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Certain Properties of k-Hypergeometric Functions

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ABSTRACT : *In this paper, we present k-hypergeometric and confluent k-hypergeometric functions in the form of certain k-beta functions. Then Laplace transform is investigated. Further, we established differentiation formulas and several integral representations of k-hypergeometric and confluent k-hypergeometric functions. The results derived here are of general in nature and can yield a number of new and known results.*
KEYWORDS : *k-Gamma function, k-Beta function, k-Hypergeometric function.*

I. INTRODUCTION

The k-gamma function, k-beta function and k-hypergeometric functions based on Pochhammer k-symbols for factorial function was introduced by Diaz et al. [1,2,3]. Mubeen et al. [9,10] gave the definition of some k-hypergeometric and confluent k-hypergeometric function in the form of integrals. Several properties and results of these k-functions have been studied by many authors [4,5,6,7,8,11].
The integral representation of k-gamma function [2] is given by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \mathrm{Re}(x)>0, k>0, \tag{1.1}$$

and $\Gamma_k(x+k)=x\Gamma_k(x)$; $\Gamma_k(k)=1$; (1.2)

$$\Gamma_k(x)=k^{\frac{x}{k}}\Gamma\left(\frac{x}{k}\right) \tag{1.3}$$

The integral representation of k-beta function [2] is as follows :

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,\ x>0\ ,\ y>0,\ k>0 \tag{1.4}$$

and $B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right)$

The Pochhammer k-Symbol [2] is defined by

$$(x)_{n,k}=x(x+k)(x+2k)\ldots\ldots\ldots.(x+(n-1)k) \tag{1.5}$$

where , $x\in C,\ k\in R^+,\ n\in\mathbf{N}$

The relations between k-gamma, k-beta and Pochhammer k-symbol is as follows [2] :

$$(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)}\ ,\ \mathrm{Re}(x)>0,\ n\geq 0,\ k>0 \tag{1.6}$$

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)},\ \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0,\ k>0 \tag{1.7}$$

The k-hypergeometric function [2] is given by

$${}_pF_{q,k}\left[\begin{matrix}(a_1,k),\ldots\ldots\ldots,(a_p,k)\\(b_1,k),\ldots\ldots\ldots,(b_q,k)\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}(a_2)_{n,k}\ldots\ldots\ldots(a_p)_{n,k}}{(b_1)_{n,k}(b_2)_{n,k}\ldots\ldots\ldots(b_q)_{n,k}}\cdot\frac{z^n}{n!} \tag{1.8}$$

where k > 0 ; c ≠ 0, –1, –2, –3……; $a_i,b_j\in C;(i=1,\ldots.,p),\ (j=1,\ldots.,q)$; |z|<1.

The k-hypergeometric function ${}_2F_{1,k}$ *and* ${}_1F_{1,k}$ are defined as [2]:

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}.\frac{z^n}{n!} \tag{1.9}$$

$$ {}_1F_{1,k}\left[(b,k);(c,k);z\right]=\sum_{n=0}^{\infty}\frac{(b)_{n,k}}{(c)_{n,k}}.\frac{z^n}{n!} \tag{1.10}$$

The integral representation of k-Hypergeometric function defined by Mubeen et al. [10] is as follows :

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}$$

$$\times(1-k\ zt)^{-\frac{a}{k}}\ \mathrm{dt} \tag{1.11}$$

$\left(\operatorname{Re}(a)>0,\ \operatorname{Re}(c)>R(b)>0,\ k>0, |z|<1\right)$

and $${}_1F_{1,k}\left[(b,k);(c,k);z\right]=\frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}e^{zt}\,dt \tag{1.12}$$

(Re (c) > Re (b) > 0 , k > 0, |z| < 1)

In this paper, we present k-hypergeometric functions and confluent k-hypergeometric functions in the new form involving k-beta functions as follows :

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\sum_{n=0}^{\infty}(a)_{n,k}\frac{B_k(b+nk,\ c-b)}{B_k(b,c-b)}\cdot\frac{z^n}{n!} \tag{1.13}$$

(Re (a) > 0, Re (c) > Re (b) > 0, k > 0, |z| < 1)

and $${}_1F_{1,k}\left[(b,k);(c,k);z\right]=\sum_{n=0}^{\infty}\frac{B_k(b+nk,\ c-b)}{B_k(b,c-b)}\cdot\frac{z^n}{n!} \tag{1.14}$$

(Re (c) > Re (b) > 0, k > 0 , |z| < 1)

For k = 1, we get definitions and results for classical special functions.

II. LAPLACE TRANSFORM OF k-HYPERGEOMETRIC AND CONFLUENT k-HYPERGEOMETRIC FUNCTIONS

Definition : The Laplace transform of f (z) is defined as [12]:

$$L\{f(z)\}=\int_0^{\infty}e^{-sz}f(z)dz,\ \operatorname{Re}(s)>0 \tag{2.1}$$

Theorem 2.1. If Re (s) > 0, k > 0, Re (a) > 0, Re (c) > Re (b) > 0 and $\left|\frac{y}{ks}\right|<1$ then

$$L\left\{z^{\frac{d}{k}-1}{}_2F_{1,k}\left[(a,k),(b,k);(c,k);yz\right]\right\}$$

$$=\frac{\Gamma_k(d)}{s^{\frac{d}{k}}k^{\frac{d}{k}-1}}\,{}_3F_{1,k}\left[(a,k)(b,k),(d,k);(c,k);\frac{y}{ks}\right] \tag{2.2}$$

provided both sides of (2.2) exist.

Proof : On using (2.1) and applying (1.13), we get

$$L\left\{z^{\frac{d}{k}-1}{}_2F_{1,k}\left[(a,k),(b,k);(c,k);yz\right]\right\}$$

$$=\int_0^{\infty}e^{-sz}z^{\frac{d}{k}-1}{}_2F_{1,k}\left[(a,k),(b,k);\ (c,k);yz\right]dz$$

$$=\int_0^\infty e^{-sz} z^{\frac{d}{k}-1}\left[\sum_{n=0}^{\infty}(a)_{n,k}\frac{B_k\left(b+nk,c-b\right)}{B_k\left(b,c-b\right)}\frac{\left(y\ z\right)^n}{n!}\right]dz \qquad (2.3)$$

Changing the order of integration and summation in (2.3), we get

$$L\left\{z^{\frac{d}{k}-1}{}_2F_{1,k}\left[(a,k),(b,k);(c,k);yz\right]\right\}$$

$$=\sum_{n=0}^{\infty}(a)_{n,k}\frac{B_k\left(b+nk,c-b\right)}{B_n\left(b,c-b\right)}.\frac{y^n}{n!}\int_0^\infty e^{-sz}.z^{\frac{d}{k}+n-1}\ dz \qquad (2.4)$$

On taking $sz=\frac{t^k}{k}$ in the inner integral of (2.4) and using the definition of k-gamma function (1.1) , we have

$$L\left\{z^{\frac{d}{k}-1}{}_2F_{1,k}\left[(a,k),(b,k);(c,k);yz\right]\right\}$$

$$=\sum_{n=0}^{\infty}(a)_{n,k}\frac{B_k\left(b+nk,c-b\right)}{B_k\left(b,c-b\right)}\frac{\Gamma_k\left(d+nk\right)}{k^{\frac{d}{k}-1}s^{\frac{d}{k}}}\left(\frac{y}{ks}\right)^n\cdot\frac{1}{n!}$$

$$=\frac{\Gamma_k\left(d\right)}{s^{\frac{d}{k}}k^{\frac{d}{k}-1}}\sum_{n=0}^{\infty}(a)_{n,k}\left(d\right)_{n,k}\frac{B_k\left(b+nk,c-b\right)}{B_k\left(b,c-b\right)}\left(\frac{y}{ks}\right)^n\cdot\frac{1}{n!} \qquad (2.5)$$

which upon simplification and interpreting the result with (1.8), yields the required result (2.2).

Theorem 2. 2. If k > 0, Re (s) > 0, Re (c) > Re (b) > 0 and $\left|\frac{y}{ks}\right|<1$ then

$$L\left\{z^{\frac{d}{k}-1}{}_1F_{1,k}\left[(b,k),(c,k)\ ;yz\right]\right\}$$

$$=\frac{\Gamma_k\left(d\right)}{s^{\frac{d}{k}}k^{\frac{d}{k}-1}}\ {}_2F_{1,k}\left[(b,k),(d,k);(c,k);\frac{y}{ks}\right] \qquad (2.6)$$

provided both sides of (2.6) exist.

Proof : On applying the similar procedure as discussed in theorem 2.1, we get the required result (2.6).

III. INTEGRAL REPRESENTATION OF THE k-HYPERGEOMETRIC AND CONFLUENT k-HYPERGEOMETRIC FUNCTIONS

In this section, various integral representations of k-Hypergeometric and confluent k-hypergeometric functions are established.

Theorem 3. 1. For the k-Hypergeometric functions, the following integral representations hold true :

$${}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{1}{k.B_k\left(b,c-b\right)}\int_0^\infty u^{\frac{b}{k}-1}\left(1+u\right)^{\frac{a-c}{k}}$$

$$\times\left[1+k\left(1-z\right)u\right]^{-\frac{a}{k}}du\ , \qquad (3.1)$$

$${}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{2}{k.B_k\left(b,c-b\right)}\int_0^{\frac{\pi}{2}}\sin^{\frac{2b}{k}-1}u\ \cos^{\frac{2(c-b)}{k}-1}u$$

$$\times\left(1-k\,z\sin^2 u\right)^{-\frac{a}{k}}du\quad , \qquad (3.2)$$

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{1}{k.B_k(b,c-b)}\int_0^\infty u^{\frac{c-b}{k}-1}(1+u)^{\frac{a-c}{k}} \times(1+u-kz)^{-\frac{a}{k}}\,du \quad , \tag{3.3}$$

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{2}{k.B_k(b,c-b)}\int_0^{\frac{\pi}{2}}\sin^{\frac{2(c-b)}{k}-1}u\,\cos^{\frac{2b}{k}-1}u \times\left(1-k\,z\cos^2 u\right)^{-\frac{a}{k}}du \tag{3.4}$$

where Re (a) > 0, Re (c) > Re (b) > 0, k > 0, |z|<1.

Proof : Using (1.4) in (1.13), we have

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right] = \frac{1}{k.B_k(b,c-b)}.\sum_{n=0}^\infty (a)_{n,k}\int_0^1 t^{\frac{b+kn}{k}-1}(1-t)^{\frac{c-b}{k}-1}.\frac{z^n}{n!}dt$$

$$=\frac{1}{k.B_k(b,c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}\left\{\sum_{n=0}^\infty (a)_{n,k}\frac{(zt)^n}{n!}\right\}dt \tag{3.5}$$

Using the following identity [2 , pp 189, equation (6)] :

$$\sum_{n=0}^\infty \frac{(a)_{n,k}}{n!}x^n = (1-kx)^{-\frac{a}{k}} \tag{3.6}$$

From (3.5), we get

$$ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]=\frac{1}{k.B_k(b,c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1} \times(1-kzt)^{-\frac{a}{k}}\,dt \tag{3.7}$$

Using the transformations $t=\frac{u}{1+u}$, $t=\sin^2 u$, $t=\frac{1}{1+u}$, $t=\cos^2 u$ respectively in (3.7), we yield the desired results (3.1) – (3.4)

Theorem 3.2. For k > 0, Re (c) > Re (b) > 0,|z|<1,

$$ {}_1F_{1,k}\left[(b,k),(c,k);\,z\right]=\frac{1}{k.B_k(b,c-b)}\int_0^1 u^{\frac{c-b}{k}-1}(1-u)^{\frac{b}{k}-1}e^{z(1-u)}\,du \tag{3.8}$$

Proof : Setting t = 1 – u and following the similar procedure as discussed in Theorem 3.1, we get the desired result (3.8).

Remark : Putting k = 1, we get the integral representations of the classical hypergeometric and confluent hypergeometric functions.

IV. DIFFERENTIATION FORMULAS FOR k-HYPERGEOMETRIC AND CONFLUENT k-HYPERGEOMETRIC FUNCTIONS

In this section, we obtained the differentiation formulas for k-Hypergeometric and confluent k-hypergeometric functions by differentiating (1.13) and (1.14) with respect to the variable z in terms of a parameter by using the following results

$$(a)_{n+1,k}=\frac{\Gamma_k\left[a+(n+1)k\right]}{\Gamma_k(a)}=\frac{\Gamma_k(a+k+nk)}{\Gamma_k(a+k)}.\frac{\Gamma_k(a+k)}{\Gamma_k(a)}= a(a+k)_{n,k} \tag{4.1}$$

and $B_k(b,c-b)=\frac{c}{b}B_k(b+k,c-b)$ (4.2)

Theorem 4.1 For k-hypergeometric function, the following differentiation formula holds true :

$$\frac{d^m}{dz^m}\left\{ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]\right\}$$

$$= \frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\ {}_2F_{1,k}\left[(a+mk,k),(b+mk,k);(c+mk,k);z\right] \qquad (4.3)$$

for m > 0 , k > 0 .

Proof : Taking derivative of both sides of (1.13) with respect to z, we have

$$\frac{d}{dz}\left\{{}_2F_{1,k}\left[(a,k),(b,k),(c,k);z\right]\right\} = \frac{d}{dz}\left[\sum_{n=0}^{\infty}(a)_{n,k}\frac{B_k(b+nk,\ c-b)}{B_k(b,c-b)}\cdot\frac{z^n}{n!}\right]$$

$$= \sum_{n=1}^{\infty}(a)_{n,k}\frac{B_k(b+nk,\ c-b)}{B_k(b,c-b)}\cdot\frac{z^{n-1}}{(n-1)!} \qquad (4.4)$$

Changing index n to n+1 in (4.4) we get

$$\frac{d}{dz}\left\{{}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]\right\} = \sum_{n=0}^{\infty}(a)_{n+1,k}\frac{B_k\left[b+(n+1)k,c-b\right]}{B_k(b,c-b)}\frac{z^n}{n!} \qquad (4.5)$$

Using (4.1), (4.2) in (4.5) and interpreting the result with (1.13), we get

$$\frac{d}{dz}\left\{{}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]\right\}$$

$$= \frac{ab}{c}\ {}_2F_{1,k}\left[(a+k,k),(b+k,k);(c+k,k);z\right] \qquad (4.6)$$

Again, differentiating both sides of (4.6) with respect to z, we get

$$\frac{d^2}{dz^2}\left\{{}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]\right\}$$

$$= \frac{ab(a+k)(b+k)}{c(c+k)}\ {}_2F_{1,k}\left[(a+2k,k),(b+2k,k);(c+2k,k);z\right] \qquad (4.7)$$

On recursive application of this procedure ‘m’ times, we get the general form as :

$$\frac{d^m}{dz^m}\ {}_2F_{1,k}\left[(a,k),(b,k);(c,k);z\right]$$

$$= \frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\ {}_2F_{1,k}\left[(a+mk,k),(b+mk,k);(c+mk,k);z\right]$$

Theorem 4.2. For k-confluent hypergeometric function, we have the following differentiation formula :

$$\frac{d^m}{dz^m}\ {}_1F_{1,k}\left[(b,k);(c,k);z\right] = \frac{(b)_{m,k}}{(c)_{m,k}}\ {}_1F_{1,k}\left[(b+mk,k);(c+mk,k);z\right] \qquad (4.8)$$

For m > 0 , k > 0.

Remark : Putting k = 1 in (4.2) and (4.7), we get the differentiation formulas for the usual hypergeometric and confluent hypergeometric functions as follows :

$$\frac{d^m}{dz^m}\left\{{}_2F_{1,1}\left[(a,1),(b,1);(c,1);z\right]\right\} = \frac{(a)_{m,1}(b)_{m,1}}{(c)_{m,1}}\ {}_2F_{1,1}\left[(a+m,1),(b+m,1);(c+m,1);z\right]$$

and

$$\frac{d^m}{dz^m}\left\{{}_1F_{1,1}\left[(b,1);(c,1);z\right]\right\} = \frac{(b)_{m,1}}{(c)_{m,1}}\ {}_1F_{1,1}\left[(b+m,1);(c+m,1);z\right]$$

V. CONCLUSION

In the present paper, we have established the Laplace transform and differentiation formulas of k-hypergeometric and confluent k-hypergeometric functions. Also, derived several integral representation of these k-functions using various transformations. It is observed that whenever k-hypergeometric functions are expressed in the form of k-beta functions which further reduces to a gamma function, the results become very important from application point of view. Therefore, the k-hypergeometric functions in the form of k-beta functions are expected to be useful.

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