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Different types of convergence for a set-indexed stochastic processes on increasing sequences

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ABSTRACT: *The purpose of this article is to extend the concept of convergence of random variables to set indexed framework. Several types of convergenceare presented (convergence in probability, convergence in almost surely, convergence in L^p and convergence in finite dimensional distribution) and the relations that exist among various notions of convergenceare formalized.*
In addition, some applications on set indexed Brownian motion are introduced.
KEYWORDS: *Convergence, set indexed stochastic process, flow, Brownian motion.*

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I. INTRODUCTION

In numerous applications we are interested in the long term behavior of a stochastic process. The study of these issues is fundamentally related to convergence of processes. An important concept in probability is a convergence of random variables. Since the important results in probability are the limit theorems that concern themselves with the asymptotic behavior of random processes, studying the convergence of random variables becomes necessary.

In this study, the concept of convergence is extended to set indexed framework. Set indexed processes are a natural generalization of planar processes where $\mathbf{A}$ is a collection of compact subsets of a fixed topological space (T,τ). The frame of a set-indexed stochastic process is not only a new step towards generalization of a classical stochastic process, but it has been proven to be entirely new view of stochastic process. In recent years, there have been many new results related to the dynamic properties of random processes indexed by a class of sets.

In this article, the several types of convergenceare presented(convergence in probability, convergence in almost surely, convergence in L^p and convergence in finite dimensional distribution) andthe relations that exist among the various notions of convergence are formalized.

In addition, some applications on set indexed Brownian motion are introduced. Let $W=\{W_A : A\in\mathbf{A}\}$ be aset indexed Brownian motion with variance σ then

$$|W_A| \overset{a.s.}{\Rightarrow} \infty,\ \frac{W_A}{\sigma(A)} \overset{a.s.}{\Rightarrow} 0 \text{ and } \frac{|W_A|}{\sqrt{2\sigma(A)\ln\ln(\sigma(A))}} \overset{a.s.}{\Rightarrow} 1.$$

Moreover, $W_n \overset{fdd}{\Rightarrow} W$ when W_n, W are set indexed Brownian motions with variance σ if and only if $W_{n,t}^f \to W_t^f$ in distribution, for all flows $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ and $t\in[0,\infty)$ when $W_{n,t}^f, W_t^f$ are time-change Brownian motions.

II. PRELIMINIARIES

The set-indexed framework:

Let (T,τ) denote a non-void σ -compact connected topological space. In set indexed works (see [Iv], [Sa], [Yo]), processes and filtrations will be indexed by a nonempty class $\mathbf{A}$ of compact connected subsets of T is called an indexed collection if it satisfies the following:

1. $\varnothing\in\mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T=\bigcup_{n=1}^{\infty} B_n^{\circ}$.

2. $\mathbf{A}$ is closed under arbitrary intersections and if $A, B \in \mathbf{A}$ are nonempty, then $A \bigcap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i \subseteq B_n$ for every i, then $\overline{\bigcup_i A_i} \in \mathbf{A}$.
3. $\sigma(\mathbf{A}) = \mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T .
4. There exist an increasing sequence of finite sub-classes $\mathbf{A}_n = \{A_1^n, ..., A_{k_n}^n\} \subseteq \mathbf{A}$ closed under intersection with $\varnothing, B_n \in \mathbf{A}_n(\mathbf{u})$ ($\mathbf{A}_n(\mathbf{u})$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n : \mathbf{A} \to \mathbf{A}_n(\mathbf{u}) \cup T$ such that:

i. g_n preserves arbitrary intersections and finite unions.

ii. For each $A \in \mathbf{A},\ A \subseteq g_n(A)^{\circ}$ and $A = \cap_n g_n(A)$, $g_n(A) \subseteq g_m(A)$ if $n \geq m$

iii. $g_n(A) \cap A' \in \mathbf{A}$ if $A, A' \in \mathbf{A}$ and $g_n(A) \cap A' \in \mathbf{A}_n$ if $A \in \mathbf{A}$ and $A' \in \mathbf{A}_n$.

iv. $g_n(\varnothing) = \varnothing$ for all n .

(Note: $\overline{(\cdot)}$ and $(\cdot)^{\circ}$ denote respectively the closure and the interior of a set).

Examples of topological spaces T and indexed collections $\mathbf{A}$:

a. The classical example is $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0, x] : x \in \Re_+^d\}$.

b. The example (a) may be generalized as follows. Let $T = \Re_+^d$ and take $\mathbf{A}$ to be the class of compact lower sets, i.e. the class of compact subsets A of T satisfying $t \in A$ implies $[0, t] \subseteq \mathbf{A}$ (We denote the class of compact lower sets by $\mathbf{A}(Ls)$).

We define some extensions of $\mathbf{A}$: $\mathbf{A}(\mathbf{u})$ which consists of all finite unions in $\mathbf{A}$, $\mathbf{C}$ which consists of all set differences of the form $A \setminus B$ ($A \in \mathbf{A}, B \in \mathbf{A}(\mathbf{u})$) and $\mathbf{C}(\mathbf{u})$ which consists of all finite unions in $\mathbf{C}$. We note that $\mathbf{A}(\mathbf{u})$ is itself a lattice with the partial order induced by set inclusion.

Let (Ω, F, P) be any complete probability space. A set indexed filtration is a class $\{F_A : A \in \mathbf{A}\}$ of complete sub-σ-algebras of F which satisfies the following conditions:

a. $\forall A, B \in \mathbf{A}$, $F_A \subseteq F_B$, if $A \subseteq B$

b. Monotone outer-continuity: $F_{\cap A_i} = \bigcap F_{A_i}$ for any decreasing sequence (A_i) in $\mathbf{A}$.

For consistency in what follows, if $T \notin \mathbf{A}$ define $F_T = F$ .Any such filtration can be extended to $\mathbf{A}(\mathbf{u})$-indexed family by definition:

$$F_B^{\circ} = \vee_{A \in \mathbf{A}, A \subseteq B} F_A .$$

If $C \in \mathbf{C}(\mathbf{u}) \setminus \mathbf{A}$ ($\mathbf{C}(\mathbf{u})$ - class of finite unions of sets in $\mathbf{C}$) then denote:

$$\mathbf{G}_C^* = \vee_{A \in \mathbf{A}(\mathbf{u}), A \cap C} F_A .$$

In addition, let A^{ss} be any finite sub-semilattice of $\mathbf{A}$ closed under intersection. For $A \in A^{ss}$, define the left neighborhood of A in A^{ss} to be a set

$$C_A = A \setminus \bigcup_{B \in A^{ss}, B \subset A} B .$$

We note that $\bigcup_{A \in A^{ss}} A = \bigcup_{A \in A^{ss}} C_A$ and that the latter union is disjoint. The sets in A^{ss} can always be numbered in the following way: $A_0 = \varnothing'$, ($\varnothing' = \bigcap_{A \in \mathbf{A}, A \neq \varnothing} A$, note that $\varnothing' \neq \varnothing$) and given $A_0, ..., A_{i-1}$, choose A_i to be any set in A^{ss} such that $A \subset A_i$ implies that $A = A_j$, some $j = 1, ..., i-1$. Any such numbering $A^{ss} = \{A_0, ..., A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left-neighborhood of A_i in A^{ss}, then $C_i = \bigcup_{j=0}^{i} A_j \setminus \bigcup_{j=0}^{i-1} A_j$ and $C_i \bigcap A_j = \varnothing$, for all $j = 0, ..., i-1,\ i = 1, ..., k$).

Any $\mathbf{A}$ -indexed function which has a (finitely) additive extension to $\mathbf{C}$ will be called additive and is easily seen to be additive on $\mathbf{C(u)}$ as well. For stochastic processes, we do not necessarily require that each sample path be additive, but additivity will be imposed in an almost sure sense:

A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to $\mathbf{C}: X_{\varnothing} = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \bigcup C_2$ and $C_1 \bigcap C_2 = \varnothing$ then almost surely

$$X_C = X_{C_1} + X_{C_2}.$$

In particular, if $C \in \mathbf{C}$ and $C = A \setminus \bigcup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \bigcap_{i=1}^{n} A_i} .$$

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to $\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C(u)}$.

Convergence of a set-indexed stochastic processes

Definition 1.

a. Let $\{A_n\}$ be an increasing sequence in $\mathbf{A}$. We write $A_n \uparrow T$ if $A_n \neq T$ for all n and $\overline{\bigcup_n A_n} = T$.

b. We write $A_n \square\ T$ if $A_n \uparrow T$ for all an increasing sequence $\{A_n\}$.

Definition 2.

a. (Convergence in probability) A set indexed stochastic process $\{X_A : A \in \mathbf{A}\}$ is said to converge to a set indexed stochastic process $\{Y_A : A \in \mathbf{A}\}$ in probability if :

$$\text{For any } 0 < \varepsilon,\ \lim_{A_n \square\ T} P\left(\left|X_{A_n} - Y_{A_n}\right| \geq \varepsilon\right) = 0 \text{ ,and denoted } X_A \overset{P}{\Rightarrow} Y_A$$

b. (Convergence in $L^P(\mathbf{A})$) A set indexed stochastic process $\{X_A : A \in \mathbf{A}\}$ is said to converge to a set indexed stochastic process $\{Y_A : A \in \mathbf{A}\}$ in $L^P(\mathbf{A})$ if :

$$|X_{A_n}|, |Y_{A_n}| \text{ in } L^P(\mathbf{A}),\ \lim_{A_n \square\ T} E\left(\left|X_{A_n} - Y_{A_n}\right|^p\right) = 0 \text{ and denoted } X_A \overset{L^P}{\Rightarrow} Y_A .$$

c. (Convergence almost surely) A set indexed stochastic process $\{X_A : A \in \mathbf{A}\}$ is said to converge to a set indexed stochastic process $\{Y_A : A \in \mathbf{A}\}$ in almost surely if :

$$P\left(\lim_{A_n \square\ T} \left|X_{A_n} - X\right| \neq 0\right) = 0 \text{, and denoted } X_A \overset{a.s.}{\Rightarrow} Y_A .$$

Theorem 1: Let $X = \{X_A : A \in \mathbf{A}\}$ and $\{Y_A : A \in \mathbf{A}\}$ be aset indexed stochastic processes. Then the following relationships hold:

a. $X_A \overset{P}{\Rightarrow} Y_A \Leftrightarrow \lim_{A_n \square\ T} E\left(f\left(\left|X_{A_n} - Y_{A_n}\right|\right)\right) = 0$ for any function f on $\Re_+$ which is bounded, strictly increasing, continuous and $f(0) = 0$.

b. If $X_A \overset{a.s.}{\Rightarrow} Y_A$ then $X_A \overset{P}{\Rightarrow} Y_A$

c. If $X_A \overset{L^P}{\Rightarrow} Y_A$ then $X_A \overset{P}{\Rightarrow} Y_A$.

d. If $X_A \overset{L^P}{\Rightarrow} Y_A$ then $X_A \overset{L^q}{\Rightarrow} Y_A$ for $1 \leq q \leq p$.

e. If $E[X_A^2] < \infty$ for all $A \in \mathbf{A}$ and $\sum_{A\in\mathbf{A}} E[X_A^2] < \infty$ then $X_A \overset{a.s.}{\Rightarrow} 0$.

f. Let f be a continuous function. If $X_A \overset{a.s.}{\Rightarrow} Y_A$ then $f(X_A) \overset{a.s.}{\Rightarrow} f(Y_A)$

g. Let $Z = \{Z_A : A \in \mathbf{A}\}$ and $W = \{W_A : A \in \mathbf{A}\}$ be aset indexed stochastic processes. Suppose that $X_A \overset{P}{\Rightarrow} W_A$ and $Y_A \overset{P}{\Rightarrow} Z_A$. Then,

(1) $X_A + Y_A \overset{P}{\Rightarrow} W_A + Z_A$, (2) $X_A Y_A \overset{P}{\Rightarrow} W_A Z_A$, (3) $\frac{X_A}{Y_A} \overset{P}{\Rightarrow} \frac{W_A}{Z_A}$ ($Y_A \neq 0, Z_A \neq 0$).

Proof.

a. Enough to prove that $U_A \overset{P}{\Rightarrow} 0 \Leftrightarrow \lim_{A_n \uparrow T} E\left(f(|U_{A_n}|)\right) = 0$, when $U_{A_n} = X_{A_n} - Y_{A_n}$.

($\Rightarrow$) Let $0 < \varepsilon$,

$f(|U_{A_n}|) = f(|U_{A_n}|)\mathbf{1}_{[f(|U_{A_n}|)>\varepsilon]} + f(|U_{A_n}|)\mathbf{1}_{[f(|U_{A_n}|)\le\varepsilon]} \le M\mathbf{1}_{[f(|U_{A_n}|)>\varepsilon]} + \varepsilon$

where $\mathbf{1}_B$ is the indicator function of the event B and f bounded by M, then $E\left(f(|U_{A_n}|)\right) \le$ $M \cdot E\left(\mathbf{1}_{[f(|U_{A_n}|)>\varepsilon]}\right) + \varepsilon = M \cdot P\left(f(|U_{A_n}|) > \varepsilon\right) + \varepsilon$. Thus, if $A_n \uparrow T$ and $\varepsilon \to 0^+$ we have $\lim_{A_n \uparrow T} E\left(f(|U_{A_n}|)\right) = 0$.

($\Leftarrow$) It is clear that, if f is a strictly increasing, continuous, bounded and $f(0) = 0$ then there exists a $M(\varepsilon) \overset{\varepsilon\to 0^+}{\to} 0$ such that $M(\varepsilon)\mathbf{1}_{[f(|U_{A_n}|)>\varepsilon]} \le f(|U_{A_n}|)\mathbf{1}_{[f(|U_{A_n}|)>\varepsilon]} \le f(|U_{A_n}|)$. Then by taking expectations, we derive $M(\varepsilon)P\left(\left|U_{A_n}\right| > \varepsilon\right) \le E\left(f(|U_{A_n}|)\right)$. Thus, if $A_n \uparrow T$ and $\varepsilon \to 0^+$ we have $U_{A_n} \overset{P}{\Rightarrow} 0$.

b. Let f be a bounded, strictly increasing, continuous function on $\Re_+$. Since $f(|X_{A_n} - Y_{A_n}|)$ is a bounded, based on (a) and by Lebegue's Dominated Convergence Theorem, we derive

$$\lim_{A_n \uparrow T} E\left(f\left(\left|X_{A_n} - Y_{A_n}\right|\right)\right) = E\left(\lim_{A_n \uparrow T} f\left(\left|X_{A_n} - Y_{A_n}\right|\right)\right) = 0.$$

c. Obvious, $P\left(|X_{A_n} - Y_{A_n}| \ge \varepsilon\right) = E\left(\mathbf{1}_{[|X_{A_n} - Y_{A_n}|\ge\varepsilon]}\right)$. Note that $\frac{|X_{A_n} - Y_{A_n}|^p}{\varepsilon^p} \ge 1$ on the event $[|X_{A_n} - Y_{A_n}| \ge \varepsilon]$, thus

$$E\left(\mathbf{1}_{[|X_{A_n} - Y_{A_n}|\ge\varepsilon]}\right) \le E\left(\frac{|X_{A_n} - Y_{A_n}|^p}{\varepsilon^p}\mathbf{1}_{[|X_{A_n} - Y_{A_n}|\ge\varepsilon]}\right) = \frac{E\left(|X_{A_n} - Y_{A_n}|^p \mathbf{1}_{[|X_{A_n} - Y_{A_n}|\ge\varepsilon]}\right)}{\varepsilon^p} \le \frac{E\left(|X_{A_n} - Y_{A_n}|^p\right)}{\varepsilon^p}.$$

Then,

$$0 \le \lim_{A_n \uparrow T} P\left(\left|X_{A_n} - Y_{A_n}\right| \ge \varepsilon\right) \le \lim_{A_n \uparrow T} \frac{E\left(|X_{A_n} - Y_{A_n}|^p\right)}{\varepsilon^p} = 0.$$

d. According to Lyapunov inequalities, $E\left(\left|X_{A_n} - Y_{A_n}\right|^q\right)^{\frac{1}{q}} \le E\left(\left|X_{A_n} - Y_{A_n}\right|^p\right)^{\frac{1}{p}}$ then

$$0 \le \lim_{A_n \uparrow T} E\left(\left|X_{A_n} - Y_{A_n}\right|^q\right) \le \lim_{A_n \uparrow T} E\left(\left|X_{A_n} - Y_{A_n}\right|^p\right)^{\frac{q}{p}} = 0.$$

e. Based on the Chebychev inequality, $P\left(\left|X_{A_n}\right| \ge \varepsilon\right) \le \frac{E\left(|X_{A_n}|^2\right)}{\varepsilon^2}$. Therefore,

$$\sum_n P\left(\left|X_{A_n}\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\sum_n E\left(|X_{A_n}|^2\right) \le \frac{1}{\varepsilon^2}\sum_{A\in\mathbf{A}} E\left(|X_A|^2\right) < \infty.$$

Thus, by the Borel-Cantelli lemma, the probability that $\left\{\left|X_{A_n}\right| \geq \varepsilon,\ i.o.\right\}$ is zero or $\left|X_{A_n}\right| \geq \varepsilon$ only for a finite number of sets A_n. Since $0 < \varepsilon$ is arbitrary it implies that $X_A \overset{a.s.}{\Rightarrow} 0$.

f. Let $\Lambda = \{\omega : \lim_{A_n \uparrow T} X_{A_n}(\omega) - Y_{A_n}(\omega) \neq 0\}$, then $P(\Lambda) = 0$ by hypothesis. Based on the continuity of f, $\lim_{A_n \uparrow T} f(X_{A_n}(\omega) - Y_{A_n}(\omega)) = f(\lim_{A_n \uparrow T}(X_{A_n}(\omega) - Y_{A_n}(\omega)) = f(0)$ when $\omega \notin \Lambda$. Since $\lim_{A_n \uparrow T} f(X_{A_n}(\omega) - Y_{A_n}(\omega)) = f(0)$ for any $\omega \notin \Lambda$, $P(\Lambda) = 0$, we get that $f(X_A) - f(Y_A) \overset{a.s.}{\Rightarrow} 0$.

g. (1). $\lim_{A_n \uparrow T} P\left(\left|(X_{A_n} + Y_{A_n}) - (W_{A_n} + Z_{A_n})\right| \geq \varepsilon\right) = \lim_{A_n \uparrow T} P\left(\left|(X_{A_n} - W_{A_n}) + (Y_{A_n} - Z_{A_n})\right| \geq \varepsilon\right)$

$\leq \lim_{A_n \uparrow T} P\left(|X_{A_n} - W_{A_n}| + |Y_{A_n} - Z_{A_n}| \geq \varepsilon\right)$

$\leq \lim_{A_n \uparrow T} P(|X_{A_n} - W_{A_n}| \geq \frac{\varepsilon}{2}) + \lim_{A_n \uparrow T} P(|Y_{A_n} - Z_{A_n}| \geq \frac{\varepsilon}{2}) = 0 + 0 = 0$.

Similarly, (2) and (3) can be proved. □

Definition 3: A strict flow (shortly, flow) is defined to be a continuous increasing function $f : [a,b] \to \mathbf{A(u)}$ where $0 \leq a < b$, i.e. such that

a. $\forall s,t \in [a,b];\ s < t \Rightarrow f(s) \subset f(t)$

b. $\forall s,t \in [a,b];\ f(s) = \bigcap_{v>s} f(v)$

c. $\forall s,t \in (a,b];\ f(s) = \overline{\bigcup_{u<s} f(u)}$.

The notion of flow was introduced in [Ca] and used by several authors[Da], [He].

Given a set indexed stochastic process X and the flow $f : [0,\infty) \to \mathbf{A(u)}$, we define a process X^f indexed by $[0,\infty)$ as follows: $X_{f(s)} = X_s^f$ for all $s \in [0,\infty)$.

Lemma 1: Let $A^{SS} = \{\varnothing' = A_0, ..., A_k\}$ be any finite sub-semilattice of $\mathbf{A}$ equipped with a numbering consistent with the strong past.

Then there exists a continuous (strict) flow $f : [0,k] \to \mathbf{A(u)}$ such that the following are satisfied:

1. $f(0) = \varnothing'$, $f(k) = \bigcup_{j=0}^{k} A_j$
2. Each left-neighbourhood C generated by A^{SS} is of the form $C = f(i) \setminus f(i-1)$ for all $1 \leq i \leq k$.
3. If $C = f(t) \setminus f(s)$ then $C \in \mathbf{C(u)}$ and $F_{f(s)} \in \mathbf{G}_C^*$.

The proof appears in [Iv].

Based on Lemma 1, we derive:

Theorem 2:

a. $X_A \overset{P}{\Rightarrow} Y_A \Leftrightarrow X_n^f - Y_n^f \overset{P}{\to} 0$ for all (strict continuous) flows $f : [0,\infty) \to \mathbf{A(u)}$

b. $X_A \overset{a.s.}{\Rightarrow} Y_A \Leftrightarrow X_n^f - Y_n^f \overset{a.s}{\to} 0$ for all (strict continuous) flows $f : [0,\infty) \to \mathbf{A(u)}$

c. $X_A \overset{L^P}{\Rightarrow} Y_A \Leftrightarrow X_n^f - Y_n^f \overset{L^P}{\to} 0$ for all (strict continuous) flows $f : [0,\infty) \to \mathbf{A(u)}$

when $\overset{P}{\to}$, $\overset{a.s}{\to}$ and $\overset{L^P}{\to}$ are one-dimensional convergences.

Proof.For the proof, we need auxiliary proposition:

Proposition:

1. If $\{A_i\}_{i=1}^{k}$ be an increasing sequence in $\mathbf{A}$ then there exists a strict continuous flow $f : [0,k] \to \mathbf{A(u)}$, $f(0) = \varnothing'$ and $f(i) = A_i$ for all $1 \leq i \leq k$.

2. If $\{A_i\}_{i=1}^{\infty} \uparrow T$ then there exists a strict continuous flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$, $f(0)=\emptyset'$ and $f(i)=A_i$ for all $1 \le i$.

Proofof Proposition:

1. Let $\{A_i\}_{i=1}^{k}$ be an increasing sequence in $\mathbf{A}$. Without loss of generality, we may assume that the sets $\{C_i\}_{i=1}^{k}$ are the left-neighbourhoods of the sub-semilattice $\mathbf{A}^{ss}$ of $\mathbf{A}$ equipped with a numbering consistent with the strong past when $C_1 = A_1$ and $C_i = A_i \setminus A_{i-1}$ for all $2 \le i \le k$. According to Lemma 1, there exists a strict continuous flow $f_k:[0,k] \to \mathbf{A}(\mathbf{u})$ such that each left-neighbourhood generated by $\mathbf{A}^{ss}$ is of the form $C_i = f(i) \setminus f(i-1), 1 \le i \le k$ and $F_{f_k(i)} \subseteq \mathbf{G}_C^*$.
2. Notice that for each k, $f_k = f_{k+1}$ on $[0,k]$. Then, We can define the function $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$ by $f(t) = f_{[t]+1}(t)$ for all t.

Based on (2), if $\{A_n\}_{n=1}^{\infty} \uparrow T$ then there exists a strict continuous flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$, $f(n) = A_n$ for all $1 \le n$. Then

a. $X_A \overset{P}{\Rightarrow} Y_A \Leftrightarrow \lim_{A_n \uparrow T} P\left(\left|X_{A_n} - Y_{A_n}\right| \ge \varepsilon\right) = 0 \Leftrightarrow \lim_{n\to\infty} P\left(\left|Z_n^f\right| \ge \varepsilon\right) = 0 \Leftrightarrow Z_n^f \overset{P}{\to} 0$.

b. $X_A \overset{a.s.}{\Rightarrow} Y_A \Leftrightarrow P\left(\lim_{A_n \uparrow T} \left|X_{A_n} - Y_{A_n}\right| \ne 0\right) = 0 \Leftrightarrow P\left(\lim_{n\to\infty}\left(\left|Z_n^f\right| \ne 0\right)\right) = 0 \Leftrightarrow Z_n^f \overset{a.s}{\to} 0$

c. $X_A \overset{L^P}{\Rightarrow} Y_A \Leftrightarrow \lim_{A_n \uparrow T} E\left(\left|X_{A_n} - Y_{A_n}\right|^p\right) = 0 \Leftrightarrow \lim_{n\to\infty} E\left(\left|Z_n^f\right|^p\right) = 0 \Leftrightarrow Z_n^f \overset{L^P}{\to} 0$.

Where $Z_n^f = X_n^f - Y_n^f$. □

Definition 4. A positive measure σ on $(T,\mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if: $\sigma_{\emptyset'} = 0$ and $\sigma_A < \sigma_B$ for all $A \subset B$, $A, B \in \mathbf{A}$. The collection of these measures is denoted by $M(\mathbf{A})$.

Definition 5. Let $\sigma \in M(\mathbf{A})$. We say that the $\mathbf{A}$-indexed process X is a Brownian motion with variance σ if X can be extended to a finitely additive process on $\mathbf{C}(\mathbf{u})$ and if for disjoint sets $C_1,\ldots,C_n \in \mathbf{C}$, $X_{C_1},\ldots,X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\sigma_{C_1},\ldots,\sigma_{C_n}$, respectively. (For any $\sigma \in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [Iv]).

Theorem 3(The characterization of set-indexed Brownian motion by flows): Let $X = \{X_A : A \in \mathbf{A}\}$ be a square-integrable set-indexed stochastic process. Let $\sigma \in M(\mathbf{A})$ then

X is set-indexed Brownian motion with variance σ if and only if the process X^f is time-change Brownian motion for all strict continuous flows $f:[a,b] \to \mathbf{A}(\mathbf{u})$.

The proof appears in [Me].

Theorem 4: Let $W = \{W_A : A \in \mathbf{A}\}$ be aset indexed Brownian motion with variance σ. Then,

a. $X_A \overset{a.s.}{\Rightarrow} +\infty$ when $X_A = |W_A|$ for all $A \in \mathbf{A}$.

b. $X_A \overset{a.s.}{\Rightarrow} 0$ when $X_A = \frac{W_A}{\sigma(A)}$ for all $A \in \mathbf{A}$.

c. $X_A \overset{a.s.}{\Rightarrow} 1$ when $X_A = \frac{|W_A|}{\sqrt{2\sigma(A)\ln\ln(\sigma(A))}}$ for all $A \in \mathbf{A}$.

Proof.

According to [Me], there exists a flow $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ and there exists $A_n\in\mathbf{A}$ such that W^f is a time-change Brownian motion and $A_n=f(n)$. (In other words, there exists a $\theta:[0,\infty)\to[0,\infty)$ and $0\le\alpha_n$ such that $W^{f\circ\theta}$ is a Brownian motion and $A_n=f(n)=f(\theta(\alpha_n))$.

a. We recall that, if $B=\{B_t:t\ge 0\}$ is a one-parameter Brownian motion, then $\lim\limits_{t\to\infty}|B_t|=+\infty$. Thus, $\lim\limits_{A_n\uparrow T}|W_{A_n}|=\lim\limits_{\alpha_n\to\infty}|W^f_{\alpha_n}|=+\infty$, almost surely.

b. We recall that, if $B=\{B_t:t\ge 0\}$ is a one-parameter Brownian motion, then $\lim\limits_{t\to\infty}\frac{B_t}{t}=0$. Thus, $\lim\limits_{A_n\uparrow T}\frac{W_{A_n}}{\sigma(A_n)}=\lim\limits_{\alpha_n\to\infty}\frac{W^f_{\alpha_n}}{\sigma(f(\theta(\alpha_n))}=0$, almost surely.

c. We recall that, if $B=\{B_t:t\ge 0\}$ is a one-parameter Brownian motion, then $\lim\limits_{t\to\infty}\frac{|B_t|}{\sqrt{2t\ln\ln t}}=1$. Thus, $\lim\limits_{A_n\uparrow T}\frac{|W_{A_n}|}{\sqrt{2\sigma(A_n)\ln\ln(\sigma(A_n))}}=\lim\limits_{\alpha_n\to\infty}\frac{|W_{\alpha_n}|}{\sqrt{2\alpha_n\ln\ln(\alpha_n)}}=1$, almost surely. $\square$

Definition 6.[Iv] Let $X_n=\{X_{n,A}:A\in\mathbf{A}\}$ and $X=\{X_A:A\in\mathbf{A}\}$ be a set-indexed stochastic processes.The sequence $\{X_n\}$ converges in finite dimensional distribution to X, denoted $X_n\overset{fdd}{\Rightarrow}X$ if $(X_{n,A_1},X_{n,A_2},\dots,X_{n,A_m})\to(X_{A_1},X_{A_2},\dots,X_{A_m})$ in distribution (as random vector) for any $m\in\mathbb{N}$ and $A_1,A_2,\dots,A_m\in\mathbf{A}$.

Theorem 5: Let $W_n=\{W_{n,A}:A\in\mathbf{A}\}$ and $W=\{W_A:A\in\mathbf{A}\}$ be aset indexed stochastic processes.Then $W_n\overset{fdd}{\Rightarrow}W$ when W_n,W areset indexed Brownian motions with variance σ if and only if $W^f_{n,t}\to W^f_t$ in distribution, for all flows $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ and $t\in[0,\infty)$ when $W^f_{n,t},W^f_t$ are time-change Brownian motions.

Proof.

($\Rightarrow$) Obvious.

($\Leftarrow$)Let $W^f_{n,t},W^f_t$ are time-change Brownian motions, for all flows $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$. Based on [Me], W_n,W are set indexed Brownian motions. Enough to prove that $(W_{n,A_1},W_{n,A_2},\dots,W_{n,A_m})\to(W_{A_1},W_{A_2},\dots,W_{A_m})$ in distribution (as random vector) for any $m\in\mathbb{N}$ and $A_1,A_2,\dots,A_m\in\mathbf{A}$.Let $\{A_i\}_{i=1}^m$ be an increasing sequence in $\mathbf{A}$. Without loss of generality, we may assume that the sets $\{C_i\}_{i=1}^k$ are the left-neighbourhoods of the sub-semilattice $\mathbf{A}^{ss}$ of $\mathbf{A}$ equipped with a numbering consistent with the strong past when $C_1=A_1$ and $C_i=A_i\setminus A_{i-1}$ for all $2\le i\le k$. According to [Me], there exists a strict continuous flow $\eta:[0,\infty)\to\mathbf{A}(\mathbf{u})$ such that each left-neighbourhood generated by $\mathbf{A}^{ss}$ is of the form $C_i=\eta(i)\setminus\eta(i-1)$, $1\le i\le k$ and $F_{\eta(i)}\subseteq\mathbf{G}^*_C$. But $W^\eta_{n,t}\to W^\eta_t$ in distributionthen $W_n\overset{fdd}{\Rightarrow}W$. $\square$

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