Properties Of Elements In Ternary-Semirings

G. Srinivasa Rao¹,², D. Madhusudhana Rao³, P. Sivaprasad⁴, G. Srinivasa Rao⁵
¹Research Scholar, Department of Mathematics, Acharya Nagarluna University, Nagarluna Nagar, Guntur, A.P. India.
²PGT Mathematics, APSWRS Lr College, Karampudi, Guntur(Dt), A.P. India.
³Associate Professor, Department of Mathematics, VSR & NVR College, Tenali, A.P. India.
⁴Department of BSH, VFSTR’S University, Vadlamudi, Guntur, A.P. India.
⁵Associate Professor, Department of Mathematics, Tirumala Engineering College, Narasarao Pet, A.P.
Corresponding Author: D. Madhusudhana Rao.

ABSTRACT: In this paper we made a study on zeroed, regular & idempotent elements in ternary semirings and we proved some properties of such elements in ternary semirings.

KEYWORDS: Zeroed, Regular, Idempotent, Ordered ternary semiring, Simple.
AMS Subject Classification: 16Y60, 03G25

I. INTRODUCTION

In the year 2011 Madhusudhana Rao, Anjaneyulu and Gangadhara rao investigatied and studied about gamma semigroups. In the year 2013, Y. Sarala, D. Madhusudhana Rao, A. Anjaneyulu made a study on ternary semigroups. Further in the year 2013 Subramaneswara rao seetamraj, D. Madhusudhana rao and A. Anjaneyulu introduce the some notions in partially ordered gamma semigroups. P. Siva Prasad, D. Madhusudhana Rao and G. Srinivasa rao developed some properties in partially ordered ternary semirings. In this paper we introduce the properties of ternary semirings.

1. Preliminaries:

Definition 1.1: A ternary semi ring is a nonempty set U on which operations of addition and ternary multiplication have been defined such that the following conditions are satisfied:
(1) (U,+) is a commutative monoid with identity element 0;
(2) (U,[]) is a ternary monoid with identity element 1T,
(3) Ternary Multiplication distributes over addition,
(4) 00r = 0r0 = r00 = 0 for all r ∈ U.
For more preliminaries refer the references.

2. Zeroed Elements in ternary semi rings:

Def 2.1: U is said to be multiplicatively left cancellative (MLC) if pqd = pqf implies that d = f for all p, q, d, f ∈ U.

Def 2.2: An element f ∈ U is known as left (right) zeroed element of U if ∃ h, g, l ∈ U ∃ fgh + l = l or l + ghf = l. An element f ∈ U is known as zeroed element of U if ∃ l ∈ U ∃ fgh + l = l or fgh + l = l. Every idempotent element of U is a zeroed element of U. The zeroed of U is the set of all f ∈ U ∃ fgh + l = l or l + ghf = l for some l ∈ U and it is denoted by Z.

Note 2.3: Some of the authors were defined the zeroed elements as left zeroed f + u = u as well as right zeroed as u + f = u, but both the conditions are equivalent, because if f is zero of U.

Theorem 2.4: Let U be a ternary semi ring with unity element, 1 + 1 = 1 ∀ 1 ∈ U and 1 + 1 = 1 for all l ∈ U. If ternary semi ring U is left cancellative then every element of U is a zeroed.

Proof: Let l ∈ U ⇒ l + 1 = 1. Since 1.1 = 1.
1 + 1 = 1 ⇒ l + 1 = l for v ∈ U
⇒ l + l = l + l = l
⇒ l + 1 = l + l = l + 1
Therefore, every element of U is a zeroed.
Def 2.5 [ ]: A nonempty subset V of a ternary semi ring U is known as k-ideal of U if \( l \in U, l + v \in V, v \in V \Rightarrow l \in V \).

Theorem 2.6: Let U be the additively commutative ternary semi ring. The zeroed Z is a k-ideal of U.

Proof: Let \( l, v \in Z \), then \( \exists f, g, p, q \in U \) \( lf + p = p, p + fgl = p & vfg + q = q, q + fgl = q \).
Suppose, \( ifl + p = l & vfg + q + p = q \), then
\[
(l + v)fg + p + q = ifg + vfg + p + q = ifg + vfg + q + p
\]
= \( ifg + q + p = ifg + p + q = p + q \Rightarrow 1 + v \in Z \).

Now, \( l, v, w \in Z \), then \( ifg + p = p, p + fgl = p, vfg + q = q, q + fgl = q \).
Therefore, \( Z \) is a ternary subsemiring of U.

To show that \( Z \) is a k-ideal of U, let \( t \in U \) and \( l \in Z \) such that \( l + t \in Z \), therefore there exist \( x, y, z \in U \) such that \((l + t)yz = x = x \). But \( lqr + p = p \) \& \( vfg + q = q \).
Suppose \( v + l = a + v + l \), since U is singular w. r. t ternary multiplication i.e. \( vvl = 1, lvv = 1 \).
Therefore, \( v + wxvvl = v \Rightarrow v + wxvl = v \).

Theorem 2.8: Let U be the additively commutative ternary semi ring. The zeroed Z is a \( l \)-ideal of U.

Proof: Let \( l \in U \Rightarrow \exists w, x, v \in U \) \( lwv + v = v \) or \( v + wxl = v \). Suppose, \( v + wxl = v \), since U is singular w. r. t ternary multiplication i.e. \( vvl = 1, lvv = 1 \).
Therefore, \( v + vxl = v \Rightarrow v + wxvl = v \).

Theorem 2.9: Let U be a ternary semiring and \( (U, +) \) is right cancellative. If \( l \) is right zeroed of U iff \( l^3 + l^4 = l^4 \).

Proof: Let \( l \) is a right zeroed of U, then \( v \in U \) \( \exists v + III = v \) or \( III + v = v \).
\[
\Rightarrow II(III + v) = IIV \Rightarrow IIIII + IIV = III + IIIII + IIV = III + IIv
\]
\[
\Rightarrow III + IIIII = III \Rightarrow I^3 + I^4 = I^4.
\]
Conversely suppose that \( 1 \in U \Rightarrow I^3 + I^4 = I^4 & 1.1.1 = 1 \Rightarrow I^3 + I^4 = I^4 \Rightarrow I^3(1 + I^3) = I^4 \Rightarrow 1 + III = 1 \Rightarrow l \in U \) is a zeroed of U and hence every element of U is zeroed.

Corollary 2.10: Let U be a ternary semiring and \( (U, +) \) is left cancellative. If \( l \) is left zeroed of U iff \( l^5 + I^4 = I^4 \).

Theorem 2.11: In a ternary semiring U with unity element and unit is zeroed then every element of U is zeroed.

Proof: Suppose U is a ternary semiring with unity element and \( 1 \in U \) so \( \exists v \in U \) \( 1 + v = v + 1 \) & \( 1.1.1 = 1 \).
Let \( v + 1 = v \Rightarrow 1(l + v + 1) = 1.1.1 \Rightarrow l + v + 1 = 1.1.1 \Rightarrow 1lv = v + 1 \Rightarrow 1lv = 1 lv = 1lv \).

Def 2.12: A ternary semiring U is said to be semi-subtractive if for any elements \( a, b \in T \); there is always some \( x \in Tor \) some \( y \in U \) \( a + y = b \) or \( b + x = x \). Every ring is a semi-subtractive ternary semiring.

Theorem 2.13: Let the zeroed of a semi subtractive ternary semiring U is empty, then \((U, +)\) is cancellative.

Proof: Suppose \( v + 1 = w + 1 \Rightarrow v \neq w \) for \( l, v, w \in U \).
\[
\exists some a \in U \Rightarrow v = a + w or w = a + v.
\]
Suppose \( v = a + w \Rightarrow v + 1 = a + w + 1 \Rightarrow v + 1 = a + v + 1 \Rightarrow v + 1 = awx + (v + 1) \), since \( awx = a \) and hence a is zeroed element of U, it is a contradiction and hence \((U, +)\) is a right cancellative. Therefore \((U, +)\) is cancellative.

Def 2.14: A ternary semiring U is known as zero cube ternary semiring if \( l^3 = 0 \) \( \forall l \in U \).

Theorem 2.15: Suppose U is a zero cube ternary semiring and \( l \) is a zeroed of U then \( \exists v, w, x \in U \) \( lwvxx = 0 \) or \( xxwlv = 0 \) or \( xvwv = 0 \).
Proof: Suppose \( l \in U, \exists v, w, x \in U \) \( uvw + x = x \) or \( x + vwl = 0 \). Let \( lwv + x = x \).
\( \Rightarrow (lw + x)xx = xxx \Rightarrow lwxx + x^3 = x^3 \Rightarrow lwxx + 0 = 0 \Rightarrow lwxx = 0. \) Similarly we can prove the other parts.

3. Idempotent and Regular Elements in Ternary Semirings:

**Def 3.1:** An element \( l \in U \) is known as **+ve idempotent** if \( l + l = l \). The set of all additive idempotent of a ternary semiring \( U \) is denoted by \( E'(U) \).

**Example 3.2:** \( RU \{–\infty \} \) is a commutative, additive idempotent ternary \( \Gamma \)-semiring with the addition and multiplication operations where \( \Gamma = R \) defined as: \( 1 \oplus v = \max(l,v) \) and \( 1 \otimes v \otimes w = l + v + w \) where \( + \) is the ordinary addition as ternary semiring multiplication. Clearly, \( –\infty \) is the zero element, and 0 is the unity.

**Def 3.3:** An element \( l \in U \) is known as an **idempotent** if \( l^3 = l \). The set of all idempotent elements in a ternary semiring \( U \) is denoted by \( E(U) \). Every identity, zero elements are idempotent elements.

**Def 3.4:** An element \( l \in U \) is known as a **proper dempotent** element provided \( l \) is an idempotent which is not the identity of \( U \) if identity exists. A ternary semiring \( U \) is said to be an **idempotent ternary semiring** provided every element of \( U \) is an idempotent.

**Def 3.5:** An element \( l \in U \) is known as **ternary multiplicatively sub-idempotent** provided \( l + l^3 = l \). A ternary semiring \( U \) is said to be a **sub-idempotent ternary semiring** provided each of its element is sub-idempotent.

**Def 3.6:** A ternary semiring \( U \) is said to be **Viterbi ternary semiring** if \( U \) is +vely idempotent and multiplicatively sub-idempotent.

**Theorem 3.7:** Let \( U \) be a ternary semiring satisfying the identity \( l + lv + l = l \) \( \forall l, v \in U \). If \( U \) contains the ternary multiplicative identity which is also an additive identity, then \( U \) is a multiplicatively sub-idempotent ternary semiring.

**Proof:** Since \( u + lv + l = 1 \) \( \forall l, v \in U \). Let \( e \) be the multiplicative identity in \( e \) is also an additive identity. i.e., \( lee = ece = e = 1 \). Given \( l + lv + l = 1 \) \( \forall l, v \in U \). Taking \( l = v \). Then \( l + l^3 + l = l \Rightarrow l(l^2 + e^3) = 1 \Rightarrow l + l^3 = 0 \Rightarrow l + l^3 = 1 \) \( \forall l \in U \). Therefore, \( U \) is multiplicatively sub-idempotent ternary semiring.

**Theorem 3.8:** Let \( U \) be a multiplicatively idempotent ternary semiring. If \( U \) satisfying the identity \( l + lv + l = 1 \) \( \forall l, v \in U \). Then \( U \) is additively idempotent.

**Proof:** Let \( l \in U \). Consider \( l + 1 = (1 + l)^3 \) \( \forall l \in U \). \( 1 + 1 = (1 + 1)^3 = l^3 + l^3 + l^3 + l^3 + l^3 + l^3 = (1 + l + l) + l + l + l + l + l + l + l + l \). Therefore, \( U \) is additively idempotent.

**Theorem 3.9:** Every ternary semiring \( U \) in which \( U \) is left singular semigroup w.r.t addition and left singular ternary semigroup w.r.t ternary multiplication, then \( U \) is a Viterbi ternary semiring.

**Proof:** By the hypothesis \( U \) is left singular semigroup w.r.t addition and left singular ternary semigroup w.r.t ternary multiplication. i.e., \( 1 + v = 1 \) and \( lv = 1 \) \( \forall l, v \in U \). Consider \( l + v = 1 \). Taking \( l = v \Rightarrow 1 + l = 1 \) \& \( lv = 1 \). Taking \( l = v \Rightarrow l^3 = 1 \Rightarrow l^3 = 1 + l + l \Rightarrow 1 + l = 1 \Rightarrow l = 1 \Rightarrow l = 1 \) \( \Rightarrow (1) \) and \( (2) \). Therefore, from (1) and (2), \( U \) is a Viterbi ternary semiring.

**Note 3.10:** The converse of the theorem 3.9, is not necessarily true. This is evident from the following example.

**Example 3.11:**

\[
\begin{array}{ccc}
+ & l & v \\
l & l & l \\
v & l & v \\
\end{array}
\]

\[
\begin{array}{ccc}
[ & l & v \\
l & l & l \\
v & v & v \\
\end{array}
\]

**Def 3.12:** An element \( l \) of a ternary semiring \( U \) is known as **+vely regular** if there exist \( v \in U \) \( \exists 1 + v + 1 = 1 \). An element \( l \in U \) is known as **ternary multiplicatively regular** if \( \exists v, w \in U \) \( lvwl = l \). \( U \) is known as **regular ternary semiring** if every element is regular.

**Example 3.13:** Let \( U = \{0, u, v\} \) be any nonempty set. If we define a binary operation + and ternary multiplication on \( U \) as the following Cayley table, then \( U \) is a regular ternary semiring.
Lemma 3.14: Every idempotent element is regular.

Def 3.15: A ternary semiring U is known as an Ordered Ternary Semiring if U is partially ordered set such that l ≤ u then (1) l + v ≤ u + v & v + l ≤ v + u, (2) lvw ≤ uvw, vlw ≤ vwu & vul ≤ vwu ∀ l, u, v, w ∈ U.

Th 3.16: Let U be a ordered ternary semiring in which (U, +) is +vely ordered ternary semiring. If u ∈ U & v be a idempotent of (U, +), uzx ≤ v implies u is zeroed for x ∈ U.

Proof: Let u, z, x ∈ U & v be a idempotent, uzx ≤ v ⇒ uzx + v ≤ v + v ⇒ uzx + v ≤ v, but uzx + v ≥ v and therefore, uzx + v = v. Thus u ∈ Z, hence u is zeroed of U.

Corollary 3.17: Let U be a ordered ternary semiring in which (U, +) is -vely ordered ternary semiring. If u ∈ U & v be a idempotent of (U, +), uzx ≤ v implies v is zeroed for x ∈ U.

Theorem 3.18: Let U be a ternary semiring, every element U is zeroed. If ternary semigroup U is band then for each u ∈ U, ∃ v, z, x ∈ U uzx + uzxvv + v = v.


Theorem 3.19: Let U be an ordered ternary semiring in which ternary semigroup is –vely ordered. If u is a regular ternary semiring iff u is an idempotent of U.

Proof: Suppose u is regular in U, then ∃ x, y ∈ U, u = uxu + v ⇒ u = idempotent of U. Conversely, suppose u is a idempotent of U, then we have u = u ⇒ u = uuuu. Thus u is regular.

Theorem 3.20: Let U be an ordered ternary semiring in which ternary semigroup is –vely ordered. If u is a regular ternary semiring iff u is an idempotent of U.


Def 3.21: An ordered ternary semiring is known as sum ordered ternary semiring if u ≤ v there exist w ∈ U, ∃ v = u + u + v = v.

Theorem 3.22: Let U be a ternary semiring with unity 1. If 1 + 1 = 1 holds implies that (U, +) is a idempotent semigroup of U.

Proof: Let U be an ordered ternary semiring as well as u ∈ U in which ternary semigroup is –vely ordered. Then u = uxu = xuy for x, y ∈ U.

Def 3.22: An ordered ternary semiring is known as sum ordered ternary semiring if u ≤ v there exist w ∈ U, ∃ u + w = v.

Theorem 3.23: Let U be an +vely idempotent ordered ternary semiring. Then
(i) 0 is the least element of U.
(ii) 1 + v ∀ 1, v ∈ U.
(iii) l ≤ v iff l + v = v
(iv) U is sum ordered ternary semiring.

Proof: Let U be an +vely idempotent ordered ternary semiring.
(i) ⇒ (ii): Suppose 0 is the least element of U & l, v ∈ U. 0 ≤ l ⇒ 0 + 1 + l + v ⇒ v ≤ 1 + v. Similarly, 1 ≤ 1 + v.
(ii) ⇒ (iii): Suppose assume (ii) & l ≤ v ⇒ l + v ≤ v ⇒ 1 + v ≤ 1 + v ⇒ 1 + v = v.
(iii) ⇒ (iv): obvious.
(iv) ⇒ (i): Suppose U is a sum ordered ternary semiring. Then ∃ v ∈ U, ∃ 0 + v = l ⇒ l = v.
Now, 0 + 1 = 1 ∀ l ∈ U ⇒ 0 ≤ 1. Therefore 0 is the least element of U.

Theorem 3.24: Let U be an ordered ternary semiring with unity 1. If 1 + 1 = 1 holds implies that (U, +) is a idempotent semigroup of U.
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Theorem 3.25: Let U be an ordered ternary semiring in which (U, +) cancellative, +ve ordered commutative semigroup. If E'(U) is nonempty as well as ternary semigroup U is cancellative implies that E'(U) is k-prime ideal of an ordered ternary semiring U.

Proof: Let l ∈ E'(U) ⇒ l + l + l ∈ E'(U).

Similarly, v, w ∈ E'(U), suppose l, v, w ∈ E'(U) ⇒ l + l + l & v + v + v = v then

Now l + v + l + v = l + v & l + l = l & l + l = l + v.

Therefore, v = l + v ⇒ v = l + v ⇒ v = v = l ∈ E'(U).

Now l + v + l + v = l + v ⇒ l + v = l + v = l + v = l + v.

Therefore, v + v = v + v = v = v ∈ E'(U) ⇒ E'(U) is a k-ideal of U.

Corollary 3.26: Suppose U is a simple ternary semiring in which (U, +) cancellative, +ve ordered commutative semigroup. If E'(U) is nonempty as well as ternary semigroup U. Then every element of U is additive idempotent.

Proof: By the theorem 3.24, E'(U) is an ideal of ordered ternary semiring and since U is simple, therefore U = E'(U). Hence every element of U is additive idempotent.

Corollary 3.27: Let U be an ordered simple ternary semiring and (U, +) cancellative, +ve ordered commutative ternary semigroup. If E'(U) = 1, then |U| = 1

Theorem 3.28: Let U be the ordered regular ternary semiring in which ternary semigroup U is negatively ordered with unity. If l, v ∈ U & l ≤ v ⇒ l = l

Proof: Let l, v ∈ U & l ≤ v, since U is regular ⇒ l = lxy for x, y ∈ U.

We have xly ≤ l ⇒ lxy ≤ l ⇒ l ≤ l ⇒ l = l.

Again, l ≤ v ⇒ l ≤ l ⇒ l ≤ l ≤ l ⇒ l = l.

II. CONCLUSION:
Here, the notion of zeroed in ternary semiring, regular & idempotent elements in ordered ternary semiring are introduced. We investigated the properties of zeroed, regular & idempotent elements and relations between them.

REFERENCES


