On intuitionistic fuzzy $\beta$ generalized closed sets

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ABSTRACT

In this paper, we have introduced the notion of intuitionistic fuzzy $\beta$ generalized closed sets, and investigated some of their properties and characterizations.

KEYWORDS: Intuitionistic fuzzy topology, intuitionistic fuzzy $\beta$ closed sets, intuitionistic fuzzy $\beta$ generalized closed sets.

I. Introduction

The concept of fuzzy sets was introduced by Zadeh [12] and later Atanasov [1] generalized this idea to intuitionistic fuzzy sets using the notion of fuzzy sets. On the other hand Coker [3] introduced intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. In this paper, we have introduced the notion of intuitionistic fuzzy $\beta$ generalized closed sets, and investigated some of their properties and characterizations.

II. Preliminaries

Definition 2.1: [1] An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form

$A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}$

where the functions $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS $(X)$, the set of all intuitionistic fuzzy sets in $X$. An intuitionistic fuzzy set $A$ in $X$ is simply denoted by $A=(x, \mu_A, \nu_A)$ instead of denoting $A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}$.

Definition 2.2: [1] Let $A$ and $B$ be two IFSs of the form $A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}$ and $B = \{ (x, \mu_B(x), \nu_B(x)) : x \in X \}$. Then,

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,
(b) $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$,
(c) $A^c = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}$,
(d) $A \cup B = \{ (x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) : x \in X \}$,
(e) $A \cap B = \{ (x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) : x \in X \}$.

The intuitionistic fuzzy sets $0^\sim = (x, 0, 1)$ and $1^\sim = (x, 1, 0)$ are respectively the empty set and the whole set of $X$.

Definition 2.3: [3] An intuitionistic fuzzy topology (IFT in short) on $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

(i) $0^\sim, 1^\sim \in \tau$
(ii) $G_i \cap G_j \in \tau$ for any $G_i, G_j \in \tau$
(iii) $\cup G_i \in \tau$ for any family $\{ G_i : i \in J \} \subseteq \tau$.

In this case the pair $(X, \tau)$ is called intuitionistic fuzzy topological space (IFTS in short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS in short) in $X$. The complement $A^c$ of an IFOS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS in short) in $X$. 

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\textbf{Definition 2.4:} [5] An IFS \( A = (x, \mu, \upsilon) \) in an IFTS \((X, \tau)\) is said to be an
\begin{enumerate}[\item]
  \item intuitionistic fuzzy \( \beta \) closed set (IF\( \beta \)CS for short) if \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \),
  \item intuitionistic fuzzy \( \beta \) open set (IF\( \beta \)OS for short) if \( A \subseteq \text{cl}(\text{int}(A)) \).
\end{enumerate}

\textbf{Definition 2.5:} [6] Let \( A \) be an IFS in an IFTS \((X, \tau)\). Then the \( \beta \)-interior and \( \beta \)-closure of \( A \) are defined as
\[ \text{\( \beta \)}\text{int}(A) = \bigcup \{ G / G \text{ is an IF} \beta \text{OS in } X \text{ and } G \subseteq A \}, \]
\[ \text{\( \beta \)}\text{cl}(A) = \bigcap \{ K / K \text{ is an IF} \beta \text{CS in } X \text{ and } A \subseteq K \}. \]

Note that for any IFS \( A \) in \((X, \tau)\), we have \( \text{\( \beta \)}\text{cl}(A^c) = (\text{\( \beta \)}\text{int}(A))^c \) and \( \text{\( \beta \)}\text{int}(A^c) = (\text{\( \beta \)}\text{cl}(A))^c \).

\textbf{Result 2.6:} Let \( A \) be an IFS in \((X, \tau)\), then
\begin{enumerate}[\item]
  \item \( \beta \text{cl}(A) \supseteq A \cup \text{int}(\text{cl}(\text{int}(A))) \)
  \item \( \beta \text{int}(A) \subseteq A \cap \text{cl}(\text{int}(\text{cl}(A))) \)
\end{enumerate}

\textbf{Proof:} (i) Now \( \text{int}(\text{cl}(\text{int}(A))) \subseteq \text{int}(\text{\( \beta \)}\text{int}(\text{\( \beta \)}\text{cl}(A))) \subseteq \beta \text{cl}(A) \), since \( A \subseteq \beta \text{cl}(A) \) and \( \beta \text{cl}(A) \) is an IF\( \beta \)CS. Therefore \( A \cup \text{int}(\text{cl}(\text{int}(A))) \subseteq \beta \text{cl}(A) \).

(ii) can be proved easily by taking complement in (i).

\section{III. Intuitionistic fuzzy \( \beta \)-generalized closed sets}

In this section we have introduced intuitionistic fuzzy \( \beta \) generalized closed sets and studied some of their properties.

\textbf{Definition 3.1:} An IFS \( A \) in an IFTS \((X, \tau)\) is said to be an \textit{intuitionistic fuzzy \( \beta \) generalized closed set} (IF\( \beta \)GCS for short) if \( \beta \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an IF\( \beta \)OS in \((X, \tau)\).

The complement \( A^c \) of an IF\( \beta \)GCS \( A \) in an IFTS \((X, \tau)\) is called an intuitionistic fuzzy \( \beta \) generalized open set (IF\( \beta \)GOS in short) in \( X \).

The family of all IF\( \beta \)GCSs of an IFTS \((X, \tau)\) is denoted by \( \text{IF} \beta \text{GCS}(X) \).

\textbf{Example 3.2:} Let \( X = \{a, b\} \) and \( G = (x, (0.5, 0.4, 0.6)) \). Then \( \tau = \{0, 1\} \) is an IFT on \( X \). Let \( A = (x, (0.4, 0.3, 0.6)) \) be an IFS in \( X \).

Then, \( \beta \text{\( \beta \)}\text{GCS}(X) = \{0, 1\} \), \( \mu_\varepsilon \in [0, 1] \), \( \upsilon_\varepsilon \in [0, 1] \), \( \mu_\upsilon \in [0, 1] \), \( \upsilon_\mu \in [0, 1] \), \( 0 \leq \mu_\mu + \upsilon_\upsilon \leq 1 \) and \( 0 \leq \mu_\upsilon + \upsilon_\mu \leq 1 \).

We have \( \beta \text{cl}(A) \subseteq G \). As \( \beta \text{cl}(A) = A \), \( \beta \text{cl}(A) \subseteq G \), where \( G \) is an IF\( \beta \)OS in \( X \). This implies that \( A \) is an IF\( \beta \)GCS in \( X \).

\textbf{Theorem 3.3:} Every IF\( \beta \)RCS \((X, \tau)\) is an IF\( \beta \)GCS in \((X, \tau)\) but not conversely.

\textbf{Proof:} Let \( A \) be an IFCS. Therefore \( \text{cl}(A) = A \). Let \( A \subseteq U \) and \( U \) be an IF\( \beta \)OS. Since \( \beta \text{cl}(A) \subseteq \text{cl}(A) = A \subseteq U \), we have \( \beta \text{cl}(A) \subseteq U \). Hence \( A \) is an IF\( \beta \)GCS in \((X, \tau)\).

\textbf{Example 3.4:} Let \( X = \{a, b\} \) and \( G = (x, (0.5, 0.4, 0.6)) \). Then \( \tau = \{0, 1\} \) is an IFT on \( X \). Let \( A = (x, (0.4, 0.3, 0.6)) \) be an IFS in \( X \).

Then, \( \beta \text{\( \beta \)}\text{GCS}(X) = \{0, 1\} \), \( \mu_\varepsilon \in [0, 1] \), \( \upsilon_\varepsilon \in [0, 1] \), \( \mu_\upsilon \in [0, 1] \), \( \upsilon_\mu \in [0, 1] \), \( 0 \leq \mu_\mu + \upsilon_\upsilon \leq 1 \) and \( 0 \leq \mu_\upsilon + \upsilon_\mu \leq 1 \).

We have \( A \subseteq G \). As \( \beta \text{cl}(A) = A \), \( \beta \text{cl}(A) \subseteq G \), where \( G \) is an IF\( \beta \)OS in \( X \). This implies that \( A \) is an IF\( \beta \)GCS in \( X \), but not an IFCS, since \( \text{cl}(A) = G \).

\textbf{Theorem 3.5:} Every IF\( \beta \)RCS \((X, \tau)\) is an IF\( \beta \)GCS in \((X, \tau)\) but not conversely.

\textbf{Proof:} Let \( A \) be an IF\( \beta \)RCS. Since every IF\( \beta \)RCS is an IFCS, by theorem 3.3, \( A \) is an IF\( \beta \)GCS.
Example 3.8: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b))$. Then $\tau = \{\{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))$ be an IFS in $X$.

Then, $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ 0 \leq \mu_a + \nu_a \leq 1$ and $0 \leq \mu_b + \nu_b \leq 1\}$. We have $A \subseteq G$. As $\beta cl(A) = A,:\beta cl(A) \subseteq G$, where $G$ is an IF $\beta$ OS in $X$. This implies that $A$ is an IF $\beta$ GCS in $X$, but not an IFCS, since $cl(cl(A)) = G \not\subseteq A$.

Theorem 3.9: Every IF $\beta$ CS in $(X, \tau)$ is an IF $\beta$ GCS in $(X, \tau)$ but not conversely.

Proof: Assume $A$ is an IF $\beta$ CS. Since $\beta cl(A) \subseteq \alpha cl(A) = A$ and $A \subseteq U$, by hypothesis, we have $hcl(c) \not\subseteq U$. Hence $A$ is an IF $\beta$ GCS.

Example 3.10: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b))$. Then $\tau = \{\{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))$ be an IFS in $X$.

Then, $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ 0 \leq \mu_a + \nu_a \leq 1$ and $0 \leq \mu_b + \nu_b \leq 1\}$. We have $A \subseteq G$. As $\beta cl(A) = A, \beta cl(A) \subseteq G$, where $G$ is an IF $\beta$ OS in $X$. This implies that $A$ is an IF $\beta$ GCS in $X$, but not an IFCS, since $cl(cl(A)) = cl(G) = G \not\subseteq A$.

Theorem 3.11: Every IF $\beta$ FCS in $(X, \tau)$ is an IF $\beta$ GCS in $(X, \tau)$ but not conversely.

Proof: Assume $A$ is an IF $\beta$ FCS. Since $\beta cl(A) \subseteq \alpha cl(A) = A$ and $A \subseteq U$, by hypothesis, we have $\beta cl(A) \not\subseteq U$. Hence $A$ is an IF $\beta$ GCS.

Example 3.12: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.6_b), (0.5_a, 0.4_b))$. Then $\tau = \{\{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.5_a, 0.7_b), (0.5_a, 0.3_b))$ be an IFS in $X$.

Then, $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ 0 \leq \mu_a + \nu_a \leq 1$ and $0 \leq \mu_b + \nu_b \leq 1\}$. Now $A \subseteq U$. As $\beta cl(A) = 1\sim \subseteq 1\sim$, we have $A$ is an IF $\beta$ GCS in $X$, but not an IFCS since $cl(cl(A)) = cl(G) = 1\sim \not\subseteq A$.

Remark 3.13: Every IF $\beta$ CS and every IF $\beta$ GCS are independent to each other.

Example 3.14: Let $X = \{a, b\}$ and $G_1 = (x, (0.5_a, 0.5_b), (0.5_a, 0.5_b))$ and $G_2 = (x, (0.3_a, 0.1_b), (0.7_a, 0.8_b))$. Then $\tau = \{\{0\sim, G_1, G_2, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))$ be an IFS in $X$. Then $A \subseteq G_1$ and $cl(A) = G_1 \subseteq G_1$, Therefore $A$ is an IF $\beta$ FCS.

Now $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ \mu_a \geq 0.5 \text{ and } \mu_b \geq 0.5 \text{ or } \mu_a < 0.3 \text{ and } \mu_b < 0.1, \nu_a + \nu_b \leq 1$ and $0 \leq \mu_a + \nu_a \leq 1\}$. Since $A \subseteq G_1$ where $G_1$ is an IF $\beta$ GCS in $X$ but $\beta cl(A) = (x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)) \not\subseteq A$, $A$ is not an IF $\beta$ GCS.

Example 3.15: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b))$. Then $\tau = \{\{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))$ be an IFS in $X$.

Then, $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ 0 \leq \mu_a + \nu_a \leq 1$ and $0 \leq \mu_b + \nu_b \leq 1\}$. We have $A \subseteq G$. As $\beta cl(A) = A, \beta cl(A) \subseteq G$, where $G$ is an IF $\beta$ OS in $X$. This implies that $A$ is an IF $\beta$ GCS in $X$, but not an IF $\beta$ CS, since $cl(cl(A)) = cl(G) = G \not\subseteq A$.

Theorem 3.16: Every IF $\beta$ CS in $(X, \tau)$ is an IF $\beta$ GCS in $(X, \tau)$ but not conversely.

Proof: Assume $A$ is an IF $\beta$ CS then $\beta cl(A) = A$. Let $A \subseteq U$ and $U$ be an IF $\beta$ OS. Then $\beta cl(A) \subseteq U$, by hypothesis. Therefore $A$ is an IF $\beta$ GCS.

Example 3.17: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.7_b), (0.5_a, 0.3_b))$, then $\tau = \{\{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.5_a, 0.8_b), (0.5_a, 0.2_b), (0.5_a, 0.7_b))$ be an IFS in $X$.

Then, $\beta C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1]/ \mu_a < 0.5 \text{ whenever } \mu_a \geq 0.5, \mu_b \geq 0.5 \text{ whenever } \mu_a < 0.5 \text{ whenever } \mu_b \geq 0.5 \text{ and } \mu_b \geq 0.5 \text{ or } \mu_a < 0.3 \text{ and } \mu_b < 0.1, \nu_a + \nu_b \leq 1$ and $0 \leq \mu_a + \nu_a \leq 1\}$. Now $A \subseteq U$ and $\beta cl(A) = 1\sim \subseteq 1\sim$. This implies that $A$ is an IF $\beta$ GCS in $X$, but not an IF $\beta$ CS since $cl(cl(A)) = cl(cl(G)) = int(1\sim) = 1\sim \not\subseteq A$.

Theorem 3.18: Every IF $\beta$ PCS in $(X, \tau)$ is an IF $\beta$ GCS in $(X, \tau)$ but not conversely.

Proof: Assume $A$ is an IF $\beta$ PCS[11]. Since every IF $\beta$ PCS is an IF $\beta$ CS[7], by theorem 3.16, $A$ is an IF $\beta$ GCS.
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**Example 3.19:** Let $X = \{a, b\}$ and $G = (x, (0.5_\alpha, 0.4_\beta), (0.5_\alpha, 0.6_\beta))$. Then $\tau = \{0\sim, G, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_\alpha, 0.5_\beta) (0.6_\alpha, 0.7_\beta))$ be an IFS in $X$.

Then, $\beta \mathcal{C}(X) = \{0\sim, 1\sim, \mu_\alpha \varepsilon [0,1], \mu_\beta \varepsilon [0,1], \nu_\alpha \varepsilon [0,1], \nu_\beta \varepsilon [0,1] / 0 \leq \mu_\alpha + \nu_\alpha \leq 1 \text{ and } 0 \leq \mu_\beta + \nu_\beta \leq 1\}$. Here $A$ is an IFS in $X$. As $\text{int}(\text{cl}(A)) = 0\sim \subseteq A$. Therefore $A$ is an IFS in $X$.

Since $\beta \mathcal{C}(X) = \{0\sim, 1\sim, \mu_\alpha \varepsilon [0,1], \mu_\beta \varepsilon [0,1], \nu_\alpha \varepsilon [0,1], \nu_\beta \varepsilon [0,1] / \text{either } \mu_\alpha \geq 0.6 \text{ or } \mu_\beta < 0.4$ whenever $\mu_\alpha, \mu_\beta \varepsilon [0.5, 0.6] \leq \mu_\alpha + \nu_\alpha \leq 1$ and $0 \leq \mu_\beta + \nu_\beta \leq 1\}$.

But $A$ is not an IFPCS in $X$, as we cannot find any IFPCS $B$ such that $\text{int}(B) \subseteq A \subseteq B$ in $X$.

In the following diagram, we have provided relations between various types of intuitionistic fuzzy closedness.

The reverse implications are not true in general in the above diagram.

**Remark 3.20:** The union of any two IFS in GCS is not an IFS in GCS in general as seen from the following example.

**Example 3.21:** Let $X = \{a, b\}$ and $\tau = \{0\sim, G_1, G_2, 1\sim\}$ where $G_1 = (x, (0.7_\alpha, 0.8_\beta), (0.3_\alpha, 0.2_\beta))$ and $G_2 = (x, (0.6_\alpha, 0.7_\beta), (0.4_\alpha, 0.3_\beta))$. Then the IFSS $A = (x, (0.6_\alpha, 0.5_\beta), (0.4_\alpha, 0.3_\beta))$ and $B = (x, (0.4_\alpha, 0.8_\beta), (0.4_\alpha, 0.2_\beta))$ are IFS in $GCS(X)$ but $A \cup B$ is not an IFS in $GCS(X)$.

Then $\beta \mathcal{C}(X) = \{0\sim, 1\sim, \mu_\alpha \varepsilon [0,1], \mu_\beta \varepsilon [0,1], \nu_\alpha \varepsilon [0,1], \nu_\beta \varepsilon [0,1] / \text{provided } \mu_\beta < 0.7 \text{ whenever } \mu_\alpha \geq 0.6$, $\mu_\alpha < 0.6 \text{ whenever } \mu_\alpha \geq 0.7$, $0 \leq \mu_\alpha + \nu_\alpha \leq 1$ and $0 \leq \mu_\beta + \nu_\beta \leq 1\}$.

As $\beta \mathcal{C}(A) = A$, we have $A$ is an IFPCS in $X$ and $\beta \mathcal{C}(B) = B$, we have $B$ is an IFS in $X$. Now $A \cup B = (x, (0.6_\alpha, 0.8_\beta), (0.4_\alpha, 0.2_\beta)) \subseteq G_1$ where $G_1$ is an IFS in $GCS(X)$, but $\beta \mathcal{C}(A \cup B) = 1\sim \not\subseteq G_1$.

**Theorem 3.22:** Let $(X, \tau)$ be an IFTS. Then for every $A \in \beta \mathcal{C}(X)$ and for every $B \in \beta \mathcal{C}(X)$, $A \subseteq \beta \mathcal{C}(A) \Rightarrow B \in \beta \mathcal{C}(X)$.

**Proof:** Let $B \subseteq U$ and $U$ be an IFS. Then, $A \subseteq B$, $A \subseteq U$. By hypothesis, $B \subseteq \beta \mathcal{C}(A)$. Therefore $\beta \mathcal{C}(B) \subseteq \beta \mathcal{C}(\beta \mathcal{C}(A)) = \beta \mathcal{C}(A) \subseteq U$, since $A$ is an IFS in $X$. Hence $B \in \beta \mathcal{C}(X)$.

**Theorem 3.23:** An IFS $A$ of an IFTS $(X, \tau)$ is an IFS if and only if $A \subseteq F$ if for every $\beta \mathcal{C}(X) F$ for every $\beta \mathcal{C}(X) F$. $X$.

**Proof:** (Necessity): Let $F$ be an IFS and $A \subseteq F$ [9], then $F$ is an IFS. Therefore $\beta \mathcal{C}(A) \subseteq F$, by hypothesis. Hence again [9], $\beta \mathcal{C}(A) \subseteq F$.

**Sufficiency:** Let $U$ be an IFS such that $A \subseteq U$. Then $U$ is an IFS and $A \subseteq (U F)$. By hypothesis, $A \subseteq U \Rightarrow \beta \mathcal{C}(A) \subseteq (U F)$. Hence by [9], $\beta \mathcal{C}(A) \subseteq (U F)$. Hence $A$ is an IFS.

**Theorem 3.24:** Let $(X, \tau)$ be an IFTS. Then every IFS in $(X, \tau)$ is an IFS if and only if $\beta \mathcal{C}(X) = \beta \mathcal{C}(X)$. $X$. $X$.

**Proof:** (Necessity): Suppose that every IFS in $(X, \tau)$ is an IFS. Let $U \in \beta \mathcal{C}(X)$, and by hypothesis, $\beta \mathcal{C}(U) \subseteq U \subseteq \beta \mathcal{C}(U)$. This implies $\beta \mathcal{C}(U) = U$. Therefore $U \in \beta \mathcal{C}(X)$. Hence $\beta \mathcal{C}(X) \subseteq \beta \mathcal{C}(X)$. Let $A \subseteq \beta \mathcal{C}(X)$, then $A \subseteq \beta \mathcal{C}(X) \subseteq \beta \mathcal{C}(X)$. That is, $A \subseteq \beta \mathcal{C}(X)$. Therefore $\beta \mathcal{C}(X) \subseteq \beta \mathcal{C}(X)$. Thus $\beta \mathcal{C}(X) = \beta \mathcal{C}(X)$.

**Sufficiency:** Suppose that $\beta \mathcal{C}(X) = \beta \mathcal{C}(X)$. Let $A \subseteq U$ and $U$ be an IFS. By hypothesis $\beta \mathcal{C}(A) \subseteq \beta \mathcal{C}(U) = U$, since $U \in \beta \mathcal{C}(X)$. Therefore $A$ is an IFS in $X$. $X$.
**Theorem 3.25:** If $A$ is an IFβOS and an IFβGCS in $(X, \tau)$ then $A$ is an IFβCS in $(X, \tau)$.

**Proof:** Since $A \subseteq A$ and $A$ is an IFβOS, by hypothesis, $\beta \text{cl}(A) \subseteq A$. But $A \subseteq \text{cl}(A)$. Therefore $\beta \text{cl}(A) = A$. Hence $A$ is an IFβCS.

**Theorem 3.26:** Let $A$ be an IFβGCS in $(X, \tau)$ and $p_{\alpha, \beta}$ be an IFP in $X$ such that $\text{int}(p_{\alpha, \beta}) \beta \text{cl}(A)$, then $\text{int}(\text{cl}(p_{\alpha, \beta}))$ $\subseteq A$.

**Proof:** Let $A$ be an IFβGCS and let $\text{int}(p_{\alpha, \beta}) \beta \text{cl}(A)$. Suppose $\text{int}(\text{cl}(p_{\alpha, \beta}))$ $\subseteq A$. Since by [9] $A \subseteq [\text{int}(\text{cl}(p_{\alpha, \beta}))]$. This implies $[\text{int}(\text{cl}(p_{\alpha, \beta}))]$ is an IFβOS. Then by hypothesis, $\beta \text{cl}(A) \subseteq [\text{int}(\text{cl}(p_{\alpha, \beta}))]$. This implies $\text{cl}(\text{int}(\text{cl}(p_{\alpha, \beta})))$ $\subseteq A$. Hence $\text{cl}(\text{int}(\text{cl}(p_{\alpha, \beta})))$ $\subseteq A$. Therefore $A$ is an IFROS.

**Theorem 3.27:** Let $F \subseteq A \subseteq X$ where $A$ is an IFβOS and an IFβGCS in $X$. Then $F$ is an IFβGCS in $A$ if and only if $F$ is an IFβGCS in $X$.

**Proof:** Necessity: Let $U$ be an IFβOS in $X$ and $F \subseteq U$. Also let $F$ be an IFβGCS in $A$. Then clearly $F \subseteq A \cap U$ and $A \cap U$ is an IFβOS in $A$. Hence the $\beta$ closure of $F$ in $A$, $\beta \text{cl}(F) \subseteq A \cap U$. By theorem 3.25, $A$ is an IFβCS. Therefore $\beta \text{cl}(A) = A$ and $\beta$ closure of $F$ in $X$, $\beta \text{cl}(F) \subseteq \beta \text{cl}(F) \cap \beta \text{cl}(A) = \beta \text{cl}(F) \cap A = \beta \text{cl}(F) \subseteq A \cap U \subseteq U$. That is, $\beta \text{cl}(F) \subseteq U$ whenever $F \subseteq U$. Hence $F$ is an IFβGCS in $X$.

Sufficiency: Let $V$ be an IFβOS in $A$ such that $F \subseteq V$. Since $A$ is an IFβOS in $X$, $V$ is an IFβOS in $X$. Therefore $\beta \text{cl}(F) \subseteq V$, since $F$ is an IFβGCS in $X$. Thus $\beta \text{cl}(F) = \beta \text{cl}(F) \cap A \subseteq V \cap A \subseteq V$. Hence $F$ is an IFβGCS in $A$.

**Theorem 3.28:** For an IFS $A$, the following conditions are equivalent:

(i) $A$ is an IFOS and an IFβGCS

(ii) $A$ is an IFRO

**Proof:** (i) $\Rightarrow$ (ii) Let $A$ be an IFOS and an IFβGCS. Then $\beta \text{cl}(A) \subseteq A$ and $A \subseteq \beta \text{cl}(A)$ this implies that $\beta \text{cl}(A) = A$. Therefore $A$ is an IFβCS, since $\text{int}(\text{cl}(A)) \subseteq A$. Since $A$ is an IFOS, $\text{int}(A) = A$. Therefore $\text{int}(\text{cl}(A)) \subseteq A$. Since $A$ is an IFOS, it is an IFPOS. Hence $A \subseteq \text{int}(\text{cl}(A))$. Therefore $A = \text{int}(\text{cl}(A))$. Hence $A$ is an IFRO.

(ii) $\Rightarrow$ (i) Let $A$ be an IFRO. Therefore $A = \text{int}(\text{cl}(A))$. Since every IFRO in an IFOS and $A \subseteq A$. This implies $\text{int}(\text{cl}(A)) \subseteq A$. Therefore $A$ is an IFβCS. Hence $A$ is an IFβGCS.

**Theorem 3.29:** For an IFOS $A$ in $(X, \tau)$, the following conditions are equivalent.

(i) $A$ is an IFCS

(ii) $A$ is an IFβGCS and an IFQ-set

**Proof:** (i) $\Rightarrow$ (ii) Since $A$ is an IFCS, it is an IFβGCS. Now $\text{int}(\text{cl}(A)) = \text{int}(A) = A = \text{cl}(A) = \text{cl}(\text{int}(A))$, by hypothesis. Hence $A$ is an IFQ-set[8].

(ii) $\Rightarrow$ (i) Since $A$ is an IFOS and an IFβGCS, by theorem 3.28, $A$ is an IFRO. Therefore $A = \text{int}(\text{cl}(A)) = \text{cl}(\text{int}(A)) = \text{cl}(A)$, by hypothesis. $A$ is an IFCS.

**Theorem 3.30:** Let $(X, \tau)$ be an IFTS, then for every $A \in \text{IFSPC}(X)$ and/or every $B$ in $X$, $\text{int}(A) \subseteq B \subseteq A \Rightarrow B \in \text{IFβGC}(X)$.

**Proof:** Let $A$ be an IFSPCS in $X$. Then there exists an IFPSC, (say) $C$ such that $\text{int}(C) \subseteq A \subseteq C$. By hypothesis, $B \subseteq A$. Therefore $B \subseteq C$. Since $\text{int}(C) \subseteq A$, $\text{int}(C) \subseteq \text{int}(A)$ and $\text{int}(C) \subseteq B$, by hypothesis. Thus $\text{int}(C) \subseteq B \subseteq C$. Thus $B \in \text{IFβGC}(X)$.
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