth *cyclotomic field to be the field* $Q[x] / (\Phi_m(x))$ Where $\Phi_m(x)$ is the mth cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ *has degree* $\varphi(m)$. *The roots of* $\Phi_m(x)$ *are just the primitive*

m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*
$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \le k \prec m, (k,m) = 1, \quad where$$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m *being our fixed choice of primitive* m^{th} *root of* unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another, or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n , then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. *Since* $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n),$ *so the result is clear*

LEMMA 1.6 If m *and* n *are relatively prime, then*
$$Q(\xi_m, \xi_n) = Q(\xi_{nm})$$
and
$$Q(\xi_m) \cap Q(\xi_n) = Q$$
(Recall the $Q(\xi_m, \xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m \xi_n$ is a primitive mn^{th} root of unity, so that
$$Q(\xi_{mn}) \subseteq Q(\xi_m, \xi_n)$$
$$[Q(\xi_m, \xi_n) : Q] \le [Q(\xi_m) : Q][Q(\xi_n : Q]$$
$$= \varphi(m)\varphi(n) = \varphi(mn);$$

Since $\left[Q(\xi_{mn}):Q\right]=\varphi(mn);$ this implies that $Q(\xi_m,\xi_n)=Q(\xi_{nm})$ We know that $Q(\xi_m,\xi_n)$ has degree $\varphi(mn)$ over Q, so we must have

$$\left[Q(\xi_m,\xi_n):Q(\xi_m)\right]=\varphi(n)$$

and

$$\left[Q(\xi_m,\xi_n):Q(\xi_m)\right]=\varphi(m)$$

$$\left[Q(\xi_m):Q(\xi_m)\cap Q(\xi_n)\right]\geq\varphi(m)$$

And thus that $Q(\xi_m)\cap Q(\xi_n)=Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m,\xi_n)=Q(\xi_{[m,n]})$$

And

$$Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)});$$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m=p_1^{e_1}......p_k^{e_k}$ and $p_1^{f_1}....p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i or f_i to be zero)

$$Q(\xi_m)=Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n)=Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$$

Thus

$$Q(\xi_m,\xi_n)=Q(\xi_{p^{e_1}})........Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}})$$
$$=Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}})$$
$$=Q(\xi_{p_1^{\max(e_1,f_1)}})......Q(\xi_{p_1^{\max(e_k,f_k)}})$$
$$=Q(\xi_{p_1^{\max(e_1,f_1)}........p_1^{\max(e_k,f_k)}})$$
$$=Q(\xi_{[m,n]});$$

An entirely similar computation shows that $Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i,y_i)=\log_2\frac{P(x_i/y_i)}{P(x_i)}\ bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i, and so they constitute an input –output pair (x_i,y_i) for which

$$P(x_i/y_j)=1\ and\ I(x_i,y_j)=\log_2\frac{1}{P(x_i)}\quad bits;$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j)=P(x_i)$ and also $I(x_i,y_j)=0;$ that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y)=\sum_{i,j}P(x_i,y_j)I(x_i,y_j)=\sum_{i,j}P(x_i,y_j)\log_2\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i,y_j)=P(x_i/y_j)P(y_j)=P(y_j/x_i)P(x_i)$$

$$P(y_j)=\sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i)=\sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y) = \sum_{i.j} P(x_i, y_j)$$

$$= \sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right]$$

$$-\sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i/y_j)} \right]$$

$$\sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right]$$

$$= \sum_{i} \left[P(x_i/y_j) P(y_j) \right] \log_2 \frac{1}{P(x_i)}$$

$$\sum_{i} P(x_i) \log_2 \frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H(X/Y)$$

Where $H(X/Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)}$

is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j) P(y_j) = P(y_j/x_i) P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_{j} P(y_j) \log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol

after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i) P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j) \frac{P(x_i) P(y_j)}{P(x_i, y_j)} \le 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i \log_2 \left(\frac{Q_i}{P_i} \right) \le 0$$

The above expression can be applied due to the factor $P(x_i) P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \le 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i) P(y_j)}{P(x_i, y_j)}$$

$$+\sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i) P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ *and* $P(x_2) = 1 - \alpha$, and transition probabilities

$P(y_3/x_2) = 1 - p$ *and* $P(y_2/x_1) = 0,$

and $P(y_3/x_1) = 0$

and $P(y_1/x_2) = p$

and $P(y_3/x_2) = 1 - p$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose

restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean n-space. $p(y \mid x)$ for the density $p_n(y_1,...,y_n \mid x_1,...x_n)$ and F for F_n. For any real number a, let

$$A = \left\{ (x,y) : \log \frac{p(y \mid x)}{p(y)} > a \right\} \qquad (1)$$

Then for each positive integer u, there is a code (u,n,λ) such that

$$\lambda \le ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \qquad (2)$$

Where

$$P\{(X,Y) \in A\} = \int_A \cdots \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y \mid x)$$

and

$$P\{X \in F\} = \int_F \cdots \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$$P\left\{ Y \in A_{x^1} \mid X = x^{(1)} \right\} \ge 1 - \varepsilon$$

where $A_x = \{ y : (x,y)\varepsilon A \};$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)},........,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\left\{ Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i \mid X = x^{(k)} \right\} \ge 1 - \varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets B_i, $i = 1,2,...,u$, form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \ge u$ by showing that $\varepsilon \le te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (If \ t = 0, \ take \ B = \phi). \ Then$$

$$P\{(X,Y) \in A\} = \int_{(x,y)\in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y \mid x)dy\,dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y \mid x)dy\,dx + \int_x p(x)$$

E. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1,.....,s_m\}$, we shall write $(s_1,.....,s_m)$ for the ideal it generates. Let a and b be ideals in A. The set $\{a+b \mid a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab \mid a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1,...,a_m)$ and $b = (b_1,...,b_n)$, then $ab = (a_1b_1,...,a_ib_j,...,a_mb_n)$. Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a \mapsto a+a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a. An ideal p if *prime* if $p \ne A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \ne 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \ne\mid A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and

$(a,b) \in c$, then $(a,0) = (a,b)(1,0) \in c$ and $(0,b) = (a,b)(0,1) \in c$. This shows that $c = a \times b$ with

$$a = \{a \mid (a,b) \in c \ \text{some} \ b \in b\}$$

and

$$b = \{b \mid (a,b) \in c \ \text{some} \ a \in a\}$$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A \to B$. A *homomorphism of* A-*algebra* $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1, ..., x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1, ..., X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k-algebra. If $1 \neq 0$ in A , then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$. **Polynomial rings.** Let k be a field. A *monomial* in $X_1, ..., X_n$ is an expression of the form $X_1^{a_1} ... X_n^{a_n}, \quad a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^α, $\alpha = (a_1, ..., a_n) \in \square^n$. The elements of the polynomial ring $k[X_1, ..., X_n]$ are finite sums

$$\sum c_{a_1 ... a_n} X_1^{a_1} ... X_n^{a_n}, \quad c_{a_1 ... a_n} \in k, \quad a_j \in \square$$

With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1, ..., X_n]$ as a k-vector space. The ring $k[X_1, ..., X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1, ..., X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist

unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or $\deg r < \deg g$. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) **lexicographic** *ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha = (a_1, ... a_n)$ and $\beta = (b_1, ... b_n)$ be two elements of $\square^n$; then $\alpha > \beta$ *and* $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha - \beta \in \square$, the left most nonzero entry is positive. For example, $XY^2 > Y^3Z^4; \quad X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha - \beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ *(total degree greater)*
$XY^5Z^2 > X^4YZ^3, \quad X^5YZ > X^4YZ^2$.

Orderings on $k[X_1, ... X_n]$ **.** Fix an ordering on the monomials in $k[X_1, ... X_n]$. Then we can write an element f of $k[X_1, ... X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$$f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$$

as

$$f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2 \quad (lex)$$

or

$$f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2 \quad (grevlex)$$

Let $\sum a_\alpha X^\alpha \in k[X_1, ..., X_n]$, in decreasing order:

$$f = a_{\alpha_0} X^{\alpha_0} +_{\alpha_1} X^{\alpha_1} + ..., \quad \alpha_0 > \alpha_1 > ..., \quad \alpha_0 \neq 0$$

Then we define.

• The *multidegree* of f to be multdeg(f) $= \alpha_0$;

- The *leading coefficient of* f to be $LC(f) = a_{\alpha_0}$;

- The *leading monomial of* f to be $LM(f) = X^{\alpha_0}$;

- The *leading term of* f to be $LT(f) = a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + ...,$ the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in** $k[X_1,...X_n]$. Fix a monomial ordering in $\square^2$. Suppose given a polynomial f and an ordered set $(g_1,...g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,...a_s$ and r such that $f = a_1 g_1 + ... + a_s g_s + r$ Where either $r = 0$ or no monomial in r is divisible by any of $LT(g_1),...,LT(g_s)$ **Step 1:** If $LT(g_1)\,|\,LT(f)$, divide g_1 into f to get

$$f = a_1 g_1 + h, \qquad a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1,...,X_n]$$

If $LT(g_1)\,|\,LT(h)$, repeat the process until $f = a_1 g_1 + f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f = a_1 g_1 + ... + a_s g_s + r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),...LT(g_s)$ **Step 2:** Rewrite $r_1 = LT(r_1) + r_2$, and repeat Step 1 with r_2 for f:

$$f = a_1 g_1 + ... + a_s g_s + LT(r_1) + r_3 \qquad \text{(different}$$

$a_i\,'s$) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a = (Y^2 - X^3)$ contains $Y^2 - X^3$ but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if

$$\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$$

all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal,* and let $A = \{\alpha\,|\,X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A,\ \beta \in \square^n \Rightarrow \alpha + \beta \in$ (∗) And a is the k-subspace of $k[X_1,...,X_n]$ generated by the $X^\alpha, \alpha \in A$. Conversely, of A is

a subset of $\square^n$ satisfying $(∗)$, then the k-subspace a of $k[X_1,...,X_n]$ generated by $\{X^\alpha\,|\,\alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,...,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,...,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)},...,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!}N(n,c) = 1\left\{\sum_{j=1}^{n} jc_j = n\right\}\prod_{j=1}^{n}(\frac{1}{j})^{c_j}\frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \square_+^n$.

Lemma1.7 For nonnegative integers $m_1,...,m_n$,

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right) = \left(\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{m_j}\right)1\left\{\sum_{j=1}^{n} jm_j \leq n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]}/c_j! = 1/(c_j - m_j)!$ when $c_j \geq m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum jm_j$. Then, with the first sum indexed by $c = (c_1,...c_n) \in \square_+^n$ and the last sum indexed by $d = (d_1,...,d_n) \in \square_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right) = \sum_c P[C^{(n)} = c]\prod_{j=1}^{n}(c_j)^{[m_j]}$$

$$= \sum_{c:c_j \geq m_j\ for\ all\ j} 1\left\{\sum_{j=1}^{n} jc_j = n\right\}\prod_{j=1}^{n}\frac{(c_j)^{[m_j]}}{j^{c_j}c_j!}$$

$$= \prod_{j=1}^{n}\frac{1}{j^{m_j}}\sum_d 1\left\{\sum_{j=1}^{n} jd_j = n - m\right\}\prod_{j=1}^{n}\frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m \leq n)$, corresponding to the fact that if $n - m \geq 0$, then

$d_j = 0$ for $j > n - m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1, ..., d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]} = j^{-r} 1\{jr \le n\} \qquad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right) = E\left(\prod_{j=1}^{n} Z_j^{[m_j]}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\}, \qquad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1, 2, ...n\}$, so that $|I| = n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha| = (n-j)!$, since the elements of $\{1, 2, ..., n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n-rj)! 1(rj \le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n - rj)! 1(rj \le n)$$

$$\times \frac{n^{[rj]}}{j^r r!} \frac{1}{n!} = 1(rj \le n) \frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \ge 0} (-1)^l \binom{k+l}{l} S_{k+l,}$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0, 1, ..., n$,

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1$. In particular, for $n \ge 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)}, ..., C_b^{(n)})$ for any $1 \le b \le n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1, ..., c_b) \in \square_+^b$ with $m = \sum ic_i$,

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c]$$

$$= \left\{\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{c_i} \frac{1}{c_i!}\right\} \sum_{\substack{l \ge 0 \ with \\ \sum il_i \le n-m}} (-1)^{l_1 + ... + l_b} \prod_{i=1}^{b}\left(\frac{1}{i}\right)^{l_i} \frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)}, ..., C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = ... = m_n = 0$

The limit distribution of cycle counts
It follows immediately from Lemma 1.2 that for each fixed j, as $n \to \infty$,

$$P[C_j^{(n)} = k] \to \frac{j^{-k}}{k!} e^{-1/j}, \quad k = 0, 1, 2, ...,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)} \to_d Z_j$ where $Z_j \square P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\square$ with intensity j^{-1}. That is, as $n \to \infty$,

$$(C_1^{(n)}, C_2^{(n)}, ...) \to_d (Z_1, Z_2, ...) \qquad (1.1)$$

Where the $Z_j, j = 1, 2, ...,$ are independent Poisson-distributed random variables with

$$E(Z_j) = \frac{1}{j}$$

Proof. To establish the converges in distribution one shows that for each fixed $b \geq 1$, as $n \to \infty$,

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c] \to P[(Z_1, ..., Z_b) = c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b = 1$. Using properties of alternating series with decreasing terms, for $k = 0, 1, ..., n$,

$$\frac{1}{k!}(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!}) \leq |P[C_1^{(n)} = k] - P[Z_1 = k]|$$

$$\leq \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!} \frac{n}{n+2} \leq \sum_{k=0}^{n} |P[C_1^{(n)} = k] - P[Z_1 = k]| \leq \frac{2^{n+1} - 1}{(n+1)!} \quad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ...) < \frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $P\left[A_n(C^{(n)})\right]$ under conditions (A_0) and (B_{01}), where

$$A_n(C^{(n)}) = \bigcap_{1 \leq i \leq n} \bigcap_{r_i' + 1 \leq j \leq r_i} \{C_{ij}^{(n)} = 0\},$$

and $\zeta_i = (r_i' / r_{id}) - 1 = O(i^{-g'})$ as $i \to \infty$, for some $g' > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0m}(Z') = n]}{P[T_{0m}(Z) = n]}$$

$$\prod_{\substack{1 \leq i \leq n \\ r_i' + 1 \leq j \leq r_i}} \left\{1 - \frac{\theta}{ir_i}(1 + E_{i0})\right\} \quad (1.1)$$

$$P[T_{0n}(Z') = n]$$

$$= \frac{\theta d}{n} \exp\left\{\sum_{i \geq 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}'(n))\right\} \quad (1.2)$$

and

$$P[T_{0n}(Z') = n]$$

$$= \frac{\theta d}{n} \exp\left\{\sum_{i \geq 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}'(n))\right\} \quad (1.3)$$

Where $\varphi_{\{1,2,7\}}'(n)$ refers to the quantity derived from Z'. It thus follows that $P[A_n(C^{(n)})] \square Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the r_i' and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\varphi_{\{1,2,7\}}'(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \leq b \leq n/8$ and $n \geq n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$

$$\leq d_{TV}(L(\check{C}[1,b]), L(\check{Z}[1,b]))$$

$$\leq \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0), (D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\check{C}[1,b] \mid T_{0b}(C) = l) = L(\check{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$$d_{TV}(L(\check{C}[1,b]), L(\check{Z}[1,b]))$$

$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$

$$= \max_A \sum_{r \in A} P[T_{0b}(Z) = r]$$

$$\left\{1 - \frac{P[T_{bn}(Z) = n-r]}{P[T_{0n}(Z) = n]}\right\} \quad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$

$$=\sum_{r\geq 0} P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_{+}$$

$$\leq \sum_{r>n/2} P[T_{0b}=r]+\sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0b}=n]}$$

$$\times\left\{\sum_{s=0}^{n} P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\right\}_{+}$$

$$\leq \sum_{r>n/2} P[T_{0b}=r]+\sum_{r=0}^{[n/2]} P[T_{0b}=r]$$

$$\times\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{\{P[T_{bn}=n-s]-P[T_{bn}=n-r]\}}{P[T_{0n}=n]}$$

$$+\sum_{s=0}^{[n/2]} P[T_{0b}=r]\sum_{s=[n/2]+1}^{n} P[T=s]P[T_{bn}=n-s]/P[T_{0n}=n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2<s\leq n} P[T_{0b}=s])/P[T_{0n}=n]$$

$$\leq \frac{2\varepsilon_{\{10.5(1)\}}(n/2,b)}{n}\frac{3n}{\theta P_{\theta}[0,1]},$$

$$\frac{3n}{\theta P_{\theta}[0,1]}4n^{-2}\phi_{\{10.8\}}^{*}(n)\sum_{r=0}^{[n/2]} P[T_{0b}=r]\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{1}{2}|r-s|$$

$$\leq \frac{12\phi_{\{10.8\}}^{*}(n)}{\theta P_{\theta}[0,1]}\frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b)=2n^{-1}ET_{0b}(Z)\left\{1+\frac{6\phi_{\{10.8\}}^{*}(n)}{\theta P_{\theta}[0,1]}\right\}P$$

$$+\frac{6}{\theta P_{\theta}[0,1]}\varepsilon_{\{10.5(1)\}}(n/2,b) \qquad (1.5)$$

Required order under Conditions $(A_0),(D_1)$ and (B_{11}), if $S(\infty)<\infty$. If not, $\phi_{\{10.8\}}^{*}(n)$ can be replaced by $\phi_{\{10.11\}}^{*}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty)<\infty$. Examining the Conditions $(A_0),(D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l\geq 2} l\varepsilon_{il}=O(i^{-a_1})$ to hold for some $a_1>1$. A first observation is that a similar problem arises with

the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\overset{\sim}{n_1}$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l\geq 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)}=s]-P[T_{bn}^{(m)}=s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from $\overset{\sim}{n_1}$ of the form $\phi_1^{\theta}(n)+u_1^{*}(n)$, which is only small if $a_1>1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta>0$, since $a_2>1$ is in any case assumed. For $s\geq n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn}=s]-P[T_{bn}=s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn}=s]-P[T_{bn}=s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta>0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w)=1_{\{w=s\}}-1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn}=s]-P[T_{bn}=s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^{\theta}(n)+u_1^{*}(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form

$$|P[T_{bn}=s]-P[T_{bn}=s+t]|=O(n^{-2}t+n^{-1-a_1+\delta})$$

for any $\delta>0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1}+n^{-a_1+\delta})$ for any $\delta>0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left| d_{TV}(L(C[1,b]), L(Z[1,b])) - \frac{1}{2}(n+1)^{-1}|1-\theta| E|T_{0b} - ET_{0b}| \right|$$
$$\le \varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7,8\}}(n,b) = O(n^{-1}b[n^{-1}b + n^{-\beta_{12}+\delta}])$

for any $\delta > 0$ under Conditions $(A_0), (D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b]))$$
$$= \sum_{r\ge 0} P[T_{0b} = r]\left\{1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]}\right\}_+$$

Now we observe that

$$\left| \sum_{r\ge 0} P[T_{0b}=r]\left\{1 - \frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+ - \sum_{r=0}^{[n/2]} \frac{P[T_{0b}=r]}{P[T_{0n}=n]} \right|$$
$$\times \left| \sum_{s=[n/2]+1}^{n} P[T_{0b}=s](P[T_{bn}=n-s] - P[T_{bn}=n-r]) \right|$$
$$\le 4n^{-2}ET_{0b}^2 + (\max_{n/2 < s \le n} P[T_{0b}=s])/P[T_{0n}=n]$$
$$+ P[T_{0b} > n/2]$$
$$\le 8n^{-2}ET_{0b}^2 + \frac{3\varepsilon_{\{10.5(2)\}}(n/2, b)}{\theta P_\theta[0,1]}, \tag{1.1}$$

We have

$$\left| \sum_{r=0}^{[n/2]} \frac{P[T_{0b}=r]}{P[T_{0n}=n]} \right.$$
$$\times \left(\left\{ \sum_{s=0}^{[n/2]} P[T_{0b}=s](P[T_{bn}=n-s] - P[T_{bn}=n-r] \right\}_+ \right.$$
$$\left. - \left\{ \sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1} P[T_{0n}=n] \right\}_+ \right) \right|$$
$$\le \frac{1}{n^2 P[T_{0n}=n]} \sum_{r\ge 0} P[T_{0b}=r]\sum_{s\ge 0} P[T_{0b}=s]|s-r|$$
$$\times \left\{ \varepsilon_{\{10.14\}}(n,b) + 2(r\vee s)|1-\theta|n^{-1}\left\{K_0\theta + 4\phi^*_{\{10.8\}}(n)\right\}\right\}$$
$$\le \frac{6}{\theta n P_\theta[0,1]} ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$
$$+ 4|1-\theta|n^{-2}ET_{0b}^2\left\{K_0\theta + 4\phi^*_{\{10.8\}}(n)\right\}$$
$$(\frac{3}{\theta n P_\theta[0,1]}) \Big\}, \tag{1.2}$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]} P[T_{0b}=r]\left| \left\{\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+ \right.$$
$$\left. - \left\{\sum_{s=0} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+ \right|$$
$$\le \sum_{r=0}^{[n/2]} P[T_{0b}=r]\sum_{s>[n/2]} P[T_{0b}=s]\frac{(s-r)|1-\theta|}{n+1}$$
$$\le |1-\theta|n^{-1}E(T_{0b}1\{T_{0b} > n/2\}) \le 2|1-\theta|n^{-2}ET_{0b}^2, \tag{1.3}$$

and then by observing that

$$\sum_{r>[n/2]} P[T_{0b}=r]\left\{\sum_{s\ge 0} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}$$
$$\le n^{-1}|1-\theta|(ET_{0b}P[T_{0b} > n/2] + E(T_{0b}1\{T_{0b} > n/2\}))$$
$$\le 4|1-\theta|n^{-2}ET_{0b}^2 \tag{1.4}$$

Combining the contributions of (1.2) –(1.3), we thus find that

$$\left| d_{TV}(L(\overset{\square}{C}[1,b]), L(\overset{\square}{Z}[1,b])) \right.$$
$$\left. -(n+1)^{-1}\sum_{r\ge 0} P[T_{0b}=r]\left\{\sum_{s\ge 0} P[T_{0b}=s](s-r)(1-\theta)\right\}_+ \right|$$
$$\le \varepsilon_{\{7,8\}}(n,b)$$
$$= \frac{3}{\theta P_\theta[0,1]}\left\{\varepsilon_{\{10.5(2)\}}(n/2,b) + 2n^{-1}ET_{0b}\varepsilon_{\{10.14\}}(n,b)\right\}$$
$$+ 2n^{-2}ET_{0b}^2\left\{4 + 3|1-\theta| + \frac{24|1-\theta|\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\} \tag{1.5}$$

The quantity $\varepsilon_{\{7,8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0), (D_1)$ and (B_{12}), provided that $S(\infty) < \infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7,8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty) < \infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0} P[T_{0b}=r]\left\{\sum_{s\ge 0} P[T_{0b}=s](s-r)(1-\theta)\right\}_+$$
$$= \frac{1}{2}|1-\theta| E|T_{0b} - ET_{0b}|$$

Example 1.0. Consider the point $O = (0,...,0) \in \square^n$. For an arbitrary vector r, the coordinates of the point $x = O + r$ are equal to the respective coordinates of the vector $r : x = (x^1,...x^n)$ and $r = (x^1,...,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\square^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\square^n$: $\square^n$ = {points}, $\square^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\square^n$ treated in this way is called an *n-dimensional affine space.* (An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\square^n$ considered as an affine space we can precede in two opposite directions: $\square^n$ as an Euclidean space $\Leftarrow \square^n$ as an affine space $\Rightarrow \square^n$ as a manifold. Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\square^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\square^n$ a Euclidean space. Namely, we define the length of a vector $a = (a^1,...,a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + ... + (a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B) := \left| \vec{AB} \right| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B) \le d(A,C) + d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b) := a^1 b^1 + ... + a^n b^n \qquad (3)$$

Thus $|a| = \sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b = (a,b)$, and hence is often referred to as the "dot product". Now, for nonzero vectors, we define the angle between them by the equality

$$\cos \alpha := \frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π. For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2 \le |a|^2 |b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a + tb$, where $t \in R$. As $(a + tb, a + tb) \ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x) = x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i. From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + ... + \frac{\partial f}{\partial x^n} dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2,...$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2,...,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 +$$

$$\ldots + \frac{\partial f}{\partial x^n} h^n, \quad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \square^n$ at $t = t_0$ and with the velocity vector $x(t_0) = \upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \quad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h, where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$f(x(t_0 + \Delta t) - f(x_0)$$
$$= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t)$$
$$+ \beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t|$$
$$= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \ldots + \frac{\partial f}{\partial x^n} x^n \quad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \to \square$, $U \subset \square^n$,

$$d(f+g) = df + dg \quad (1)$$
$$d(fg) = df.g + f.dg \quad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $x(t_0) = \upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t = t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\square^m$ instead of $\square$. The only difference is that now the differential of a map $F : U \to \square^m$ at a point x will be a linear function taking vectors in $\square^n$ to vectors in $\square^m$ (instead of $\square$). For an arbitrary vector $h \in | \square^n$,

$$F(x+h) = F(x) + dF(x)(h)$$
$$+ \beta(h)|h| \quad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. We have $dF = (dF^1, \ldots, dF^m)$ and

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \ldots + \frac{\partial F}{\partial x^n} dx^n$$

$$= \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \ldots & \ldots & \ldots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix} \quad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\square^n$, the differential of a map $F : U \to \square^m$ (where $U \subset \square^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\square^m$:

$$\frac{dF(x(t))}{dt} = dF(x(t))(\dot{x}(t)) \quad (1)$$

Proof. By the definition of the velocity vector,

$$x(t + \Delta t) = x(t) + \dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t \quad (2)$$

Where $\alpha(\Delta t) \to 0$ when $\Delta t \to 0$. By the definition of the differential,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h| \qquad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. we obtain

$$F(x(t+\Delta t)) = F(x + \underbrace{\dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t)}_{h}$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t) +$$

$$\beta(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t\right|$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F: U \to V$ and $G: V \to W$, where $U \subset \Box^n, V \subset \Box^m, W \subset \Box^p$ (open domains). Let $F: x \mapsto y = F(x)$. Then the differential of the composite map $GoF: U \to W$ is the composition of the differentials of F and G:

$$d(GoF)(x) = dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\Box^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\Box^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t) = G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\Box^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence

$$d(GoF)(\dot{x}) = dG(dF(\dot{x})) \qquad \text{for an arbitrary}$$

vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\Box^n$ by $(x^1, ..., x^n)$ and in $\Box^m$ by $(y^1, ..., y^m)$, and write

$$dF = \frac{\partial F}{\partial x^1} dx^1 + ... + \frac{\partial F}{\partial x^n} dx^n \qquad (1)$$

$$dG = \frac{\partial G}{\partial y^1} dy^1 + ... + \frac{\partial G}{\partial y^n} dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF) = \frac{\partial G}{\partial y^1} dF^1 + ... + \frac{\partial G}{\partial y^m} dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i = dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF) = \begin{pmatrix} \frac{\partial G^1}{\partial y^1} & \cdots & \frac{\partial G^1}{\partial y^m} \\ \cdots & \cdots & \cdots \\ \frac{\partial G^p}{\partial y^1} & \cdots & \frac{\partial G^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \cdots & \frac{\partial z^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial z^p}{\partial x^1} & \cdots & \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} & \cdots & \frac{\partial z^1}{\partial y^m} \\ \cdots & \cdots & \cdots \\ \frac{\partial z^p}{\partial y^1} & \cdots & \frac{\partial z^p}{\partial y^m} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a} = \sum_{i=1}^{m} \frac{\partial z^\mu}{\partial y^i} \frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y \in \Box^m$ on $x \in \Box^n$ is given by the map F, the dependence of $z \in \Box^p$ on $y \in \Box^m$ is given by the map G, and the dependence of $z \in \Box^p$ on $x \in \Box^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U \subset \Box^n$. Consider also another copy of $\Box^n$, denoted for distinction $\Box_y^n$, with the standard coordinates

$(y^1...y^n)$. A system of coordinates in the open domain U is given by a map $F:V \to U$, where $V \subset \square_y^n$ is an open domain of $\square_y^n$, such that the following three conditions are satisfied :

(1) F is smooth;

(2) F is invertible;

(3) $F^{-1}:U \to V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \square_y^n$

In other words,

$$F:(y^1...,y^n) \mapsto x = x(y^1...,y^n) \qquad (1)$$

Here the variables $(y^1...,y^n)$ are the "new" coordinates of the point x

Example 1.2.   Consider a curve in $\square^2$ specified in polar coordinates as

$$x(t): r = r(t), \varphi = \varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t),\varphi(t)),(r,\varphi) \mapsto x(r,\varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$ are vectors depending on point in $\square^2$. We can compare this with the formula in the "standard" coordinates:

$\dot{x} = e_1 \dot{x} + e_2 \dot{y}$.   Consider the vectors $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos\varphi, \sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r\sin\varphi, r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$ are, respectively, the velocity vectors for the curves $r \mapsto x(r,\varphi)$ $(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r,\varphi)$ $(r = r_0 \ fixed)$. We can conclude

that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r},\dot{\varphi})$ if as a basis we take $e_r := \frac{\partial x}{\partial r}, e_\varphi := \frac{\partial x}{\partial \varphi}$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3.   The velocity vector has the same appearance in all coordinate systems.

Proof.   Follows directly from the chain rule and the transformation law for the basis e_i .In particular, the elements of the basis $e_i = \frac{\partial x}{\partial x^i}$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1,...,x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F:\square^n \to \square^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0): \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \square^n$ to vectors attached to the point $F(x) \in \square^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + ... + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1,...,e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ ... \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1}dx^1 + ... + \frac{\partial f}{\partial x^n}dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3   Consider a 1-form in $\square^2$ given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r\cos\varphi$, $y = r\sin\varphi$, hence

$dx = \cos\varphi dr - r\sin\varphi d\varphi$

$dy = \sin\varphi dr + r\cos\varphi d\varphi$

Substituting into A, we get

$A = -r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi)$

$+ r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi)$

$= r^2(\sin^2\varphi + \cos^2\varphi)d\varphi = r^2 d\varphi$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U :

$$\omega(\upsilon) = \omega_1\upsilon^1 + ... + \omega_n\upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i \upsilon^j$, where $e_i = \partial x / \partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta_j^i \qquad (2) \qquad \text{at}$$

every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \dfrac{dx}{dt'} \right\rangle$ and

$\left\langle \omega(x(t(t'))), \dfrac{dx}{dt'} \right\rangle$ As

$\left\langle \omega(x(t(t'))), \dfrac{dx}{dt'} \right\rangle = \left|\left\langle \omega(x(t(t'))), \dfrac{dx}{dt'} \right\rangle \cdot \dfrac{dt}{dt'}\right|$,

Let p be a rational prime and let $K = \Box(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p - 1$ over $\Box$. We wish to show that $O_K = \Box[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\Box[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not

divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, ..., \zeta^{p-1}$. Therefore

$$Tr_{K/\Box}(\zeta^j) = \zeta + \zeta^2 + ... + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\Box}(\zeta^j) = p - 1$ By linearity of the trace, we find that

$$Tr_{K/\Box}(1 - \zeta) = Tr_{K/\Box}(1 - \zeta^2) = ...$$

$$= Tr_{K/\Box}(1 - \zeta^{p-1}) = p$$

We also need to compute the norm of $1 - \zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + ... + 1 = \Phi_p(x)$$

$$= (x - \zeta)(x - \zeta^2)...(x - \zeta^{p-1});$$

Plugging in $x = 1$ shows that

$$p = (1 - \zeta)(1 - \zeta^2)...(1 - \zeta^{p-1})$$

Since the $(1 - \zeta^j)$ are the conjugates of $(1 - \zeta)$, this shows that $N_{K/\Box}(1 - \zeta) = p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1 - \zeta)O_K \cap \Box = p\Box$$

Proof. We saw above that p is a multiple of $(1 - \zeta)$ in O_K, so the inclusion $(1 - \zeta)O_K \cap \Box \supseteq p\Box$ is immediate. Suppose now that the inclusion is strict. Since $(1 - \zeta)O_K \cap \Box$ is an ideal of $\Box$ containing $p\Box$ and $p\Box$ is a maximal ideal of $\Box$, we must have $(1 - \zeta)O_K \cap \Box = \Box$ Thus we can write

$$1 = \alpha(1 - \zeta)$$

For some $\alpha \in O_K$. That is, $1 - \zeta$ is a unit in O_K.

COROLLARY 1.1   For any $\alpha \in O_K$, $Tr_{K/\Box}((1 - \zeta)\alpha) \in p\Box$

PROOF.   We have

$$Tr_{K/\Box}((1 - \zeta)\alpha) = \sigma_1((1 - \zeta)\alpha) + ... + \sigma_{p-1}((1 - \zeta)\alpha)$$

$$= \sigma_1(1 - \zeta)\sigma_1(\alpha) + ... + \sigma_{p-1}(1 - \zeta)\sigma_{p-1}(\alpha)$$

$$= (1 - \zeta)\sigma_1(\alpha) + ... + (1 - \zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of

K) with the usual ordering. Furthermore, $1 - \zeta^j$ is a multiple of $1 - \zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\square}(\alpha(1 - \zeta)) \in (1 - \zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = |\square(\zeta_p)$ be the pth cyclotomic field. Then $O_K = \square[\zeta_p] \cong \square[x]/(\Phi_p(x))$; Thus $1, \zeta_p, ..., \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write $\alpha = a_0 + a_1\zeta + ... + a_{p-2}\zeta^{p-2}$ With $a_i \in \square$. Then

$$\alpha(1 - \zeta) = a_0(1 - \zeta) + a_1(\zeta - \zeta^2) + ... + a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\square}(\alpha(1 - \zeta)) = pa_0$ We also have $Tr_{K/\square}(\alpha(1 - \zeta)) \in p\square$, so $a_0 \in \square$ Next consider the algebraic integer $(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + ... + a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \square$, and continuing in this way we find that all of the a_i are in $\square$. This completes the proof.

Example 1.4 Let $K = \square$, then the local ring $\square_{(p)}$ is simply the subring of $\square$ of rational numbers with denominator relatively prime to p. Note that this ring $\square_{(p)}$ is not the ring $\square_p$ of p-adic integers; to get $\square_p$ one must complete $\square_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta/\gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta/\gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$.

Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed. We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p}/pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha/\beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as

$$x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + ... + \frac{\alpha_0}{\beta_0}$$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1...\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma/\beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/\square}(\alpha O_K) = |N_{K/\square}(\alpha)|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\square$ is

Galois. Let σ be an element of $Gal(K/\square)$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\square}(\sigma(\alpha)O_K) = N'_{K/\square}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\square)$, we have $N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N'_{K/\square}(\alpha O_K)^n$ Since $N_{K/\square}(\alpha)$ is a rational integer and O_K is a free $\square$-module of rank n,

$O_K / N_{K/\square}(\alpha)O_K$ Will have order $N_{K/\square}(\alpha)^n$; therefore

$$N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N_{K/\square}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m.$

IV. CLOUD OPERATIONAL TRUST PROPERTIES

In this section we analyze the main properties, which help in assessing a Cloud provider operational trust.

A. Adaptability Property

Adaptability property reflects Cloud provider ability to provide timely and efficient reaction on infrastructure and application changes and events. Figure 3 identifies a function Ada(f1) that could be used to calculate Adaptability property; where f1 represents the factors affecting the Adaptability property, which are as follows.

• Adaptability as a Service — This service resembles Cloud internal employees when reacting to changes in infrastructure and its hosted applications. Example of such changes include: hardware failure, changes in user requirements, security incidents, etc. Factors that help in assessing Adaptability Service include the following: Mean Time to Discover (MTTD) an incident, Mean Time to Invoke (MTTI) an action to remedy the incident, and Mean Time to Recover (MTTR) an incident. Providing adaptability as an automated service that does not require human intervention should result in much quicker incident discovery time, invocation time, and recovery time. This in turn would reduce the values of MTTD, MTTI and MTTR because of the following: self-services do not require human being physical presence on incidents, do not require coordination amongst multiple team members, and do not require critical human observations. Trust in Adaptability as a self-managed service should also be considered as an important factor when calculating Ada(f1). In traditional enterprise infrastructure Trust is related to operational services which are provided by human beings and is assessed based on prior experience in the enterprise. Automated services, on the other hand, enable better measurement of trust, as the more mature and tested an adaptability service the higher trust would be granted.

• Tolerance to Attack — This is based on (a.) statistical figures on prior experience with the infrastructure ability to mitigate attacks, which can be caused by either insiders or outsiders; and (b.) Cloud provider proactivity which could be estimated based on the security risk management process.

B. Resilience Property

Resilience is the ability of a system to maintain its features (e.g. serviceability and security) despite a number of subsystem and components failures. High resilience requires a design which uses redundancy to eliminate any single points of failure, together with well crafted procedures (e.g. defining disaster recovery process). Resilient design helps in achieving higher availability and reliability, as its design approach focuses on tolerating and surviving the inevitable failures rather than trying to reduce them. Figure 4 identifies a function Res(f2) that could be used to calculate Resilience property; where f2 represents the factors affecting the Resilience property, which are as follows.

• Resilience as a Service — This service resembles Cloud internal employees when architecting the infrastructure to eliminate any single point of failure. Example of reliability services include the following: if a hardware component fails, the system services should not be affected; if a process fails the system should provide redundant services that support the failed services; and data should be replicated to protect against physical corruption, failures, and/or security attacks. Factors that help in assessing Resilience Service include the following: MTTD, MTTI, MTTR, and Trust. These follow the same description provided for Adaptability Service.

• Adaptability as a Property— The higher the Adaptability property the better the system can support resilience services, which in turn enhances the resilience property.
• Tolerance to Attack — This follows the same description provided for Adaptability Service.
• Architect — architecture properties of a system affect system resilience. These properties include: (a.) redundancy and replication of resources, (b.) individual component reliability as provided by the manufacturer; and (c.) process management that provides automated scripts and documents (these identify exact procedures on incidents).
• Feedback of Availability and Reliability as a Service — Availability and Reliability are the two main properties that are directly reflected by Resilience. The higher resilient a system, the higher availability and reliability would be expected. Availability and reliability services reflect the on-time real work performed by a system to maintain its

availability and reliability properties. Therefore, getting statistical figures about Availability and Reliability services should indicate the effectiveness of resilience services. This in turn affects resilience property.

C. Scalability Property

Scalability is about enabling the virtual infrastructure to scale resources up or down based on demand. For example, on peak periods the virtual layer should scale resources up, and similarly on off-peak periods the virtual layer should release unneeded resources by scaling down. These should be reflected at the application to support the addition and removal of virtual resources. Also, these should not affect fundamental system properties and should always enforce user requirements (e.g. security and privacy). Scalability at virtual layer can be: Horizontal Scalability, Vertical Scalability, or combination of both. Horizontal Scalability is about the amount of instances that would need to be added or removed to a system to satisfy increase or decrease in demand. Vertical Scalability is about increasing or decreasing the size of instances themselves to maintain increase or decrease in demand. In this regard application layer scalability reacts differently to both types of scalability. For example, Horizontal Scalability means the application will be replicated at the newly created VMs; however, Vertical Scalability means the application needs to take advantages of the additional allocated resources (e.g. increase memory usage, spawn additional child processes). Figure 5 identifies a function Scal(f3) that could be used to calculate Scalability property; where f3 represents the factors affecting the Scalability property, which are as follows.

• Scalability as a Service — This service resembles Cloud internal employees when architecting the infrastructure to scale up and down on demand. It also resembles application architects who design the application to take advantage of additional resources when scaling up and releasing resources when scaling down. Example of cases,

D. System Model

XML has played a tremendous role in resolving syntax heterogeneous, however, there are still heterogeneous of semantics, the application of ontology technology can be a good solution to semantic heterogeneity. In this paper, the combination of XML and Ontology is adopted to solve the heterogeneous of syntax and semantics of power system integration, in order to build the unified information platform under Smart Grid. The model is based on three-tier architecture of BIS, namely, the application layer, middle layer and data layer, this structure gives full play to the BIS and CIS architecture advantages, has realized thin-client,

distributed applications, and transparent access. System model is shown in Figure l . In accordance with the actual situation in power system, the application layer includes two types of users: one is the administrator workstation which designed specially for the staff of power system. Managers monitor the real-time data comes from the scence and various systems through a dedicated software, then give the corresponding operations to ensure that all the systems running nomally; the other is for people who use the facilities for query, the most important terminal is the browser, followed by telephone, voice, and SMS, also mobile phone platform (wap), that will meet the needs of the future Smart Grid. The middle layer is mainly responsible for the communication, integration and exchange of the heterogeneous data. This layer combinating middleware with component to achieve data integration, mainly consists of three main parts: ontology component, schema mapping component, and query component. In the environment of the future Smart Grid, by establishing the cloud platform of the power system, we can achieve the aims of saving source of hardware and software, scheduling reasonable, and giving balance of the source, as well as co-ordinating the electricity market. Taking the security requirements of the power system into account, the above three components are regarded as "private cloud", respectively, the entire middleware as the "private cloud" of the power system. When other applications or customizations need some of the above, or the whole functions, then you can directly access the cloud platform to call this function in order to assemble and reuse, without redeveloping, this implement reusability and Plug and Play functionality of component.

E. System Infrastructure an Net-AMI Meter Data Flow Illustration Through the Protocol Stack

The Net-AMI meter is the major component for communication of information flows in the smart grid distribution system networks. The Net-AMI meter needs to relay time of use metering, power information, HAN information for outage management, demand response, network optimization, distribution of renewable sources and controlling home or building appliances. As we discussed in Section III, the Net- AMI meter may not be capable of processing HAN protocols due to protocol incompatibilities issues and limited processing capabilities. The Net-AMI meter's principal architectural function is the fast communication of data to and from the cloud center. In this section, we describe how the Net-AMI meter combines all the data from various sources and hands off data to a cellular protocol [e.g., typically GSM, 3 GPP high speed packet access (HSPA), 3 GPP LTE] for over the air communication. Eventually, data is received by the cloud system for

processing. In order to process the data in the cloud in a systematic way, the Net-AMI meter appends a packet header to data to enable protocol identification and message handling. In Fig. 4(a), we illustrate an example of the mobile device initiating a control operation of in-building power system (i.e., HAN network) using the Net-AMI as relay portal. In our discussion the cognitive spectrum management (cognitive radio services) residing in cloud center, identifies the unused bands and sends a CTS signal including the query by the user to the Net-AMI meter indicating band availability. The Net-AMI responds to query (i.e., in this case it is remote home energy control) and retrieves home energy data from HAN network. The Net-AMI acts a relay portal device and performs uplink transmission to the cloud center. The cloud center performs a plethora of services such as waveform processing service, protocols service and energy manager and optimization algorithms. The energy control services are displaced back to Net-AMI via ROF through CRA for in-building control operation. The Net-AMI meter performs concatenation of energy application control (EAC) header for uplink transmission as shown in Fig. 4(c). The EAC header details the type of the HAN protocol (i.e., Zigbee, Zwave, Proprietary, and WiFi), the version of protocol, and the Net- AMI identifier used for source and destination identification. The Net-AMI meter waits for the CTS signal from CRA for uplink transmission. The CTS signal broadcasts specific information to Net-AMI meter, as shown in Fig. 4(c). The EAC header in downlink details first field as frame type. The Cloud sends CTS, query or both. This field identifies as binary 0 (CTS), query (1), and both (0, 1). The next field provides scheduling of time and frequency slots for data transfer with less interference of signals at CRA. If the Net-AMI is unable to perform transmission within a CTS time-out period, the system delays transmission until next CTS control signal. A field indicating the type of cellular protocol (i.e., LTE, GSM, and CDMA) is included in the RF carrier frequency signal to support uplink transmission. The cloud reference time is used to account for time out scenarios based on received time slots. Finally, a Net-AMI identifier is needed for destination and source routing. Based on all these inputs, the Net-AMI meter bandpass modulates the signal to a specific RF carrier frequency and performs uplink transmission. In Fig. 4(b), without loss of generality, we illustrate data flow through a cellular protocol stack with respect to the 3 GPP-LTE protocol. The Net-AMI meter concatenates an EAC header to the data packet in the EAC layer. It inserts as information packets in the cellular protocol frame (LTE in this example). The uplink and downlink transmission of the LTE protocol are below the EAC layer. Thus, LTE applies packet retransmission to the occasional uncorrectable packet errors due to fading and

pathloss. This is accomplished through a highly sophisticated two-layered retransmission scheme: a fast hybrid-ARQ protocol with low overhead feedback and support for soft combining with incremental redundancy. This is complemented with a highly reliable selective-repeat ARQ protocol in the MAC layer [41], [42]. A fixed bit 24 bit cyclic redundancy check (CRC) is performed on every block coming from above layers in physical layer. The calculation performed using CRC is appended to every transport block in order to check coherence of received signal. This process ensures data integrity and allows cellular protocols to handle the issues without our intervention. Since we included the CRC in the application layer, retransmission of packets is done in network layer.

F. Authors and Affiliations

Dr Akash Singh is working with IBM Corporation as an IT Architect and has been designing Mission Critical System and Service Solutions; He has published papers in IEEE and other International Conferences and Journals.
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