Progress in Intracoronary Optical Coherence Tomography

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Abstract

Understanding the etiology and evolution of the vulnerable coronary plaque is important for the early detection, treatment, and prevention of coronary artery disease. Intravascular optical coherence tomography (OCT) enables imaging of the coronary arteries in vivo with sufficient resolution to accurately differentiate arterial pathology, however, the clinical utility of this technology has been limited due to slow image acquisition rates. The development of high-speed Fourier-domain OCT techniques, including optical frequency-domain imaging, enables comprehensive microstructural imaging of long coronary artery segments. Other OCT advancements, including polarization sensitive OCT provides complementary birefringence information that is related to tissue composition. Together with new image processing, acquisition, and display techniques, these advances have enhanced the usability and utility of intracoronary OCT, bringing it closer to becoming a mainstream imaging modality in interventional cardiology.

Keywords- Cardiovascular, coronary arteries, intravascular imaging, optical coherence tomography (OCT), optical frequencydomain imaging (OFDI), polarization sensitive optical coherence tomography (PS-OCT).

I. INTRODUCTION

`ACUTE myocardial infarction (AMI), most frequently caused by the disruption of a vulnerable atherosclerotic plaque, is the leading cause of death in the western world [10]. Thin-cap fibroatheromas (TCFAs), the predominant form of vulnerable plaques resulting in sudden cardiac death [11], [12], have been defined as plaques with a large lipid pool, a thin fibrous cap (<65 μm), and activated macrophages near or within the fibrous cap [13]–[15]. The rupture of a TCFA, which may be precipitated by biomechanical stresses [16], [17], causes the bloodstream to be exposed to procoagulant factors, forming a nidus for thrombus. In some instances, the thrombus can impede blood flow to downstream myocardium, trigging an acute coronary event [18]. In addition to TCFA, coronary artery thrombosis has also be been attributed to eroded plaques and superficial calcific nodules [6], [19], [20]. Regardless of the underlying pathologic substrate, we do not understand which plaques will give rise to coronary thrombosis in any given patient. For this reason, current vulnerable plaque research is focused on the detection and study of the natural history of these high-risk lesions. Data gained from this research will provide clinicians with the information and tools required to guide pharmacologic and/or interventional management. Patients presenting with stenotic coronary lesions may be treated with stent implantation during percutaneous coronary intervention (PCI). The role of stent placement is to restore and maintain blood flow through the artery. While effective for this purpose, in-stent restenosis, caused by aggressive neointimal hyperplasia, is a significant problem with bare metal stents (BMS), leading to the need for a repeat PCI in a substantial number of patients. Drug-eluting stents (DES), coated with an agent designed to attenuate neointimal growth, reduce this problem, but may result in delayed endothelial healing and rare cases of late stent thrombosis [21]–[24] The potential risk of stent thrombosis in patients with DES mandates long-term administration of anticlotting drugs, which are expensive and have their own bleeding-associated complications. Given these difficulties encountered with coronary stenting, there is a need for a tool to evaluate the stent healing process, which may be used to tailor antiplatelet regimen durations on an individual patient basis. A number of imaging modalities have been investigated for studying vulnerable plaques in the hope of uncovering new knowledge regarding this disease process [25]. Both conventional and experimental intravascular imaging modalities include intravascular ultrasound (IVUS) [26]–[30], magnetic resonance imaging (MRI) [31]–[33], optical coherence tomography (OCT) [34], [35], angioscopy [6], [36]–[40], thermography [41], [42] and near-infrared [43], fluorescence [44], [45], and Raman spectroscopy [9], [46], [47]. Of these, OCT is the only imaging modality with sufficient resolution to visualize the majority of the pathologic features currently associated with the vulnerable plaque [4]. OCT has been utilized as an investigative imaging tool for the assessment of coronary artery pathology for a number of years [48], and has additionally been effective in evaluating the effects of coronary stenting [49]. Early OCT imaging studies were performed using systems based on timedomain OCT (TD-OCT) technology [5], [7], [35], [48]–[53]. One difficulty with conducting coronary OCT in vivo is the need to remove blood from the imaging field in order to clearly visualize the artery wall. Methods employed to displace blood

during OCT imaging include flushing the artery with saline both with and without proximal balloon occlusion. The prolonged balloon occlusion or flush times necessary for TD-OCT increased the risk of myocardial ischemia during the procedure, and together with the increased procedural complexity, may have limited the widespread clinical adoption of this technology. With extensive training expert users were able to mitigate these risks while acquiring pullbacks of up to 3–5 cm in length [54], [55]. As a result, thousands of coronary patients have been imaged with OCT at several hundred sites around the world with commercially available TD-OCT systems and over 200 studies have been published, mostly in clinical journals. With the advent of second generation Fourier-domain OCT (FD-OCT), which enables high-quality imaging at speeds up to 100× that of TD-OCT, 3-D imaging of long coronary segments during a brief transparent media flush is now possible. This paper addresses the imaging principles of OCT that make it an ideal tool for interrogating coronary microstructure, in addition to the recent developments in the technology, which have increased the likelihood that this imaging modality will be widely adopted in cardiology.

II. TIME-DOMAIN OPTICAL COHERENCE

Intravascular OCT is a structural imaging modality that is similar in principle to IVUS. With OCT, the echo time delay of the incident light, rather than acoustic waves, is measured using low-coherence interferometry [35]. In TD-OCT systems, a broadband light source is split into two arms, a reference arm and a sample arm. The reference arm light typically illuminates a reflector and the sample arm light is directed toward the coronary wall. Light returned from both arms is then recombined and detected. When the optical path length traveled by the light in each arm is within the coherence length of the source, the cross correlation of the two electromagnetic fields results in an interference pattern, the amplitude of which may be mapped to a pixel intensity value. By scanning the optical delay of the reference arm, interference fringes from discrete locations within the tissue are obtained and may be assembled to form profiles of reflectivity as a function of depth or A-lines. 2-D and 3-D OCT images are obtained by scanning the sample arm beam across the sample and recording A-lines at each scan position. OCT systems are based on fiber-optic technology, and therefore, are highly conducive to catheter-based imaging required for many clinical applications [56]. OCT imaging of the coronary artery

was first demonstrated in early in vitro studies, where investigators described the visualization of coronary microstructure including the adventitia, media, and intima [48], [50] Image criteria for the differentiation of coronary artery microstructures have been developed and validated in histopathologic correlative imaging studies, conducted on autopsy specimens ex vivo (see Table I). The classification criteria that are currently utilized to interpret lesion morphology in the clinical setting were developed and prospectively tested by Yabushita et al. [7]. In this study, 357 OCT-histology correlated images of atherosclerotic lesions were obtained from 90 cadavers. The investigators found that fibrous plaques could be identified by homogeneous signal-rich regions, fibrocalcific plaques by signal-poor regions with sharp borders, and lipid-rich plaques by signal-poor regions with diffuse borders (see Fig. 1). The sensitivity and specificity for plaque characterization based on these criteria were reported to range from 71% to 79% and 97% to 98% for fibrous plaques, 95% to 96% and 97% for fibrocalcific plaques, and 90% to 94% and 90% to 92% for lipid-rich plaques [7] In addition to the discrimination of plaque type, the capability of OCT to identify arterial macrophages has also been reported [2]. This study showed that macrophage density measured by OCT was correlated to immunohistochemical CD68 staining of macrophages from corresponding histopathologic slides ($r = 0.84, P < 0.0001$) [2]. Given that the presence of activated macrophages in the atherosclerotic plaque are thought to increase plaque vulnerability and probability of rupture [57]–[59]. The knowledge of macrophage distribution and density that may be determined by intracoronary OCT, may prove useful for evaluating arterial inflammation and plaque vulnerability. In 1999, the first in vivo intravascular OCT study was performed in the abdominal aorta of New Zealand white rabbits [51]. Using a 2.9 Fr OCT catheter in conjunction with a nonocclusive saline flush, the normal arterial wall microstructure, including the media and adventitia, were identified [51]. Following this initial demonstration, in vivo OCT imaging of coronary arteries was demonstrated in five swine [60]. The study revealed that intravascular OCT images provided superior resolution when compared to IVUS images obtained from the same locations, and enabled the visualization of features, such as the intima, including intimal flaps and defects, disruptions in the media, and stent strut apposition that could not be identified by IVUS [60] Based on the ability of OCT to discriminate between various intracoronary plaque microstructures and the potential of this imaging modality to have significant clinical impact, the first intravascular clinical studies with TD-OCT were published in 2002, demonstrating the safety and feasibility of this technique [5], [7], [52], [53]. Intracoronary OCT imaging in living patients enabled the visualization of coronary artery walls with unprecedented resolution. As in prior animal studies, in vivo OCT in the clinical setting was found to provide additional, more detailed structural information when compared to corresponding images obtained

with IVUS [5]. In the years, since this initial demonstration, OCT has been used extensively by a number of investigators in the clinical realm for assessing coronary plaque features [61]–[65], stent placement [49], [66], [67], apposition [49], [68]–[70], stent strut coverage [71]–[75], and thrombus [76] While early studies demonstrated a niche for TD-OCT during PCI, the clinical utility of the technology was hampered by relatively low-image acquisition rates (2–4 kHz, A-line rates), which is because of the need for mechanical actuation of the reference arm, and as a consequence of the inverse relationship between TD-OCT imaging speed and signal-to-noise ratio. The relatively slow image acquisition rates of TD-OCT was problematic as flushing blood from the field of view was the only practical solution to obtaining clear images of the artery wall and the duration of a bolus of saline within the coronary artery was limited to approximately 2 s. The recognition that the combination of a substantial increase in acquisition speed with a short nonocclusive flush could solve the blood problem and enable screening of long coronary segments [77], was the key advance that has taken intracoronary OCT to the next level required for widespread clinical adoption.

III. OPTICAL FREQUENCY DOMAIN IMAGING

FD-OCTwas the critical technical advance that enabled imaging at sufficient speeds for coronary screening during a brief, nonocclusive flush. One form of FD-OCT, optical frequencydomain imaging (OFDI) [78] also called swept-source OCT (SS-OCT) [79], is the particular implementation of FD-OCT, used in most state of the art intracoronary OCT systems. With OFDI, the cross correlation of the optical signal returning from the sample and reference arms is sampled as a function of wavenumber rather than time. The spectrally resolved interference between the sample and reference arms is generated using a rapidly tuned wavelength-swept light source with a narrow instantaneous linewidth. A square-law photodectector is used to acquire the interference signal between the two arms, while the optical path-length of the arms remain constant. Each frequency component of the interference signal is associated with a discrete depth location within the tissue. To generate an A-line, the Fourier transform of the interference fringe is calculated [78]. As in TD-OCT, 2-D and 3-D OFDI images are acquired by scanning the light from the sample arm over the tissue. By detecting all depths of the A-line simultaneously during a single sweep of the light source, the detection sensitivity of OFDI is theoretically increased to a maximum of several orders of magnitude over TD-OCT [78], [79] This increased sensitivity may be leveraged to increase the imaging speed, enabling 3-D imaging of coronary artery segments during a short nonocclusive saline/radiocontrast purge. An additional advantage of OFDI technology is that it is possible to double the interferometric ranging depth by creating a very narrow instantaneous linewidth [80], [81] or by utilizing both the positive and negative differential delays [82]–[86]. This extended ranging depth can be achieved by shifting the frequency of the detector signal by a constant value, using an acoustooptical frequency shifter in the interferometer [86], or by acquiring both the in-phase (real) and quadrature (imaginary) components of the interferometric signal [82]–[85]. The increased ranging depths (>7 mm) now obtained with OFDI allow imaging of even the largest human coronary arteries [77] Intracoronary OFDI was first demonstrated in swine studies in vivo in 2006, where comprehensive microscopy of long segments of coronary arteries was presented [77]. Forty-four in vivo swine intracoronary OFDI datasets were acquired at 108 frames per second with pullback speeds of 5 mm/s in segments up to 6 cm in length [77]. The imaging system used in this study had a source tuning range of 111 nm at a 54-kHz A-line rate corresponding to an image acquisition rate of 108 frames per second [77]. The system utilized dual-balanced, polarization diverse detection, in addition to frequency shifting to remove the depth degeneracy providing a ranging depth of 7.3 mm [77], [86]. To highlight the volumetric capabilities of the imaging technology, the investigators performed angioplasty followed by stenting in the circumflex artery of one swine. The corresponding cross-sectional OFDI images revealed clear stent strut visualization in addition to dissected intimia and media as a result of the balloon angioplasty [77]. By presenting the volumetric data in 3-D, a greater appreciation of the artery structure was realized. In 2008, the same group translated this technology to the clinical setting and published the first demonstration of intracoronary OFDI in three human patients undergoing PCI [87]. Intracoronary OFDI datasets from three patients were presented, with imaging rates of 100 frames per second, and pullback speeds ranging from 5 to 20 mm/s [87] highlights the long arterial segments that can be successfully imaged with OFDI. The imaging core was translated at a speed of 20 mm/s with a frame rate of 100 frames per second (frame size: 1536 axial points × 512 A-lines) resulting in a longitudinal imaging pitch of 200 μm. During image acquisition, the raw data was continuously streamed to a hard disk drive at a rate of 320 MB/s. The volumetric OFDI data of a right coronary artery was obtained during a single limited duration flush at 3 mL/s through a 7 Fr guide catheter. Using these parameters, a 7-cm longitudinal OFDI pullback of clear blood-free imaging was obtained in under 4 s. The single pullback shows a proximal BMS that was placed nine years prior to imaging, and a DES placed immediately prior to imaging. The wealth of

information obtained in the single pullback is highlighted by the volumetric renderings. These renderings were created offline by manually segmenting the images according to previously validated image criteria [1], [2], [7], [49] for the identification of the artery wall, lipid pools, calcific nodules, and stent struts. The macrophages highlighted in the renderings were automatically segmented with previously validated normalized image intensity metrics [2], together with the manual removal of outliers. Each of the segmented features was rendered in a different color according to the following scheme: red = artery wall, yellow = lipid pool, white = calcific nodule, blue = stent, grey = guide wire, and green = macrophage. The individual renderings were then recombined to form the final 3-D image. Due to the manual segmentation process, the time required to construct the final volume renderings approached a couple of hours, however, with the development of automated and semiautomated image processing algorithms, these times may be considerable reduced. The volume renderings clearly show a high degree of tissue coverage on the BMS. In addition, the placement of the DES over a lipid-rich plaque can be visualized. Together with Figs. 2 and 3 highlights the high level of detail that can be observed with OFDI. Of specific interest in coronary intervention are issues relating to the use and effectiveness of stents, particularly stent placement including individual stent strut apposition, and tissue coverage over the struts.

IV. POLARIZATION SENSITIVE OPTICAL COHERENCE TOMOGRAPHY

Polarization sensitive OCT (PS-OCT), another embodiment of OCT, provides a measure of tissue birefringence by detecting polarization changes in the light returning from the tissue sample being imaged [88], [89] When light travels through tissues that exhibit form birefringence, orthogonal polarization components of the light will undergo phase retardation with respect to one another. This degree of phase retardation is dependent on the orientation of the polarization state with respect to the organized linear structures within the tissue, such as collagen fibers [90]. The detected birefringence increases in tissues containing highly organized linear structures. PS-OCT provides complementary image information to structural OCT images that may assist in the identification of the intravascular tissue composition, and may additionally provide insight to the mechanical stability of atherosclerotic plaques [91] PS-OCT has been demonstrated in histopathologic correlative studies conducted ex vivo to provide a quantitative measure of the collagen content, collagen fiber thickness, and smooth muscle content in atherosclerotic plaques [91]. In 2006, using a spectral-domain PS-OCT imaging system, Nadkarni et al. imaged aortic plaques and compared the PS-OCT spatially averaged birefringence with the plaque collagen content and thickness, and smooth muscle cell content measured from histologic sections stained with picrosirius red and alpha-smooth muscle actin, respectively [91]. This ex vivo study revealed a high-positive correlation between the PS-OCT measured birefringence and the total collagen content ($r = 0.67$, $p = 0.001$), the thick collagen fiber content ($r = 0.76$, $p = 0.001$), and the smooth muscle cell content ($r = 0.74$, $p = 0.01$) [91], providing compelling evidence that the measurement of artery birefringence may aid in determining the tissue composition of plaques, information that may be used to assess mechanical stability. In intravascular and other catheter-based PS-OCT systems, it is necessary to use optical fibers to transmit the imaging signal to the tissue of interest. Maintaining the polarization state of the transmitted light in fiber-based systems is difficult, even with polarization maintaining fibers, as the polarization state is susceptible to stresses acting on the fiber. One method for circumventing this issue is to modulate the polarization state of the source incident on the sample tissue between two perpendicular states in successive A-line pairs. This modulation ensures that the polarization state of the light source differs for at least one of a successive pair of A-lines, from that of the linear birefringence axis of the sample. Each A-line pair is subsequently combined to form a single axial profile, using either Stokes vector [92] or Jones matrix [93] analysis. This method of fiber-based PS-OCT has been demonstrated in both spectraldomain PS-OCT [94]–[96] and OFDI systems [97]. Recently however, a novel approach to PS-OCT unique to OFDI has been demonstrated that utilizes frequency multiplexing to enable illumination and detection of two polarization states simultaneously [98]. This new implementation of PS-OFDI obviates the need for modulating the polarization state of the source between successive A-lines. PS-OFDI with frequency multiplexing has been demonstrated in ex vivo studies through an intracoronary catheter at an A-line rate of 62 kHz [98]. Structural and a PS-OFDI image acquired from a human coronary artery in vivo. The PS-OFDI image adds additional detail regarding the structural integrity of the artery that can be inferred from the tissue birefringence strength map.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r)=\sum_{j=1}^{p}\int_{\Omega}J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\ \qquad\qquad +I_i^{ext}(r,t), \qquad t\geq 0, 1\leq i\leq p, \\ V_i(t,r)=\phi_i(t,r) \qquad\qquad t\in[-T,0] \end{cases} \tag{1}$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d . The vector r and $\bar{r}$ represent points in Ω . The function $S : R \to (0,1)$ is the normalized sigmoid function:

$$S(z) = \frac{1}{1+e^{-z}} \qquad (2)$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i = 1,..., p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i = 1,..., p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p \times p$ matrix of functions $J = \{J_{ij}\}_{i,j=1,...,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i = 1,..., p,$ determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i = 1,..., p,$ determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i = 1,..., p,$ determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S : R^p \to R^p$, defined by $S(x) = [S(\sigma_1(x_1 - h_1)),..., S(\sigma_p - h_p))]$, and the diagonal $p \times p$ matrix $L_0 = diag(l_1,..., l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt} + l_i)$ is replaced by $(\frac{d}{dt} + l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt} + l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\overline{\Omega}^2, R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r}) \neq \tau_{ij}(\bar{r},r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,\bar{r}\in\Omega\times\Omega)} \tau_{i,j}(r,\bar{r}) \qquad (3)$$

Hence we choose $T = \tau_m$

A. *Mathematical Framework*

A convenient functional setting for the non-delayed packet field equations is to use the space $F = L^2(\Omega, R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U \rangle_F = \sum_{i=1}^{p} \int_\Omega V_i(r)U_i(r)dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m, 0], F)$ with $\|\phi\| = \sup_{t\in[-\tau_m,0]} \|\phi(t)\| F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t+\theta), \theta \in [-\tau_m, 0]$, we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0V(t) + L_1S(V_t) + I^{ext}(t), \\ V_0 = \phi \in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1 : C \to F, \\ \phi \to \int_\Omega J(.,\bar{r})\phi(\bar{r}, -\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \leq \|J\|_{L^2(\Omega^2, R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $J \in L^2(\Omega^2, R^{p\times p})$,
2. The external current $I^{ext} \in C^0(R, F)$,
3. $\tau \in C^0(\overline{\Omega^2}, R_+^{p\times p}), \sup_{\overline{\Omega^2}} \tau \leq \tau_m$.

Then for any $\phi \in C$, there exists a unique solution $V \in C^1([0,\infty), F) \cap C^0([-\tau_m, \infty, F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

B. *Boundedness of Solutions*

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t \in R^+} \left\| I^{ext}(t) \right\|_F < \infty.$

Proof :Let us defined $f : R \times C \to R^+$ as

$$f(t, V_t) \overset{def}{=} \left\langle -L_0 V_t(0) + L_1 S(V_t) + I^{ext}(t), V(t) \right\rangle_F = \frac{1}{2} \frac{d \left\| V \right\|_F^2}{dt}$$

We note $l = \min_{i=1,\dots p} l_i$

$$f(t, V_t) \le -l \left\| V(t) \right\|_F^2 + (\sqrt{p|\Omega|} \left\| J \right\|_F + I) \left\| V(t) \right\|_F$$

Thus, if

$$\left\| V(t) \right\|_F \ge 2 \frac{\sqrt{p|\Omega|}.\left\| J \right\|_F + I}{l} \overset{def}{=} R, f(t, V_t) \le -\frac{lR^2}{2} \overset{def}{=} -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \ge 0s$ and that $f < 0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\left\| V_0 \right\|_C < R$ and set $T = \sup\{t \mid \forall s \in [0,t], V(s) \in \overline{B_R}\}.$ Suppose that $T \in R,$ then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have $\frac{d}{dt} \left\| V \right\|_F^2 \mid_{t=T} = f(T, V_T) \le -\delta < 0$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and small enough, $V(T + \varepsilon) \in \overline{B_R}$ which contradicts the definition of T. Thus $T \notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0) \in \partial B_R$ implies that $\forall t > 0, V(t) \in B_R$. Finally we consider the case $V(0) \in C\overline{B_R}$. Suppose that $\forall t > 0, V(t) \notin \overline{B_R},$ then $\forall t > 0, \frac{d}{dt} \left\| V \right\|_F^2 \le -2\delta,$ thus $\left\| V(t) \right\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T > 0 \mid V(T) \in B_R.$

Proposition 1.1 : Let s and t be measured simple functions on X. for $E \varepsilon M$, define

$$\phi(E) = \int_E s \, d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X (s+t) d\mu = \int_X s \, d\mu + \int_X t d\mu \qquad (2)$$

Proof : If s and if $E_1, E_2, \dots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r)$$

$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \phi(E_r)$$

Also, $\varphi(\phi) = 0,$ so that φ is not identically ∞.

Next, let s be as before, let $\beta_1, \dots, \beta_m$ be the distinct values of t,and let $B_j = \{x : t(x) = \beta_j\}$ If $E_{ij} = A_i \cap B_j,$ the $\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij})$

and $\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \le i \le n, 1 \le j \le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon > 0,$ then there exists a polynomial P such that $\left| f(z) = P(z) \right| < \varepsilon$ for all $z \varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously

satisfied, and the conclusion holds for every $f \varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2) - f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2 - z_1| \le \delta$. Since f is uniformly continous, we have $\lim_{\delta \to 0} \omega(\delta) = 0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10{,}000\ \omega(\delta) \quad (z \varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi \varepsilon C_c^{'}(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \le \omega(\delta), \qquad (3)$$

$$|(\partial\Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z) = -\frac{1}{\pi}\iint_X \frac{(\partial\Phi)(\zeta)}{\zeta - z} d\zeta d\eta \qquad (\zeta = \xi + i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi\delta^2}\left(1 - \frac{r^2}{\delta^2}\right)^2 \qquad (0 \le r \le \delta), \qquad (6)$$

And define

$$A(z) = a(|z|) \qquad (7)$$

For all complex z. It is clear that $A \varepsilon C_c^{'}(R^2)$. We claim that

$$\iint_{R^s} A = 1, \qquad (8)$$

$$\iint_{R^2} \partial A = 0, \qquad (9)$$

$$\iint_{R^3} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A / \partial\theta = 0$,

$$\partial A / \partial r = -a^{'},$$

Now define

$$\Phi(z) = \iint_{R^2} f(z - \zeta) A d\xi d\eta = \iint_{R^2} A(z - \zeta) f(\zeta) d\xi d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z) = \iint_{R^2} [f(z - \zeta) - f(z)] A(\xi) d\xi d\eta \qquad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A \varepsilon C_c^{'}(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$\begin{aligned}(\partial\Phi)(z) &= \iint_{R^2} (\overline{\partial A})(z - \zeta) f(\zeta) d\xi d\eta \\ &= \iint_{R^2} f(z - \zeta)(\partial A)(\zeta) d\xi d\eta \\ &= \iint_{R^2} [f(z - \zeta) - f(z)](\partial A)(\zeta) d\xi d\eta \qquad (13)\end{aligned}$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi = 0$ in G, where G is the set of all $z \varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z \varepsilon G); \qquad (14)$$

Note that $\partial f = 0$ in G, since f is holomorphic there. Now if $z \varepsilon G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\begin{aligned}\Phi(z) &= \int_0^{\delta} a(r) r dr \int_0^{2\pi} f(z - re^{i\theta}) d\theta \\ &= 2\pi f(z) \int_0^{\delta} a(r) r dr = f(z) \iint_{R^2} A = f(z) \qquad (15)\end{aligned}$$

For all $z\ \varepsilon\ G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact

and that X can be covered by finitely many open discs $D_1,...,D_n$, of radius 2δ, whose centers are not in K. Since S^2-K is connected, the center of each D_j can be joined to ∞ by a polygonal path in S^2-K. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that S^2-E_j is connected and so that $K\cap E_j=\phi.$ with $r=2\delta$. There are functions $g_j \varepsilon H(S^2-E_j)$ and constants b_j so that the inequalities.

$$\left|Q_j(\zeta,z)\right|<\frac{50}{\delta}, \qquad (16)$$

$$\left|Q_j(\zeta,z)-\frac{1}{z-\zeta}\right|<\frac{4,000\delta^2}{\left|z-\zeta\right|^2} \qquad (17)$$

Hold for $z\notin E_j$ and $\zeta\in D_j$, if

$$Q_j(\zeta,z)=g_j(z)+(\zeta-b_j)g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1\cup...\cup E_n$. Then Ω is an open set which contains K. Put $X_1=X\cap D_1$ and $X_j=(X\cap D_j)-(X_1\cup...\cup X_{j-1}),$ for $2\le j\le n,$

Define

$$R(\zeta,z)=Q_j(\zeta,z) \qquad (\zeta\varepsilon X_j, z\,\varepsilon\,\Omega) \qquad (19)$$

And

$$F(z)=\frac{1}{\pi}\iint_X(\partial\Phi)(\zeta)R(\zeta,z)d\zeta d\eta \qquad (20)$$

$(z\ \varepsilon\ \Omega)$

Since,

$$F(z)=\sum_{j=1}\frac{1}{\pi}\iint_{X_i}(\partial\Phi)(\zeta)Q_j(\zeta,z)d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F\varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$\left|F(z)-\Phi(z)\right|<\frac{2\omega(\delta)}{\pi\delta}\iint_X\left|R(\zeta,z)\right.$$

$$\left.-\frac{1}{z-\zeta}\right|d\xi d\eta \quad (z\ \varepsilon\ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta\ \varepsilon\ X$ and $z\ \varepsilon\ \Omega$. Now fix $z\ \varepsilon\ \Omega.$, put $\zeta=z+\rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho<4\delta$, by (17) if $4\delta\le\rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta}+\frac{1}{\rho}\right)\rho d\rho=808\pi\delta \qquad (23)$$

And

$$2\pi\int_{4\delta}^{\infty}\frac{4,000\delta^2}{\rho^2}\rho d\rho=2,000\pi\delta. \qquad (24)$$

Hence (22) yields

$$\left|F(z)-\Phi(z)\right|<6,000\omega(\delta) \qquad (z\ \varepsilon\ \Omega) \qquad (25)$$

Since $F\ \varepsilon\ H(\Omega), K\subset\Omega,$ and S^2-K is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f\varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z)=-\frac{1}{\pi}\iint_{R^2}\frac{(\partial f)(\zeta)}{\zeta-z}d\xi d\eta$$

$$(\zeta=\xi+i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta)=f(z+re^{i\theta}),\ r>0,\ \theta$ real

If $\zeta=z+re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta)=\frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon\to 0,$ of

$$-\frac{1}{2}\int_\varepsilon^\infty\int_0^{2\pi}\left(\frac{\partial\varphi}{\partial r}+\frac{i}{r}\frac{\partial\varphi}{\partial\theta}\right)d\theta dr \qquad (4)$$

For each $r>0,\varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi/\partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_\varepsilon^\infty\frac{\partial\varphi}{\partial r}dr=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon,\theta)d\theta \qquad (5)$$

As $\varepsilon \to 0,\ \varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^\alpha \in a$ and $X^\beta \in k[X_1,...X_n]$, then $X^\alpha X^\beta = X^{\alpha+\beta} \in a$, and so A satisfies $(*)$. Conversely,

$$(\sum_{\alpha\in A} c_\alpha X^\alpha)(\sum_{\beta\in \square^n} d_\beta X^\beta) = \sum_{\alpha,\beta} c_\alpha d_\beta X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^\alpha, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals (X^n), $n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^\alpha \mid \alpha \in A\rangle$ for the ideal corresponding to A (subspace generated by the $X^\alpha, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^\alpha, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{\beta \in \square^n \mid \beta - \alpha \in \square^n, \quad \textit{some } \alpha \in S\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^\alpha, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^\beta \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^\beta = X^\alpha X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^\alpha \mid X^\beta \Leftrightarrow \beta - \alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that

$$A = \{\beta \in \square^n \mid \beta - \alpha_i \in \square^2, \ \textit{some} \ \alpha_i \in S\}$$

(The α_i's are the corners of A) Moreover, $a \overset{df}{=} \langle X^\alpha \mid \alpha \in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find

$$f = a_1g_1 + ... + a_sg_s + r, \qquad a_i, r \in k[X_1,...,X_n]$$

, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1),...,LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1,...,g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1, |...,g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1),...,LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1,...,X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1,...X_{n-1}][X_n] \to k[X_1,...X_n]$ is an isomorphism – this simply says that every

polynomial f in n variables $X_1,...X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,...,X_n]$:

$$f(X_1,...X_n) = a_0(X_1,...X_{n-1})X_n^r + ... + a_r(X_1,...X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0X^r + a_1X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1,...,g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write

$$a = \sum b_i a_i, \qquad b_i \in A,$$

a_i = *leading coefficient of* g_i

Now

$f - \sum b_i g_i X^{s-r_i}$, $r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \mod(g_1,...g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1},...,g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \mod(g_{d,1},...g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1},...g_{r-1,m_{r-1}},...g_{0,1},...g_{0,m_0})$ Hence

$$f_t \in (g_1,...g_m g_{r-1,1},...g_{r-1,m_{r-1}},...,g_{0,1},...,g_{0,m_0})$$

and so the polynomials $g_1,...,g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x.x$ is most conveniently defined as a function from D *to* D, which must then be regarded as an element of *D*. Let $\psi:[D \to D] \to D$ be the function that picks out elements of *D* to represent elements of $[D \to D]$ and $\phi: D \to [D \to D]$ be the function that maps elements of *D* to functions of *D*. Since $\psi(f)$ is intended to represent the function f as an element of *D*, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D \to D]}$ Furthermore, we often want to view every element of *D* as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that ϕ *and* ψ are inverses--- that is, *D* is isomorphic to the space of functions from *D to D* that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of *C*. Each element of *D* corresponds to either an element of *A* or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains *X* such that $X \cong A + [X \to X]$, and such that for any domain *Y* also satisfying this equation, there is an embedding of *X* to *Y* --- a pair of maps

$$X \underset{f^R}{\overset{f}{\rightleftarrows}} Y$$

Such that

$$f^R \, o \, f = id_X$$
$$f \, o \, f^R \subseteq id_Y$$

Where $f \subseteq g$ means that f *approximates* g in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering *F* not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, *F*.

Definition 1.3: Let ***K*** be a category and $F : K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a ***K-object*** and $a : F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a ***K-object*** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category ***K*** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$$

Recall that a cocone μ of an $\omega - chain$ Δ is a *K*-object *X* and a collection of K –*arrows* $\{\mu_i : D_i \to X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} o \, f_i$ for all $i \geq 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of $\mu's$ components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with the property that if $\nu : \Delta \to X^{'}$ is also a cocone then there exists a unique mediating arrow $k : X \to X^{'}$ such that for all $i \geq 0,, \; v_i = k \, o \, \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - co\lim its$. Dually, an $\omega^{op} - chain$ in ***K*** is a diagram of the following form:

$$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} .....$$

A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain$ Δ is a ***K***-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \geq 0\}$ such that for all $i \geq 0, \; \mu_i = f_i \, o \, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op} - chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X^{'} \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X^{'} \to X$ such that for all $i \geq 0, \mu_i \, o \, k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of ***K,*** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each ***K***-object A. It is also convenient to write $\Delta^{-} = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$ to denote all of Δ except D_o and f_0. By analogy, μ^{-} is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under ***F*** we write

$$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} .....$$

and $F(\mu) = \{F(\mu_i) \mid i \geq 0\}$

We write F^i for the ***i***-fold iterated composition of ***F*** – that is, $F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let ***K*** be a category with initial object $\perp$ and let $F : K \to K$ be a functor. Define the $\omega - chain \, \Delta$ by

$$\Delta = \perp \xrightarrow{!\perp \to F(\perp)} F(\perp) \xrightarrow{F(!\perp \to F(\perp))} F^2(\perp) \xrightarrow{F^2(!\perp \to F(\perp))} .....$$

If both $\mu : \Delta \to D$ and $F(\mu) : F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d : F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^{-}

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, ...X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, ...x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1}) ...$$
$$..P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \leq P(x_1, x_2, ...x_n) \leq 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the

Markov condition is satisfied. To do this, we need show for $1 \le k \le n$ that whenever

$$P(pa_k) \neq 0, if \;\; P(nd_k \mid pa_k) \neq 0$$
$$and \;\; P(x_k \mid pa_k) \neq 0$$
$$then \; P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, ...X_{k-1}\}$$
$$Let$$
$$D_k = \{X_{k+1}, X_{k+2}, ...X_n\}$$

follows $\sum_{d_k}$

We define the m^{th} *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ has degree $\varphi(m)$. *The roots of* $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \le k \prec m, (k,m) = 1, \quad where$$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another,or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n , then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. Since $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6 If m and n are relatively prime, then

$$Q(\xi_m, \xi_n) = Q(\xi_{nm})$$

and

$$Q(\xi_m) \cap Q(\xi_n) = Q$$

(Recall the $Q(\xi_m, \xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m \xi_n$ is a primitive mn^{th} root of unity, so that

$$Q(\xi_{mn}) \subseteq Q(\xi_m, \xi_n)$$
$$[Q(\xi_m, \xi_n) : Q] \le [Q(\xi_m) : Q][Q(\xi_n : Q]$$
$$= \varphi(m)\varphi(n) = \varphi(mn);$$

Since $[Q(\xi_{mn}) : Q] = \varphi(mn);$ this implies that $Q(\xi_m, \xi_n) = Q(\xi_{nm})$ We know that $Q(\xi_m, \xi_n)$ has degree $\varphi(mn)$ over Q , so we must have

$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(n)$$

and

$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(m)$$

$$[Q(\xi_m) : Q(\xi_m) \cap Q(\xi_n)] \ge \varphi(m)$$

And thus that $Q(\xi_m) \cap Q(\xi_n) = Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m, \xi_n) = Q(\xi_{[m,n]})$$

And

$$Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)});$$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m = p_1^{e_1}.....p_k^{e_k}$ *and* $p_1^{f_1}....p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)

$$Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$$

Thus

$$\begin{aligned} Q(\xi_m,\xi_n) &= Q(\xi_{p_1^{e_1}}).....Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{\max(e_1,f_1)}}).....Q(\xi_{p_1^{\max(e_k,f_k)}}) \\ &= Q(\xi_{p_1^{\max(e_1,f_1)}.....p_1^{\max(e_k,f_k)}}) \\ &= Q(\xi_{[m,n]}); \end{aligned}$$

An entirely similar computation shows that $Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i,y_i) = \log_2 \frac{P(x_i/y_i)}{P(x_i)} \; bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input –output pair (x_i,y_i) for which

$$P(x_i/y_j) = 1 \; and \; I(x_i,y_j) = \log_2 \frac{1}{P(x_i)} \quad \text{bits;}$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j) = P(x_i)$ and also $I(x_i,y_j) = 0$; that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y) = \sum_{i.j} P(x_i,y_j)I(x_i,y_j) = \sum_{i.j} P(x_i,y_j)\log_2 \left[\frac{P(x_i/y_j)}{P(x_i)} \right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i,y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y) = \sum_{i.j} P(x_i,y_j)$$

$$= \sum_{i.j} P(x_i,y_j)\log_2 \left[\frac{1}{P(x_i)} \right]$$

$$-\sum_{i.j} P(x_i,y_j)\log_2 \left[\frac{1}{P(x_i/y_j)} \right]$$

$$\sum_{i.j} P(x_i,y_j)\log_2 \left[\frac{1}{P(x_i)} \right]$$

$$= \sum_i \left[P(x_i/y_j)P(y_j) \right] \log_2 \frac{1}{P(x_i)}$$

$$\sum_i P(x_i)\log_2 \frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H(X/Y)$$

Where $H(X/Y) = \sum_{i,j} P(x_i,y_j)\log_2 \frac{1}{P(x_i/y_j)}$ is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j) \log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j) \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \le 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i \log_2 \left(\frac{Q_i}{P_i}\right) \le 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \le 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

$$+ \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ *and* $P(x_2) = 1 - \alpha,$ and transition probabilities

$P(y_3/x_2) = 1 - p$ *and* $P(y_2/x_1) = 0,$

and $P(y_3/x_1) = 0$

and $P(y_1/x_2) = p$

and $P(y_3/x_2) = 1 - p$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean n-space. $p(y \mid x)$ for the density $p_n(y_1, ..., y_n \mid x_1, ...x_n)$ and F *for* F_n. For any real number a, let

$$A = \left\{(x,y) : \log \frac{p(y \mid x)}{p(y)} > a\right\} \qquad (1)$$

Then for each positive integer u, there is a code (u, n, λ) such that

$$\lambda \le ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \qquad (2)$$

Where

$P\{(X,Y) \in A\} = \int_A ... \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y \mid x)$

and

$$P\{X \in F\} = \int_F ... \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$P\{Y \in A_{x^1} \mid X = x^{(1)}\} \ge 1 - \varepsilon$

where $A_x = \{y : (x,y) \varepsilon A\};$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)}, ..., x^{(k-1)}$ and $B_1, ..., B_{k-1}$, select $x^k \in F$ such that

$$P\left\{Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i \mid X = x^{(k)}\right\} \geq 1-\varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets B_i, $i = 1,2,...,u,$ form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \geq u$ by showing that $\varepsilon \leq te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (If \quad t = 0, \ take \ \ B = \phi). \ Then$$

$$P\{(X,Y) \in A\} = \int_{(x,y)\in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y \mid x)dy\,dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y \mid x)dy\,dx + \int_x p(x)$$

C. **Algorithms**

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1, ..., s_m\}$, we shall write $(s_1, ..., s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b \mid a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab \mid a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_n)$, then $ab = (a_1 b_1, ..., a_i b_j, ..., a_m b_n)$ .Let a be an ideal of A. The set of cosets of a in A forms a ring A/a , and $a \mapsto a + a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p \neq A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \neq 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \neq| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and $(a,b) \in c$, then $(a,0) = (a,b)(1,0) \in c$ and $(0,b) = (a,b)(0,1) \in c$. This shows that $c = a \times b$ with

$$a = \{a \mid (a,b) \in c \ \ some \ \ b \in b\}$$

and

$$b = \{b \mid (a,b) \in c \ \ some \ \ a \in a\}$$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A \to B$. A *homomorphism of* A -algebra $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1, ..., x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1, ..., X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k -algebra. If $1 \neq 0$ in A, then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$.

Polynomial rings. Let k be a field. A *monomial* in $X_1, ..., X_n$ is an expression of the form $X_1^{a_1} ... X_n^{a_n}, \quad a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^{α}, $\alpha = (a_1, ..., a_n) \in \square^n$. The elements of the polynomial ring $k[X_1, ..., X_n]$ are finite sums $\sum c_{a_1 ... a_n} X_1^{a_1} ... X_n^{a_n}, \quad c_{a_1 ... a_n} \in k, \quad a_j \in \square$

With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1,...,X_n]$ as a k -vector space. The ring $k[X_1,...,X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1,...,X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f=gh \Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q,r \in k[X]$ such that $f=qg+r$ with either $r=0$ or deg r < deg g. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) ***lexicographic*** *ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha=(a_1,...a_n)$ and $\beta=(b_1,...b_n)$ be two elements of $\square^n$; then $\alpha > \beta$ *and* $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha-\beta \in \square$, the left most nonzero entry is positive. For example, $XY^2 > Y^3Z^4$; $X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha-\beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ *(total degree greater)*

$XY^5Z^2 > X^4YZ^3$, $X^5YZ > X^4YZ^2$.

Orderings on $k[X_1,...X_n]$. Fix an ordering on the monomials in $k[X_1,...X_n]$. Then we can write an element f of $k[X_1,...X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$f=4XY^2Z+4Z^2-5X^3+7X^2Z^2$

as

$f=-5X^3+7X^2Z^2+4XY^2Z+4Z^2$ (lex)

or

$f=4XY^2Z+7X^2Z^2-5X^3+4Z^2$ $(grevlex)$

Let $\sum a_\alpha X^\alpha \in k[X_1,...,X_n]$, in decreasing order:

$f=a_{\alpha_0}X^{\alpha_0}+_{\alpha_1}X^{\alpha_1}+..., \quad \alpha_0>\alpha_1>..., \quad \alpha_0 \neq 0$

Then we define.

- The *multidegree* of f to be multdeg(f)= α_0;
- The *leading coefficient of* f to be $LC(f)=a_{\alpha_0}$;
- The *leading monomial of* f to be $LM(f)=X^{\alpha_0}$;
- The *leading term of* f to be $LT(f)=a_{\alpha_0}X^{\alpha_0}$

For the polynomial $f=4XY^2Z+...$, the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in** $k[X_1,...X_n]$. Fix a monomial ordering in $\square^2$. Suppose given a polynomial f and an ordered set $(g_1,...g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,...a_s$ and r such that $f=a_1g_1+...+a_sg_s+r$ Where either $r=0$ or no monomial in r is divisible by any of $LT(g_1),...,LT(g_s)$ **Step 1:** If $LT(g_1)|LT(f)$, divide g_1 into f to get

$$f=a_1g_1+h, \qquad a_1=\frac{LT(f)}{LT(g_1)} \in k[X_1,...,X_n]$$

If $LT(g_1)|LT(h)$, repeat the process until $f=a_1g_1+f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f=a_1g_1+...+a_sg_s+r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),...LT(g_s)$ **Step 2:** Rewrite $r_1=LT(r_1)+r_2$, and repeat Step 1 with r_2 for f : $f=a_1g_1+...+a_sg_s+LT(r_1)+r_3$ (different a_i's) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the

ideal $a=(Y^2-X^3)$ contains Y^2-X^3 but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if $\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$ all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal,* and let $A=\{\alpha \mid X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A,\ \beta \in \mathbb{N}^n \Rightarrow \alpha+\beta \in$ (*) And a is the k-subspace of $k[X_1,...,X_n]$ generated by the $X^\alpha, \alpha \in A$. Conversely, of A is a subset of $\mathbb{N}^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,...,X_n]$ generated by $\{X^\alpha \mid \alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,...,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,...,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)}=(C_1^{(n)},...,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)}=c]=\frac{1}{n!}N(n,c)=1\left\{\sum_{j=1}^n jc_j=n\right\}\prod_{j=1}^n\left(\frac{1}{j}\right)^{c_j}\frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \mathbb{N}_+^n$.

Lemma1.7 For nonnegative integers $m_{1,...,}m_n$,

$$E\left(\prod_{j=1}^n (C_j^{(n)})^{[m_j]}\right)=\left(\prod_{j=1}^n\left(\frac{1}{j}\right)^{m_j}\right)1\left\{\sum_{j=1}^n jm_j \le n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]}/c_j^{!}=1/(c_j-m_j)!$ when $c_j \ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m=\sum jm_j$. Then, with the first sum indexed by $c=(c_1,...c_n) \in \mathbb{N}_+^n$ and the last sum indexed by $d=(d_1,...,d_n) \in \mathbb{N}_+^n$ via the correspondence $d_j=c_j-m_j$, we have

$$E\left(\prod_{j=1}^n (C_j^{(n)})^{[m_j]}\right)=\sum_c P[C^{(n)}=c]\prod_{j=1}^n (c_j)^{[m_j]}$$

$$=\sum_{c:c_j\ge m_j \text{ for all } j} 1\left\{\sum_{j=1}^n jc_j=n\right\}\prod_{j=1}^n\frac{(c_j)^{[m_j]}}{j^{c_j}c_j!}$$

$$=\prod_{j=1}^n\frac{1}{j^{m_j}}\sum_d 1\left\{\sum_{j=1}^n jd_j=n-m\right\}\prod_{j=1}^n\frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m \le n)$, corresponding to the fact that if $n-m \ge 0$, then $d_j=0$ for $j>n-m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1,...,d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]}=j^{-r}1\{jr \le n\} \qquad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^n (C_j^{(n)})^{[m_j]}\right)=E\left(\prod_{j=1}^n Z_j^{[m_j]}\right)1\left\{\sum_{j=1}^n jm_j \le n\right\}, \qquad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j)=1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)}=k]=\frac{j^{-k}}{k!}\sum_{l=0}^{[n/j]-k}(-1)^l\frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1,2,...n\}$, so that $|I|=n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha|=(n-j)!$, since the elements of $\{1,2,...,n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There

are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n-rj)!1(rj \le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n-rj)!1(rj \le n) \times \frac{n^{[rj]}}{j^r r!}\frac{1}{n!} = 1(rj \le n)\frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \ge 0}(-1)^l \binom{k+l}{l} S_{k+l,}$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0,1,...,n,$

$$P[C_1^{(n)} = k] = \frac{1}{k!}\sum_{l=0}^{n-k}(-1)^l \frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1.$ In particular, for $n \ge 2,$ the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)},...,C_b^{(n)})$ for any $1 \le b \le n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1,...,c_b) \in \mathbb{Z}_+^b$ with $m = \sum ic_i,$

$$P[(C_1^{(n)},...,C_b^{(n)}) = c] = \left\{\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{c_i}\frac{1}{c_i!}\right\}\sum_{\substack{l \ge 0 \text{ with} \\ \sum il_i \le n-m}}(-1)^{l_1+...+l_b}\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{l_i}\frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)},...,C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = ... = m_n = 0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed $j,$ as $n \to \infty,$

$$P[C_j^{(n)} = k] \to \frac{j^{-k}}{k!}e^{-1/j}, \quad k = 0,1,2,...,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j;$ we use the notation $C_j^{(n)} \to_d Z_j$ where $Z_j \sim P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\sim$ with intensity j^{-1}. That is, as $n \to \infty,$

$$(C_1^{(n)}, C_2^{(n)},...) \to_d (Z_1, Z_2,...) \qquad (1.1)$$

Where the $Z_j, j = 1,2,...,$ are independent Poisson-distributed random variables with $E(Z_j) = \frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b \ge 1,$ as $n \to \infty,$

$$P[(C_1^{(n)},...,C_b^{(n)}) = c] \to P[(Z_1,...,Z_b) = c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b = 1$. Using properties of alternating series with decreasing terms, for $k = 0,1,...,n,$

$$\frac{1}{k!}\left(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!}\right) \le \left|P[C_1^{(n)} = k] - P[Z_1 = k]\right| \le \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!}\frac{n}{n+2} \le \sum_{k=0}^{n}\left|P[C_1^{(n)} = k] - P[Z_1 = k]\right| \le \frac{2^{n+1}-1}{(n+1)!} \qquad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ...\right) < \frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $\mathrm{P}\left[A_n(C^{(n)})\right]$ under conditions (A_0) and $(B_{01}),$ where

$$A_n(C^{(n)}) = \bigcap_{1 \le i \le n}\ \bigcap_{r_i'+1 \le j \le r_i}\left\{C_{ij}^{(n)} = 0\right\},$$

and $\zeta_i = (r_i^{'}/r_{id}) - 1 = O(i^{-g^{'}})$ as $i \to \infty$, for some $g^{'} > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0m}(Z^{'}) = n]}{P[T_{0m}(Z) = n]}$$

$$\prod_{\substack{1 \le i \le n \\ r_i^{'}+1 \le j \le r_i}} \left\{1 - \frac{\theta}{ir_i}(1 + E_{i0})\right\} \qquad (1.1)$$

$$P[T_{0n}(Z^{'}) = n]$$

$$= \frac{\theta d}{n} \exp\left\{\sum_{i \ge 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi^{'}_{\{1,2,7\}}(n))\right\} \qquad (1.2)$$

and

$$P[T_{0n}(Z^{'}) = n]$$

$$= \frac{\theta d}{n} \exp\left\{\sum_{i \ge 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}(n))\right\} \qquad (1.3)$$

Where $\varphi^{'}_{\{1,2,7\}}(n)$ refers to the quantity derived from $Z^{'}$. It thus follows that $P[A_n(C^{(n)})] \sim Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the $r_i^{'}$ and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g^{'}})$ from some $g^{'} > 0$, since, under these circumstances, both $n^{-1}\varphi^{'}_{\{1,2,7\}}(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g^{'} > 1$.

For $0 \le b \le n/8$ and $n \ge n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$

$$\le d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$

$$\le \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0), (D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\tilde{C}[1,b] \mid T_{0b}(C) = l) = L(\tilde{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$

$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$

$$= \max_A \sum_{r \in A} P[T_{0b}(Z) = r]$$

$$\left\{1 - \frac{P[T_{bn}(Z) = n - r]}{P[T_{0n}(Z) = n]}\right\} \qquad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$

$$= \sum_{r \ge 0} P[T_{0b} = r]\left\{1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]}\right\}_+$$

$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]}$$

$$\times \left\{\sum_{s=0}^{n} P[T_{0b} = s](P[T_{bn} = n - s] - P[T_{bn} = n - r]\right\}_+$$

$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r]$$

$$\times \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{\{P[T_{bn} = n - s] - P[T_{bn} = n - r]\}}{P[T_{0n} = n]}$$

$$+ \sum_{s=0}^{[n/2]} P[T_{0b} = r] \sum_{s=[n/2]+1}^{n} P[T = s]P[T_{bn} = n - s] / P[T_{0n} = n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2 < s \le n} P[T_{0b} = s]) / P[T_{0n} = n]$$

$$\le \frac{2\varepsilon_{\{10.5(1)\}}(n/2, b)}{n} \frac{3n}{\theta P_\theta[0,1]},$$

$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n) \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{1}{2}|r - s|$$

$$\le \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{1 + \frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\} P$$

$$+ \frac{6}{\theta P_\theta[0,1]} \varepsilon_{\{10.5(1)\}}(n/2, b) \qquad (1.5)$$

Required order under Conditions $(A_0), (D_1)$ and (B_{11}), if $S(\infty) < \infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be

replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty)<\infty$. Examining the Conditions $(A_0),(D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l\geq 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1>1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\bar{n}_1$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l\geq 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T^{(m)}_{bn}=s]-P[T^{(m)}_{bn}=s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from $\bar{n}_1$ of the form $\phi^{\theta}_1(n)+u^*_1(n)$, which is only small if $a_1>1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta>0$, since $a_2>1$ is in any case assumed. For $s\geq n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn}=s]-P[T_{bn}=s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn}=s]-P[T_{bn}=s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta>0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w)=1_{\{w=s\}}-1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn}=s]-P[T_{bn}=s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi^{\theta}_1(n)+u^*_1(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form $|P[T_{bn}=s]-P[T_{bn}=s+t]|=O(n^{-2}t+n^{-1-a_1+\delta})$ for any $\delta>0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1}+n^{-a_1+\delta})$ for any $\delta>0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left|d_{TV}(L(C[1,b]),L(Z[1,b]))-\frac{1}{2}(n+1)^{-1}|1-\theta|E|T_{0b}-ET_{0b}|\right| \leq \varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b)=O(n^{-1}b[n^{-1}b+n^{-\beta_{12}+\delta}])$ for any $\delta>0$ under Conditions $(A_0),(D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(\bar{C}[1,b]),L(\bar{Z}[1,b])) = \sum_{r\geq 0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+$$

Now we observe that

$$\left|\sum_{r\geq 0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+ - \sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right| \times\left|\sum_{s=[n/2]+1}^{n}P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r])\right|$$

$$\leq 4n^{-2}ET^2_{0b}+(\max_{n/2<s\leq n}P[T_{0b}=s])/P[T_{0n}=n] +P[T_{0b}>n/2]$$

$$\leq 8n^{-2}ET^2_{0b}+\frac{3\varepsilon_{\{10.5(2)\}}(n/2,b)}{\theta P_\theta[0,1]}, \qquad (1.1)$$

We have

$$\Big| \sum_{r=0}^{[n/2]} \frac{P[T_{0b}=r]}{P[T_{0n}=n]}$$

$$\times \Big(\Big\{\sum_{s=0}^{[n/2]} P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\Big\}_+$$

$$-\Big\{\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}P[T_{0n}=n]\Big\}_+\Big) \Big|$$

$$\le \frac{1}{n^2 P[T_{0n}=n]}\sum_{r\ge 0}P[T_{0b}=r]\sum_{s\ge 0}P[T_{0b}=s]|s-r|$$

$$\times \Big\{\varepsilon_{\{10.14\}}(n,b)+2(r\vee s)|1-\theta|n^{-1}\Big\{K_0\theta+4\phi^*_{\{10.8\}}(n)\Big\}\Big\}$$

$$\le \frac{6}{\theta n P_\theta[0,1]}ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$

$$+4|1-\theta|n^{-2}ET_{0b}^2\Big\{K_0\theta+4\phi^*_{\{10.8\}}(n)\Big\}$$

$$\Big(\frac{3}{\theta n P_\theta[0,1]}\Big)\ \Big\}, \qquad (1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]} P[T_{0b}=r]\Big|\Big\{\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\Big\}_+$$

$$-\Big\{\sum_{s=0} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\Big\}_+ \Big|$$

$$\le \sum_{r=0}^{[n/2]} P[T_{0b}=r]\sum_{s>[n/2]} P[T_{0b}=s]\frac{(s-r)|1-\theta|}{n+1}$$

$$\le |1-\theta|n^{-1}E(T_{0b}1\{T_{0b}>n/2\}) \le 2|1-\theta|n^{-2}ET_{0b}^2, \qquad (1.3)$$

and then by observing that

.

$$\sum_{r>[n/2]} P[T_{0b}=r]\Big\{\sum_{s\ge 0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\Big\}$$

$$\le n^{-1}|1-\theta|(ET_{0b}P[T_{0b}>n/2]+E(T_{0b}1\{T_{0b}>n/2\}))$$

$$\le 4|1-\theta|n^{-2}ET_{0b}^2 \qquad (1.4)$$

Combining the contributions of (1.2) –(1.3), we thus find tha

$$\Big| d_{TV}(L(\hat{C}[1,b]),L(\hat{Z}[1,b]))$$

$$-(n+1)^{-1}\sum_{r\ge 0}P[T_{0b}=r]\Big\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\Big\}_+ \Big|$$

$$\le \varepsilon_{\{7.8\}}(n,b)$$

$$=\frac{3}{\theta P_\theta[0,1]}\Big\{\varepsilon_{\{10.5(2)\}}(n/2,b)+2n^{-1}ET_{0b}\varepsilon_{\{10.14\}}(n,b)\Big\}$$

$$+2n^{-2}ET_{0b}^2\Big\{4+3|1-\theta|+\frac{24|1-\theta|\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\Big\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0),(D_1)$ and (B_{12}) , provided that $S(\infty)<\infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty)<\infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0}P[T_{0b}=r]\Big\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\Big\}_+$$

$$=\frac{1}{2}|1-\theta|E|T_{0b}-ET_{0b}|$$

Example 1.0. Consider the point $O=(0,...,0)\in \mathbb{R}^n$. For an arbitrary vector r, the coordinates of the point $x=O+r$ are equal to the respective coordinates of the vector $r: x=(x^1,...x^n)$ and $r=(x^1,...,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x . (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\mathbb{R}^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\mathbb{R}^n$: $\mathbb{R}^n$ = {points}, $\mathbb{R}^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\mathbb{R}^n$ treated in this way is called an *n-dimensional affine space*. (An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that

vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\mathbb{R}^n$ considered as an affine space we can precede in two opposite directions: $\mathbb{R}^n$ as an Euclidean space $\Leftarrow$ $\mathbb{R}^n$ as an affine space $\Rightarrow$ $\mathbb{R}^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\mathbb{R}^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\mathbb{R}^n$ a Euclidean space. Namely, we define the length of a vector $a = (a^1, ..., a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + ... + (a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A, B) := \left|\overrightarrow{AB}\right| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A, B) \le d(A, C) + d(C, B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a, b) := a^1 b^1 + ... + a^n b^n \qquad (3)$$

Thus $|a| = \sqrt{(a, a)}$. The scalar product is also denote by dot: $a.b = (a, b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos\alpha := \frac{(a, b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a, b)^2 \le |a|^2 |b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a + tb,$ where $t \in R$. As $(a + tb, a + tb) \ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x) = x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i .From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + ... + \frac{\partial f}{\partial x^n} dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2, ...$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2, ...,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 + ... + \frac{\partial f}{\partial x^n} h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \mathbb{R}^n$ at $t = t_0$ and with the velocity vector $x(t_0) = \upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$f(x(t_0+\Delta t)-f(x_0)$$
$$=df(x_0)(\upsilon.\Delta t+\alpha(\Delta t)\Delta t)$$
$$+\beta(\upsilon.\Delta t+\alpha(\Delta t)\Delta t).|\upsilon\Delta t+\alpha(\Delta t)\Delta t|$$
$$=df(x_0)(\upsilon).\Delta t+\gamma(\Delta t)\Delta t$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t)\to 0$ when $\Delta t\to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t=t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt}=\frac{\partial f}{\partial x^1}x^1+\ldots+\frac{\partial f}{\partial x^n}x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t=t_0$.

Theorem 1.8. For functions $f,g:U\to\mathbb{R}$, $U\subset\mathbb{R}^n$,

$$d(f+g)=df+dg \qquad (1)$$
$$d(fg)=df.g+f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0)=x_0$ and $x(t_0)=\upsilon$. Hence

$$d(f+g)(x_0)(\upsilon)=\frac{d}{dt}(f(x(t))+g(x(t)))$$

at $t=t_0$ and

$$d(fg)(x_0)(\upsilon)=\frac{d}{dt}(f(x(t))g(x(t)))$$

at $t=t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$. The only difference is that now the differential of a map $F:U\to\mathbb{R}^m$ at a point x will be a linear function taking vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (instead of $\mathbb{R}$). For an arbitrary vector $h\in|\mathbb{R}^n$,

$$F(x+h)=F(x)+dF(x)(h)$$
$$+\beta(h)|h| \qquad (3)$$

Where $\beta(h)\to 0$ when $h\to 0$. We have $dF=(dF^1,\ldots,dF^m)$ and

$$dF=\frac{\partial F}{\partial x^1}dx^1+\ldots+\frac{\partial F}{\partial x^n}dx^n$$
$$=\begin{pmatrix}\frac{\partial F^1}{\partial x^1}&\ldots&\frac{\partial F^1}{\partial x^n}\\ \ldots&\ldots&\ldots\\ \frac{\partial F^m}{\partial x^1}&\ldots&\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \ldots\\ dx^n\end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\mathbb{R}^n$, the differential of a map $F:U\to\mathbb{R}^m$ (where $U\subset\mathbb{R}^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\mathbb{R}^m$:

$$\frac{dF(x(t))}{dt}=dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t+\Delta t)=x(t)+\dot{x}(t).\Delta t+\alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t)\to 0$ when $\Delta t\to 0$. By the definition of the differential,

$$F(x+h)=F(x)+dF(x)(h)+\beta(h)|h \qquad (3)$$

Where $\beta(h)\to 0$ when $h\to 0$. we obtain

$$F(x(t+\Delta t))=F(x+\underbrace{x(t).\Delta t+\dot{\alpha}(\Delta t)\Delta t)}_{h}$$
$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t)+$$
$$\beta(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t).|x(t)\Delta t+\dot{\alpha}(\Delta t)\Delta t|$$
$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t)\to 0$ when $\Delta t\to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve

passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F:U\to V$ and $G:V\to W,$ where $U\subset \mathbb{R}^n, V\subset \mathbb{R}^m, W\subset \mathbb{R}^p$ (open domains). Let $F: x\mapsto y=F(x)$. Then the differential of the composite map $GoF:U\to W$ is the composition of the differentials of F and G:

$$d(GoF)(x)=dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\mathbb{R}^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\mathbb{R}^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t)=G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\mathbb{R}^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence $d(GoF)(\dot{x})=dG(dF(\dot{x}))$ for an arbitrary vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\mathbb{R}^n$ by $(x^1,...,x^n)$ and in $\mathbb{R}^m$ by $(y^1,...,y^m)$, and write

$$dF=\frac{\partial F}{\partial x^1}dx^1+...+\frac{\partial F}{\partial x^n}dx^n \qquad (1)$$

$$dG=\frac{\partial G}{\partial y^1}dy^1+...+\frac{\partial G}{\partial y^n}dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF)=\frac{\partial G}{\partial y^1}dF^1+...+\frac{\partial G}{\partial y^m}dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i=dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF)=\begin{pmatrix}\frac{\partial G^1}{\partial y^1}&\cdots&\frac{\partial G^1}{\partial y^m}\\ \cdots&\cdots&\cdots\\ \frac{\partial G^p}{\partial y^1}&\cdots&\frac{\partial G^p}{\partial y^m}\end{pmatrix}\begin{pmatrix}\frac{\partial F^1}{\partial x^1}&\cdots&\frac{\partial F^1}{\partial x^n}\\ \cdots&\cdots&\cdots\\ \frac{\partial F^m}{\partial x^1}&\cdots&\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \cdots\\ dx^n\end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix}\frac{\partial z^1}{\partial x^1}&\cdots&\frac{\partial z^1}{\partial x^n}\\ \cdots&\cdots&\cdots\\ \frac{\partial z^p}{\partial x^1}&\cdots&\frac{\partial z^p}{\partial x^n}\end{pmatrix}=\begin{pmatrix}\frac{\partial z^1}{\partial y^1}&\cdots&\frac{\partial z^1}{\partial y^m}\\ \cdots&\cdots&\cdots\\ \frac{\partial z^p}{\partial y^1}&\cdots&\frac{\partial z^p}{\partial y^m}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial y^1}{\partial x^1}&\cdots&\frac{\partial y^1}{\partial x^n}\\ \cdots&\cdots&\cdots\\ \frac{\partial y^m}{\partial x^1}&\cdots&\frac{\partial y^m}{\partial x^n}\end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a}=\sum_{i=1}^{m}\frac{\partial z^\mu}{\partial y^i}\frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y\in\mathbb{R}^m$ on $x\in\mathbb{R}^n$ is given by the map F, the dependence of $z\in\mathbb{R}^p$ on $y\in\mathbb{R}^m$ is given by the map G, and the dependence of $z\in\mathbb{R}^p$ on $x\in\mathbb{R}^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U\subset\mathbb{R}^n$. Consider also another copy of $\mathbb{R}^n$, denoted for distinction $\mathbb{R}^n_y$, with the standard coordinates $(y^1...y^n)$. A system of coordinates in the open domain U is given by a map $F:V\to U,$ where $V\subset\mathbb{R}^n_y$ is an open domain of $\mathbb{R}^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;

(2) F is invertible;

(3) $F^{-1}:U\to V$ is also smooth

The coordinates of a point $x\in U$ in this system are the standard coordinates of $F^{-1}(x)\in\mathbb{R}^n_y$

In other words,

$$F:(y^1...,y^n)\mapsto x=x(y^1...,y^n) \qquad (1)$$

Here the variables $(y^1...,y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\mathbb{R}^2$ specified in polar coordinates as

$$x(t): r = r(t), \varphi = \varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x/\partial r, \partial x/\partial \varphi$ are vectors depending on point in $\mathbb{R}^2$. We can compare this with the formula in the "standard" coordinates: $\dot{x} = e_1\dot{x} + e_2\dot{y}$. Consider the vectors $\partial x/\partial r, \partial x/\partial \varphi$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos\varphi, \sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r\sin\varphi, r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x/\partial r, \partial x/\partial \varphi$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$ $(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r, \varphi)$ $(r = r_0 \ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \partial x/\partial r, e_\varphi := \partial x/\partial \varphi$:

$$\dot{x} = e_r\dot{r} + e_\varphi\dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i. In particular, the elements of the basis $e_i = \partial x/\partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1, \ldots, x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F: \mathbb{R}^n \to \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0): \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + \ldots + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1, \ldots, e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \cdots \ \cdots \ \cdots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1}dx^1 + \ldots + \frac{\partial f}{\partial x^n}dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\mathbb{R}^2$ given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r\cos\varphi, y = r\sin\varphi$, hence

$$dx = \cos\varphi dr - r\sin\varphi d\varphi$$

$$dy = \sin\varphi dr + r\cos\varphi d\varphi$$

Substituting into A, we get

$$A = -r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi)$$

$$+r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi)$$

$$= r^2(\sin^2\varphi + \cos^2\varphi)d\varphi = r^2 d\varphi$$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U:

$$\omega(\upsilon) = \omega_1\upsilon^1 + \ldots + \omega_n\upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i \upsilon^i$, where $e_i = \partial x / \partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta^i_j \qquad (2)$$

at every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \frac{dx}{dt'} \right\rangle$ and $\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle$ As

$$\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle = \left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle . \frac{dt}{dt'},$$

Let p be a rational prime and let $K = \mathbb{Q}(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p-1$ over $\mathbb{Q}$. We wish to show that $O_K = \mathbb{Z}[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Z}[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, \ldots, \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + \ldots + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p-1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta) = Tr_{K/\mathbb{Q}}(1-\zeta^2) = \ldots$$
$$= Tr_{K/\mathbb{Q}}(1-\zeta^{p-1}) = p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + \ldots + 1 = \Phi_p(x)$$
$$= (x-\zeta)(x-\zeta^2)\ldots(x-\zeta^{p-1});$$

Plugging in $x = 1$ shows that

$$p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta) = p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K \cap \mathbb{Z} = p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K, so the inclusion $(1-\zeta)O_K \cap \mathbb{Z} \supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1-\zeta)O_K \cap \mathbb{Z} = \mathbb{Z}$ Thus we can write

$$1 = \alpha(1-\zeta)$$

For some $\alpha \in O_K$. That is, $1-\zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha \in O_K$, $Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) \in p\mathbb{Z}$.

PROOF. We have

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) = \sigma_1((1-\zeta)\alpha) + \ldots + \sigma_{p-1}((1-\zeta)\alpha)$$
$$= \sigma_1(1-\zeta)\sigma_1(\alpha) + \ldots + \sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha)$$
$$= (1-\zeta)\sigma_1(\alpha) + \ldots + (1-\zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = \mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, \ldots, \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write

$$\alpha = a_0 + a_1\zeta + \ldots + a_{p-2}\zeta^{p-2}$$ With $a_i \in \mathbb{Q}$.

Then

$$\alpha(1-\zeta) = a_0(1-\zeta) + a_1(\zeta - \zeta^2) + \ldots$$
$$+a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\square}(\alpha(1-\zeta)) = pa_0$ We also have $Tr_{K/\square}(\alpha(1-\zeta)) \in p\square$, so $a_0 \in \square$ Next consider the algebraic integer $(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + \ldots + a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \square$, and continuing in this way we find that all of the a_i are in $\square$. This completes the proof.

Example 1.4 Let $K = \square$, then the local ring $\square_{(p)}$ is simply the subring of $\square$ of rational numbers with denominator relatively prime to p. Note that this ring $\square_{(p)}$ is not the ring $\square_p$ of p-adic integers; to get $\square_p$ one must complete $\square_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta/\gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta/\gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed.We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p}/pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha/\beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as $x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + \ldots + \frac{\alpha_0}{\beta_0}$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1\ldots\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma/\beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then $N'_{K/\square}(\alpha O_K) = |N_{K/\square}(\alpha)|$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\square$ is Galois. Let σ be an element of $Gal(K/\square)$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\square}(\sigma(\alpha)O_K) = N'_{K/\square}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\square)$, we have $N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N'_{K/\square}(\alpha O_K)^n$ Since $N_{K/\square}(\alpha)$ is a rational integer and O_K is a free $\square$-module of rank n, $O_K/N_{K/\square}(\alpha)O_K$ Will have order $N_{K/\square}(\alpha)^n$; therefore

$$N'_{K/\square}(N_{K/\square}(\alpha)O_K) = N_{K/\square}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K]=m.$

V. CONCLUSION

Many of the early challenges faced with the use of intravascular OCT in the clinical setting were overcome with the development of OFDI, the most significant being imaging speed. While the laser and detection electronics are capable of operating faster still, the ability of acquisition electronics and data processing to keep pace, remains a challenge. Recently however, newer acquisition electronic systems have been developed enabling the acquisition and storage of data at rates approaching 1 GB/s. Additionally, solutions to alleviate CPU processing requirements have been implemented, using hardware components, such as digital signal processors (DSPs) [99] and field programmable gate arrays (FPGAs) [100]. These hardware solutions may be integrated into OFDI systems to handle much of the pre and postacquisition processing, thus enabling both real-time display and an increase in the data transfer rates achievable [99]–[102]. Bit-depth reduction with aminimal associated loss in the signalto- noise ratio of the OCT images, may also result in an increase in image acquisition rates due to the reduced bandwidth and storage requirements at lower bit-depths [80], [103] Coupled with the rapidly increasing use of intravascular OCT in catheterization laboratories, is a pressing need for automated and semiautomated image processing techniques for the evaluation of coronary features including classification based on tissue pathology, stent strut identification and quantification of strut tissue coverage. To date the vast majority of this type of evaluation is manually performed by expert intracoronary OCT readers. This process involves an extremely large time commitment and is subject to interobserver variations. In the case of quantitative feature analyses, such as stent strut coverage or lumen diameter analysis, well-defined and validated protocols are required in addition to controlled image processing steps to account for variances in the refractive indexes of both the tissue and flushing media. While preliminary studies have been conducted describing semiautomated analyses of OCT image data [104], the development of appropriate automated and semiautomated image analysis tools could improve the ease of use of intravascular OCT, particularly in nonspecialized catheterization centers that may have little or no intravascular OCT expertise. With the evolution of OFDI and PS-OFDI, there is also an increasing need for improved visual display techniques that can highlight relevant features, provide an enhanced appreciation of the 3-D morphology, and can amalgamate the complementary information into user-friendly maneuverable 3-D displays. In order to fully appreciate the complex 3-D morphology of the artery, investigators are exploring various display techniques ranging from standard longitudinal and transverse cross-sectional displays to intensity-based volume rendering, and more complex methods involving segmentation and pseudocoloring based on tissue characterization with subsequent 3-D volume rendering [87]. While preliminary work has been demonstrated by some investigators in the manipulation, analysis, and display of intravascular OCT datasets, further work in this field is needed, which may be leveraged from the extensive research performed with other imaging modalities, such as IVUS [105], [106] The potential clinical utility of intravascular OCT has no doubt increased as a direct result of the development of highspeed OFDI technology. OFDI enables imaging of long coronary segments, previously difficult with the first generation TD-OCT, during a brief flush with an optically transparent media. Based on the status of currently available imaging modalities for interrogating the coronary arteries, intravascular OCT is uniquely situated to play a critical role in improving our understanding of the vulnerable plaque, in addition to possibly guiding patient management and monitoring the response to PCI.

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