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Some Oscillation Properties of Third Order Linear Neutral Delay Difference Equations

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ABSTRACT
*In this paper, we establish **some suf**ficient conditions for the oscillation of solutions of third order linear neutral delay difference equations of the form*
$\Delta\left(a(n)\Delta^2(x(n)+p(n)x(\tau(n)))\right)+q(n)x(\sigma(n))=0$.

I. INTRODUCTION

In this paper, we consider the third order linear neutral delay difference equations from

$$\Delta\left(a(n)\right)\Delta^2\left(x(n)+p(n)x(\tau(n))\right)\big)+q(n)x(\sigma(n))=0 \qquad (1)$$

where $n\in N(n_0)=\{n_0,n_0+1..,\},n_0$ is a nonnegative integer, subject to the following conditions

$(H_1)\, a(n),p(n),q(n)$ are positive sequences.

$0\le p(n)\le p\le 1,\tau(n)\le n,\sigma(n)\le n,\lim_{n\to 0}\tau(n)=\lim_{n\to 0}\sigma(n)=\infty$ and $R(n)=\sum_{s=n_0}^{n-1}\frac{1}{a(s)}\to\infty$ as $n\to\infty$

$(H_2)\sum_{n=n_0}^{\infty}\sum_{s=n_0}^{\infty}\left(\frac{1}{a(s)}\sum_{t=n}^{\infty}q(t)\right)=\infty$.

$(H_3)\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}q(s)(1-p(\sigma(s)))\left(\frac{KM(\sigma(s))^2}{2}\right)-\frac{a^2(s+1)}{4sa(s)}=\infty$.

We set $z(n)=x(n)+p(n)x(\tau(n))$.

The oscillation theory of difference equations and their applications have received more attention in the last few decades, see [[1]-[4]], and the references cited therein. Especially the study of oscillatory behavior of second order equations of various types occupied a great deal of interest. However the study of third order difference equations have received considerably less attention even though such equations have wide applications. In [[5]-[10]] the authors investigated the oscillatory properties of solutions of third order delay difference equations and in [[11]-[15]]. Motivated by the above observations, in this paper, we investigate the oscillatory behavior of solutions of equation (1).

Let $\theta=\max\left\{\lim_{\delta x\to 0}\sigma(n),\tau(n)\right\}$. By a solution of equation (1) we mean a real sequence *x(n)* which is defined for all $n\ge n_0-\theta$ satisfying (1) for all $n\ge n_0$. A non-trivial solution *x(n)* is said to be oscillatory if it is neither eventually positive or eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

II. MAIN RESULTS

Lemma 2.1. Let *x(n)* be a positive solution of equation (1) for all $n\ge n_0$ such $x(n)>0,\Delta x(n)\ge 0,$ and $\Delta^2x(n)\le 0$ on $[n_1,\infty)$ for some $n_1\ge n_0$. Then for each k with 0 < k < 1, there exists $n_2\ge n_1$ such that

$$\frac{x(n-\sigma)}{x(n)}\ge k\frac{n-\sigma}{n},n\ge n_2. \qquad (2)$$

Proof. From the Lagrange's Mean value theorem, we have for $n\ge n_1$, for some k

$$\Delta x(k)=\frac{x(n)-x(\sigma(n))}{n-\sigma(n)} \text{; for some } k \qquad (3)$$

such that $\sigma(n)<k<n.\Delta^2 x(n)\le 0$ and $\Delta x(n)$ is non-increasing, which implies that $\Delta x(k)<\Delta x(\sigma(n))$ and hence, using equation (3)

$$x(n)\le x(\sigma(n))+\Delta x(\sigma(n))(n-\sigma(n))$$

$$\frac{x(n)}{x(\sigma(n))}\le 1+\frac{\Delta x(\sigma(n))}{x(\sigma(n))}(n-\sigma(n)) \qquad (4)$$

Apply Lagrange's Mean value theorem once again for *x(n)* on $[n_1,\sigma(n)]$ for $n\ge n_1+\sigma(n)$. Now

$\Delta x(c)=\dfrac{x(\sigma(n))-x(n_1)}{\sigma(n)-n_1}$ for some c such that $n_1<c<\sigma(n)$ and $\Delta x(c)>\Delta x(\sigma(n))$ which implies $x(\sigma(n))\ge \Delta x(\sigma(n))(\sigma(n)-n_1)$. Hence

$$\frac{x(\sigma(n))}{\Delta x(\sigma(n))}\ge \sigma(n)-n_1$$

For $K\in(0,1),$ we can find $n_2\ge n_1+\sigma$

$$\frac{x(\sigma(n))}{\Delta x(\sigma(n))}\ge K\sigma(n) \text{ for } n\ge n_2 \qquad (5)$$

From equation (4) and for all $n\ge n_2$, we have

$$\begin{aligned}\frac{x(n)}{x(\sigma(n))}&\le 1+\frac{1}{K\sigma(n)}(n-\sigma(n))\\ &\le 1+\frac{n}{K\sigma(n)}-\frac{\sigma(n)}{K\sigma(n)}\\ &\le \frac{n}{K\left(\sigma(n)\right)}\end{aligned}$$

Hence,

$$\frac{x(\sigma(n))}{x(n)}\ge \frac{K\left(\sigma(n)\right)}{n} \qquad (6)$$

Lemma 2.2 Let *x(n)* be a positive solution of equation (1), then the corresponding sequence *z(n)* satisfies the following condition *z(n)* > 0, $\Delta z\left(n\right)>0,\ \Delta^2 z\left(n\right)>0,\ \Delta^3 z\left(n\right)>0$ for some $n_1\ge n_0$. Then there exits $n_2\ge n_1$ such that $\dfrac{z(n)}{\Delta z(n)}\ge \dfrac{Mn}{2},\quad n\ge n_2$ for each M, 0 < M < 1.

Proof. We define a function H(n) for $n\ge n_2\ge n_1$, as

$$H(n)=(n-n_2)z(n)-\frac{M(n-n_2)^2}{2}\Delta z(n) \qquad (7)$$

$$\Delta H(n)\ge z(n)+(n-n_2)\Delta z(n)-\frac{M(n-n_2)^2}{2}\Delta^2 z(n) \qquad (8)$$

By Taylor's Theorem,

$$z(n)\ge z(n_2)+(n-n_2)\Delta z(n_2)+\frac{(n-n_2)^2}{2}\Delta^2 z(n)$$

From (8)

$$\Delta H(n)\ge z(n_2)+(n-n_2)\Delta z(n_2)+\frac{(n-n_2)^2}{2}\Delta^2 z(n)+(n-n_2)\Delta z(n)-\frac{M(n-n_2)^2}{2}\Delta^2 z(n)$$

which implies $\Delta H(n)>0$ and $H(n+1)>H(n)>H(n_2)=0$ for every $n\ge n_2$ from (7)

$$(n-n_2)z(n)-\frac{M(n-n_2)^2}{2}\Delta z(n)>0$$

which implies $\frac{z(n)}{\Delta z(n)}\geq\frac{Mn}{2}$ for $n\geq n_2$.

Theorem 2.3. Assume that (H_1) to (H_2) hold, then equation (1) is oscillatory.

Proof: Suppose, if possible that the equation (1) has a nonoscillatory solution. Without loss of generality suppose that x(n) is a positive solution of equation (1). We shall discuss the following cases for *z(n)*.

(i) $z(n)>0,\ \Delta z(n)<0,\ \Delta^2 z(n)>0,\ \Delta^3 z(n)\leq 0,$

(ii) $z(n)>0,\ \Delta z(n)>0,\ \Delta^2 z(n)>0,\ \Delta^3 z(n)\leq 0,$

Case 1. $z(n)>0,\ \Delta z(n)<0,\ \Delta^2 z(n)>0,\ \Delta^3 z(n)\leq 0,$

Since z(n) > 0and $\Delta z(n)<0,$ then there exists finite limits $\lim_{n\to\infty} z(n)=k$. We shall prove that k = 0. Assume that k > 0. Then for any $\in>0$, we have $k+\in>z(n)>k$. Let $0<\in<\frac{k(1-p)}{p}$, we have $k+\in>x(n)>k-p(n)x(\tau(n))$.

$$x(n)>k-p(k+\in)=m(k+\in)$$
$$x(n)>mz(n)$$

When $m=\frac{k-p(k+\in)}{(k+\in)}$. Now from the equation (1) we have

$$\Delta\Big(a(n)\Delta^2\big(\big(x(n)+p(n)x(\tau(n))\big)\big)\Big)=-q(n)x(\sigma(n))-\Delta\Big(a(n)\Delta^2 z(n)\Big)\geq q(n)mz(\sigma(s))$$

Summing the above inequality from n to ∞ we get,

$$-\sum_{t=n}^{\infty}\Delta\Big(a(t)\Delta^2 z(t)\Big)\geq m\sum_{s=n}^{\infty}q(s)z(\sigma(s))$$

$$a(n)\Delta^2 z(n)\geq m\sum_{s=n}^{\infty}q(s)z(\sigma(s))$$

Using the fact that $z(\sigma(n))\geq k$ we obtain, $a(n)\Delta^2 z(n)\geq mk\sum_{s=n}^{\infty}q(s)$ which implies $\Delta^2(z(n))\geq mk\left(\frac{1}{a(n)}\sum_{s=n}^{\infty}q(s)\right)$. Summing from n to ∞ we have,

$$\sum_{s=n}^{\infty}\Delta^2 z(s)\geq mk\sum_{s=n}^{\infty}\left(\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)\right)$$

$$-\Delta z(n)\geq mk\sum_{s=n}^{\infty}\left(\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)\right)$$

Summing the last inequality n_1 to ∞

$$z(n_1)\geq mk\sum_{n=n_1}^{\infty}\sum_{s=n}^{\infty}\left(\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)\right)$$

This contradicts (H_2). Thus k = 0. Moreover, the inequality, $0\leq x(n)\leq z(n)$ implies $\lim_{n\to\infty}x(n)=0$.

Case 2. $z(n)>0,\ \Delta z(n)>0,\ \Delta^2 z(n)>0,\ \Delta^3 z(n)\leq 0,$

We have x(n) = z(n) - p(n) $x(\tau(n))$, we obtain further

$$\begin{aligned}x(\sigma(n))&=z(\sigma(n))-p(\sigma(n))x(\sigma(n)-\tau)\\&\geq z(\sigma(n))-p(\sigma(n))x(\sigma(n))\end{aligned}$$

$$\geq (1-p(\sigma(n)))z(\sigma(n)).$$

From equation (1) we have,

$$\Delta\left(a(n)\Delta^2 z(n)\right) \leq -q(n)x(\sigma(n))$$

$$\Delta\left(a(n)\Delta^2 z(n)\right) \leq -q(n)(1-p(\sigma(n)))z(\sigma(n))$$

$$w(n) = n\frac{a(n)\Delta^2 z(n)}{\Delta z(n)}, n \geq n_1 \qquad (9)$$

$$\Delta w(n) = \left(\frac{a(n+1)\Delta^2 z(n+1)}{\Delta z(n+1)}\right) + n\left(\Delta\left(\frac{a(n)\Delta^2 z(n)}{\Delta z(n)}\right)\right)$$

$$= \frac{w(n+1)}{n+1} + n\left(\frac{\Delta(a(n)\Delta^2 z(n))}{\Delta z(n+1)} - \frac{a(n)\Delta^2 z(n)\Delta^2 z(n)}{\Delta z(n)\Delta z(n+1)}\right)$$

$$\leq \frac{w(n+1)}{n+1} + n\left(\frac{\Delta(a(n)\Delta^2 z(n))}{\Delta z(n)} - a(n)\frac{(\Delta^2 z(n+1))^2}{(\Delta z(n+1))^2}\right)$$

$$\leq \frac{w(n+1)}{n+1} - \frac{nq(n)(1-p(\sigma(n)))z(\sigma(n))}{\Delta z(n)} - \frac{na(n)}{(n+1)^2 a^2(n+1)} w^2(n+1) \qquad (10)$$

Also from Lemma (2.1) with x(n) = $\Delta z(n)$

$$\frac{x(\sigma(n))}{x(n)} \geq \frac{K\sigma(n)}{n}, \sigma(n) \geq n$$

$$\frac{\Delta z(\sigma(n))}{\Delta z(n)} \geq \frac{K\sigma(n)}{n} \qquad (11)$$

$$\frac{1}{\Delta z(n)} \geq \frac{K\sigma(n)}{n}\frac{1}{\Delta z(\sigma(n))} \text{ for } \sigma(n) \geq n_1 \geq n_2.$$

By Lemma (2.2)

$$\frac{z(\sigma(n))}{\Delta z(n)} \geq \frac{K\sigma(n)}{n}\frac{z(\sigma(n))}{\Delta z(\sigma(n))}$$

$$\geq \frac{K\sigma(n)}{n}\frac{M\sigma(n)}{2} \qquad (12)$$

$$\frac{z(\sigma(n))}{\Delta z(n)} \geq \frac{KM}{2}\frac{(\sigma(n))^2}{n}$$

Using (11) and (12) in (10)

$$\Delta w(n) \leq -q(n)(1-p(\sigma(n)))\left(\frac{KM\sigma^2(n)}{2}\right) + \frac{w(n+1)}{n+1} - \frac{na(n)}{(n+1)^2 a^2(n+1)} w^2(n+1) \qquad (13)$$

Using the inequality

$$Vx - Ux^2 \leq \frac{1}{4}\frac{V^2}{U}, U > 0$$

And put $x = w(n+1), V = \dfrac{1}{n+1}, U = \dfrac{na(n)}{(n+1)^2 a^2(n+1)}$, we have

$$\frac{w(n+1)}{(n+1)} - \frac{na(n)}{(n+1)^2 a^2(n+1)} w^2(n+1) \leq \frac{a^2(n+1)}{4na(n)} \qquad (14)$$

From equation (13)

$$\Delta w(n) \le -nq(n)(1-p(\sigma(n)))\left(\frac{KM}{2}\frac{(\sigma(n))^2}{n}\right)+\frac{a^2(n+1)}{4na(n)} \tag{15}$$

Summing the last inequality from n_2 to $n-1$ we obtain

$$\sum_{n=n_2}^{n-1} q(s)(1-p(\sigma(s)))\left(\frac{KM(\sigma(s))^2}{2}\right)-\frac{a^2(s+1)}{4sa(s)} \le w(n_2)$$

Taking lim sup in the above inequality, we obtain contradiction with H_3.

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