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Caldron Reproducing Formula For Legendre Convolution

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ABSTRACT

Calderon-type reproducing formula for Legendre convolution is established using the theory of Legendre transform.

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I. INTRODUCTION

Calderon formula [5] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$f(x) = \int_0^\infty (\phi_t * \phi_t * f)(x)\frac{dt}{t}, \qquad (1.1)$$

where $\phi : \mathbf{R}^n \to C$ and $\phi_t(x) = t^{-n}\phi(x/t),\ t > 0.$ For conditions of validity of identity (1.1), we may refer to [5].

We follow the notation and terminology used in [2].

Let X denote the space $L^p(-1,1),\ 1 \le p < \infty,\ \text{or}\ C[-1,1]$ endowed with the norms

$$\| f \|_p = \left[\frac{1}{2}\int_{-1}^{1} | f(x) |^p \, dx\right]^{1/p} < \infty, \quad 1 \le p < \infty, \qquad (1.2)$$

$$\| f \|_C = \sup_{-1\le x\le 1} | f(x) |. \qquad (1.3)$$

An inner product on X is given by

$$< f, g > = \frac{1}{2}\int_{-1}^{1} f(x)\,\overline{g(x)}\, dx \quad . \qquad (1.4)$$

As usual we denote the Legendre polynomial of degree $n \in \mathbf{N}_0$ by $P_n(x)$, i.e.

$$P_n(x) = (2^n\, n!)^{-1}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n\ ; \quad x \in [-1,1].$$

For these polynomials one has

(i) $| P_n(x) | \le P_n(1) = 1 \quad ; \qquad x \in [-1,1]$ (1.5)

(ii) $(1 - x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0\,;$ (1.6)

(iii) $P_n^{'}(1) = \frac{n(n+1)}{2}$. (1.7)

The Legendre transform of a function $f \in X$ is defined by

$$L[f](k) = \hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x)P_k(x)dx;\ \ k \in \mathbf{N}_0 \quad . \tag{1.8}$$

The operator L associates to each $f \in X$, a sequence of real (complex) numbers $\left\{\hat{f}(k)\right\}_{k=0}^{\infty}$, called the Fourier Legendre coefficients.

The inverse Legendre transform is given by

$$L[f]^{v}(x) = f(x) = \sum_{k=0}^{\infty}(2k+1)\ \hat{f}(k)\, P_k(x)\,. \tag{1.9}$$

II. PRELIMINARIES

Lemma 2.1. Assume f, g $\in X$, k $\in \mathbf{N}_0$ and c $\in \mathbf{R}$, then

(i) $\left|L[f](k)\right| \le \|f\|_{X;}$

(ii) $L[f+g](k) = L[f](k) + L[g](k)$, $L[cf](k) = cL[f](k);$

(iii) $L[f](k) = 0$ for all $k \in \mathbf{N}_0$ iff f(x) = 0 a.e ;

(iv) $L[P_k](j) = \begin{cases} \frac{1}{2k+1}, & k = j \\ 0 & , \ k \ne j,\ (k,j) \in N_0 \end{cases}$

Let us now define the basic function K(x,y,z) which plays role in our investigation

$$K(x,y,z) = \begin{cases} 1 - x^2 - y^2 - z^2 + 2xyz, & z_1 < z < z_2 \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

where $z_1 = xy - [(1-x^2)(1-y^2)]^{1/2}$ and $z_2 = xy + [(1-x^2)(1-y^2)]^{1/2}$.

Then the function K(x,y,z) possesses the following properties;

(i) K(x,y,z) is symmetric in all the three variables

(ii) $\int_{-1}^{1} K(x,y,z)dz = \pi$.

<u>Also it has been shown in [2] that</u>

$$P_k(x)\,P_k(y) = \frac{1}{\pi}\int_{-1}^{1} P_k(z)K(x,y,z)dz \tag{2.2}$$

Applying (1.9) to (2.2), we have

$$K(x,y,z) = \frac{\pi}{2}\sum_{k=0}^{\infty}(2k+1)\,P_k(x)\,P_k(y)\,P_k(z)\,.$$

The Legendre translation τ_y for $y \in [-1,1]$ of a function $f \in X$ is defined by

$$(\tau_y f)(x) = f(x,y) = \frac{1}{\pi}\int_{-1}^{1} f(z)\ K(x,y,z)dz \quad . \tag{2.3}$$

Using Hölder's inequality it can be shown that

$$\|\tau_y f\|_X \le \|f\|_X \tag{2.4}$$

and the map $y \to \tau_y f$ is a positive linear operator from X into itself.

As in [2], for functions f,g defined on [-1,1] the Legendre convolution is given by

$$(f * g)\ (x) = \frac{1}{2}\int_{-1}^{1} (\tau_y f)(x)\ g(y)\, dy$$

$$= \frac{1}{2}\int_{-1}^{1} (\tau_x f)(y)\, g(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(z)\, g(y)\, K(x, y, z)\, dy\, dz \qquad (2.5)$$

Lemma 2.2 If $f \in X, g \in L^1(-1,1)$, then the convolution (f*g) (x) exists (a.e.) and belongs to X. Moreover,

$$\| f * g \|_X \le \| f \|_X \| g \|_1, \qquad (2.6)$$

$$(f * g)^\wedge(k) = \hat{f}(k)\hat{g}(k). \qquad (2.7)$$

For any $f \in L^2(-1,1)$ the following Parseval identity holds for Legendre transform,

$$\sum_k (2k+1)\,|\hat{f}(k)|^2 = \| f \|_2^2. \qquad (2.8)$$

III. MAIN RESULTS

Calderon's formula

In this section, we obtain Calderon's reproducing identity using the properties of Legendre transform and Legendre convolution.

Theorem 3.1 Let ϕ and $\psi \in L^1$(-1, 1)be such that following admissibility condition holds:

$$\int_{-1}^{1} \hat{\phi}(\lambda)\hat{\psi}(\lambda)\frac{d\lambda}{\lambda} = 1 \qquad (3.1)$$

for all $\lambda \in (-1,1)$. Then the following Calderon's reproducing identity holds:

$$.f(x) = \int_{-1}^{1} (f * \phi_a * \psi_a)(x)\frac{da}{a} \quad \forall\ f \in L^1(-1,1) \qquad (3.2)$$

Proof: Taking Legendre transform of the right hand side of (3.2), we get

$$L\int_{-1}^{1} (f * \phi_a * \psi_a)(x)\frac{da}{a}$$

$$= \int_{-1}^{1} \hat{f}(\lambda)\hat{\phi}_a(\lambda)\hat{\psi}_a(\lambda)\frac{da}{a}$$

$$= \hat{f}(\lambda)\int_{-1}^{1} \hat{\phi}_a(\lambda)\hat{\psi}_a(\lambda)\frac{da}{a} \qquad (3.3)$$

$$= \hat{f}(\lambda)\int_{-1}^{1} \hat{\phi}_a(a\lambda)\hat{\psi}_a(a\lambda)\frac{da}{a}$$

$$= \hat{f}(\lambda)$$

Now, by putting $a\lambda = \omega$

$$\int_{-1}^{1} \hat{\phi}_a(a\lambda)\hat{\psi}_a(a\lambda)\frac{da}{a} = \int_{-1}^{1} \hat{\phi}_a(\omega)\hat{\psi}_a(\omega)\frac{d\omega}{\omega} \qquad (3.4)$$

$$= 1.$$

Hence the result follows.

Theorem 3.2 Suppose $\phi \in L^1$(-1, 1) is real valued and satisfies

$$\int_{-1}^{1}\left[\hat{\phi}_a(a\lambda)\right]^2\frac{da}{a}=1 \qquad (3.5)$$

For $f\in L^1(-1,1)\cap L^2(-1,1)$ suppose that

$$f_{\varepsilon\delta}(x)=\int_{-1}^{1}(f*\phi_a*\phi_a)(x)\frac{da}{a} \qquad (3.6)$$

Then $\|f-f_{\varepsilon\,\delta}\|_2\to 0$ as $\varepsilon\to$ -1and $\delta\to$ -1.

Proof: Taking Legendre transform of both sides of (3.6) and using Fabini's theorem, we get

$$\hat{f}_{\varepsilon\delta}(\lambda)=\hat{f}(\lambda)\int_{-1}^{1}\left[\hat{\phi}_a(a\lambda)\right]^2\frac{da}{a} \qquad (3.7)$$

By [5], we have

$$\|\phi_a*\phi_a*f\|_2\le\|\phi_a*\phi_a\|_1\|f\|_2$$
$$\le\|\phi_a\|_1^2\|f\|_2. \qquad (3.8)$$

Now using above inequality and Minkowski'senequality [3, page 41],we get

$$\|f\|_2^2=\int_{-1}^{1}d\mu(x)\left|\int_{\varepsilon}^{\delta}(\varphi_a*\varphi_a*f)(x)\frac{da}{a}\right|^2$$
$$\le\int_{\varepsilon}^{\delta}\int_{-1}^{1}\left|(\phi_a*\phi_a*f)(x)\right|^2dx\frac{da}{a}$$
$$\le\int_{\varepsilon}^{\delta}\left\|(\phi_a*\phi_a*f)(x)\right\|_2\frac{da}{a} \qquad (3.9)$$
$$\le\|\phi_a\|_1^2\|f\|_2\int_{\varepsilon}^{\delta}\frac{dt}{t}$$
$$=\|\phi_a\|_1^2\|f\|_2\log\left(\frac{\delta}{\varepsilon}\right).$$

Hence by Parseval formula, we get

$$\lim_{\substack{\varepsilon\to-1\\ \delta\to-1}}\|f-f_{\varepsilon,\delta}\|_2^2=\lim_{\substack{\varepsilon\to-1\\ \delta\to-1}}\left\|\hat{f}-\hat{f}_{\varepsilon,\delta}\right\|_2^2$$
$$\lim_{\substack{\varepsilon\to-1\\ \delta\to-1}}\int_{-1}^{1}\left|\hat{f}(\lambda)\left(1-\int_{\varepsilon}^{\delta}\left[\hat{\phi}(a\lambda)\right]^2\frac{da}{a}\right)\right|^2dx \qquad (3.10)$$
$$=0.$$

Since $\left|\hat{f}(\lambda)\left(1-\int_{\varepsilon}^{\delta}\left[\hat{\phi}(a\lambda)\right]^2\frac{da}{a}\right)\right|\le\hat{f}(\lambda)$, therefore by the dominated convergence theorem, the result follows.

The reproducing identity (3.2) holds in the point wise sense under different set of nice conditions.

Theorem 3.3 Suppose $f,\hat{f}\in L^1(-1,1)$. $Let\,\phi\in L^1(-1,1)$ be real valued and satisfies

$$\int_{-1}^{1}\left[\hat{\phi}(a\lambda)\right]^2\frac{da}{a}=1,\quad \lambda\in R-\{0\}. \qquad (3.11)$$

Then

$$\lim_{\substack{\varepsilon\to-1\\ \delta\to-1}}\int_{-1}^{1}(f*\phi_a*\phi_a)(x)\frac{da}{a}=f(x)\triangleright. \qquad (3.12)$$

Proof: Let

$$f_{\varepsilon,\delta}(x)=\int_{\varepsilon}^{\delta}(f*\phi_a*\phi_a)(x)\frac{da}{a}. \qquad (3.13)$$

By [3], we have

$$\|\phi_a * \phi_a * f\|_1 \le \|\phi_a * \phi_a\|_1 \|f\|_1$$
$$\le \|\phi_a\|_1^2 \|f\|_1 \qquad (3.14)$$

Now

$$\|f_{\varepsilon,\delta}\|_1 = \int_{-1}^{1} dx \left| \int_{\varepsilon}^{\delta} (\phi_a * \phi_a * f)(x) \frac{da}{a} \right|$$
$$\le \int_{\varepsilon}^{\delta} \int_{-1}^{1} \left| (\phi_a * \phi_a * f)(x) \right| dx \frac{da}{a}$$
$$\le \int_{\varepsilon}^{\delta} \left\| (\phi_a * \phi_a * f)(x) \right\|_1 \frac{da}{a} \qquad (3.15)$$
$$\le \|\phi_a\|_1^2 \|f\|_1 \int_{\varepsilon}^{\delta} \frac{dt}{t}$$
$$= \|\phi_a\|_1^2 \|f\|_1 \log\left(\frac{\delta}{\varepsilon}\right).$$

Therefore, $f_{\varepsilon,\delta} \in L^1(-1,1)$. Also using Fubini's, we get theorem and taking Legendre transform of (3.13), we get

$$\hat{f}_{\varepsilon,\delta}(\lambda) = \int_{-1}^{1} \varphi_\lambda(x\lambda) \left(\int_{\varepsilon}^{\delta} (\phi_a * \phi_a * f)(x) \frac{da}{a} \right) dx$$
$$= \int_{\varepsilon}^{\delta} \int_{-1}^{1} \varphi_\lambda(x\lambda) (\phi_a * \phi_a * f)(x)\, dx \frac{da}{a} \qquad (3.16)$$
$$= \int_{\varepsilon}^{\delta} \hat{\phi}_a(\lambda) \hat{\phi}_a(\lambda)\, \hat{f}(\lambda) \frac{da}{a}$$
$$= \hat{f}(\lambda) \int_{\varepsilon}^{\delta} [\hat{\phi}(a\lambda)]^2 \frac{da}{a}.$$

Therefore, by (3.11), $\left| \hat{f}_{\varepsilon,\delta}(\lambda) \right| \le \left| \hat{f}(\lambda) \right|$.

It follows that $\hat{f}_{\varepsilon,\delta} \in L^1(-1,1)$ .By inversion, we have

$$f(x) - f_{\varepsilon,\delta}(x) = \int_{-1}^{1} \varphi_\lambda(x\lambda) [\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda)] d\xi,\ x \in (-1,1). \qquad (3.17)$$

Putting

$$h_{\varepsilon,\delta}(\lambda : x) = \varphi_\lambda(x\lambda) \left[\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda) \right]$$
$$= \hat{f}(\lambda) \varphi_\lambda(x\lambda) \left[1 - \int_{\varepsilon}^{\delta} [\hat{\phi}(a\lambda)]^2 \frac{da}{a} \right] \qquad (3.18)$$

we get

$$f(x) - f_{\varepsilon,\delta}(x) = \int_{-1}^{1} \varphi_\lambda(x\lambda) \left[\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda) \right] d\lambda \qquad (3.19)$$
$$= \int_{-1}^{1} h_{\varepsilon,\delta}(\lambda : x)\, d\mu(\xi).$$

Now using (3.11) in (3.18), we get

$$\lim_{\substack{\varepsilon \to -1 \\ \delta \to -1}} h_{\varepsilon,\delta}(\lambda : x) = 0,\ \ \lambda \in R - \{0\}. \qquad (3.20)$$

Since $\left| h_{\varepsilon,\delta}(\lambda : x) \right| \le \left| \hat{f}(\lambda) \right|$, the Lebsegue dominated convergence theorem yields

$$\lim_{\substack{\varepsilon \to -1 \\ \delta \to -1}} \left[f(x) - f_{\varepsilon,\delta}(x) \right] = 0,\ \ \forall x. \qquad (3.21)$$

REFERENCES

[1]. Braaksma, B.L.J. and Meulenbeld, B., Integral transforms with generalizedLegendre functions as Kernels, CompositoMathematica,**18**(1967), 235-287.
[2]. Stens, R.L.andWehrens, M.Legendre transform methods and best algebraic approximation,Comment.Math.PraceMat; 21(2) (1980), 351-380
[3]. H.L.Elliott and M.Loss, Analysis, Narosa Publishing House, New Delhi, 1997.
[4]. Pathak, R.S., and M.M. Dixit, Continuous and discrete Bessel wavelet transforms, J. Computational and Applied Mathematics,**160**(2003),241-250.
[5]. M.Frazier, B.Jawerth, and G.Weiss, Littlewood-Paley theory and the study of function spaces,
[6]. CBMS RegionalConfrence Series in Mathematics, Vol.79, American Mathematical Society, Rhode Island, 1991.

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