Ultraspherical Solutions for Neutral Functional Differential Equations

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ABSTRACT
This paper is concerned with the numerical solution of neutral functional differential equations (NFDEs). Based on the ultraspherical ν -stage continuous implicit Runge-Kutta method is proposed. The description and outlines algorithm of the method are introduced. Numerical results are included to confirm the efficiency and accuracy of the method.

Keywords: *Functional differential equations, Equations of neutral type, Implicit delay equations.*

I. INTRODUCTION

We are interested in the numerical solution of initial-value problem for neutral functional differential equations (NFDEs), which take the form:

$$\begin{aligned} y'(t) &= f(t, y(.), y'(.)) \qquad t_0 < t \le t_N, \\ y(t) &= g(t) \qquad \alpha \le t \le t_0 \end{aligned} \tag{1}$$

where $g(t) \in C^1[\alpha, t_0]$ and the function $f(t, y(.), y'(.))$ satisfies the following conditions:-

H_1: For any $y \in C^1[t_0, t_N]$ the mapping $t \to f(t, y(.), y'(.))$ is continuous on $[t_0, t_N]$.

H_2: There exist constants $L_1 \ge 0,\ 0 \le L_2 \le 1$ such that

$$\left\| f(t, y_1(.), z_1(.)) - f(t, y_2(.), z_2(.)) \right\| \le \Lambda$$

$$\Lambda = L_1 \left\| y_1 - y_2 \right\|_{C^1[\alpha, t_N]} + L_2 \left\| z_1 - z_2 \right\|_{C^0[\alpha, t_N]}$$

for any $t \in [t_0, t_N]$, $y_1, y_2 \in C^1[\alpha, t_N]$ and $z_1, z_2 \in C^0[\alpha, t_N]$.

Under the conditions H_1 and H_2 the problem (1) has a unique solution $y(x)$ [1]. The equations of type (1) have applications in many fields such as control theory, oscillation theory, electrodynamics, biomathematics, and medical science. Numerical methods for the problem (1) were discussed extensively by many authors; see [2-26].

This paper is concerned with the numerical solution of neutral functional differential equations (NFDEs). Based on the ultraspherical ν -stage continuous implicit Runge-Kutta method is proposed. In section 2, we will adapt a finite Ultraspherical expansion to approximate $\int_{t_i}^{t} f(s)\, ds$ and $f(.)$ on the interval I_i , $i = 0(1)N-1$. Also, an easily implemented numerical method for NFDEs will be derived. Finally, in section 3 we present some numerical examples; which show that the presented method provides a noticeable improvement in the efficiency over some previously suggested methods.

II. THE NUMERICAL METHOD

2. 1 The Description of the method

Let $\Delta := (t_0 < t_1 < \cdots < t_N)$ define a partition for $[t_0, t_N]$, with the step size $h_i = t_{i+1} - t_i$.

Each subinterval $I_i = [t_i, t_{i+1}]$ is divided by the Chebyshev collocation points:

$$t_{ij} = t_i + h_i \xi_j \ , \xi_j = \frac{1}{2}(1 - \cos\ (\frac{j\pi}{\nu})\ , \quad j = 0(1)\nu \tag{2}$$

This method is based on a finite Ultraspherical expansion in each subinterval I_i , $i = 0(1)N-1$. Consider the approximation $\tilde{f}(t)$ of $f(t)$ for $t = t_i + \theta\, h_i$, $0 \le \theta \le 1$, $i = 0(1)N-1$ as follows:

$$\tilde{f}(t) = \sum_{r=0}^{\nu} a_r\, C_r^{[\alpha]} \left(\frac{2\,(t - t_i)}{h_i} - 1 \right) , \ t \in [t_i , t_{i+1}] \tag{3}$$

where

$$a_r = \frac{\pi}{\nu\ \lambda_r^{[\alpha]}} \sum_{l=0}^{\nu}{}^{//}\ \tilde{f}\ (t_{il}) \left(\sin\ (\frac{l\pi}{\nu}) \right)^{2\alpha} C_r^{[\alpha]} \left(-\cos\ (\frac{l\pi}{\nu}) \right)$$

$$t_{il} = t_i + \frac{h_i}{2}(1 - \cos\ (\frac{l\pi}{\nu})) \tag{4}$$

Here $C_r^{[\alpha]}(x)$ is the r-th Ultraspherical polynomial. As especial cases, at $\alpha = 0$ give Chebyshev Polynomials of the first kind $C_r^{[0]}(x) = T_r(x)$, at $\alpha = \frac{1}{2}$ give Legendre Polynomials

$C_r^{[\frac{1}{2}]}(x) = P_r(x)$, at $\alpha = 1$ give Chebyshev Polynomials of the second kind $C_r^{[1]}(x) = \dfrac{1}{1+r} R_r(x)$.
A summation symbol with double prims denotes a sum with the first and last terms halved.

Now, we can easily show that the following relations are true:

$$I_r(\theta) = \int_{-1}^{2\theta-1} C_r^{[\alpha]}(s)\, ds = \begin{cases} C_1^{[\alpha]}(2\theta-1)+1 & if \quad r=0 \\ \dfrac{(2\alpha+1)}{4(\alpha+1)}[C_2^{[\alpha]}(2\theta-1)-1] & if \quad r=1 \\ \dfrac{1}{2(\alpha+r)}\left(\dfrac{2\alpha+r}{r+1}\Phi_1 - \dfrac{r}{2\alpha+r-1}\Phi_2\right) & if \quad r\geq 2 \end{cases} \tag{5}$$

where $\Phi_1 = C_{r+1}^{[\alpha]}(2\theta-1)+(-1)^r$, $\Phi_2 = C_{r-1}^{[\alpha]}(2\theta-1)+(-1)^r$

Using (3), (4) and (5) the indefinite integral $\int_{t_i}^{t_i+\theta h_i} \tilde{f}(s)\, ds$ takes the form

$$\int_{t_i}^{t_i+\theta h_i} \tilde{f}(s)\, ds = \frac{h_i}{2}\sum_{r=0}^{\nu} b_l^{[\alpha]}(\theta)\, \tilde{f}(t_{il}) \tag{6}$$

where

$$b_l^{[\alpha]}(\theta) = \frac{\pi\left[\sin\left(\frac{l\pi}{\nu}\right)\right]^{2\alpha}}{2(1+\delta_{l0}+\delta_{l\nu})}\sum_{l=0}^{\nu}\frac{I_r(\theta)\, C_r^{[\alpha]}\left(-\cos\left(\frac{l\pi}{\nu}\right)\right)}{\lambda_r^{[\alpha]}}, \quad l = 0(1)\nu \tag{7}$$

and $\delta_{l\nu}$ is the Kronecker delta.

From the relations (2), (6) and (7) , we obtained the following results:

$$\left[\int_{t_i}^{t_{ij}} \tilde{f}(s)\, ds\right] = \frac{h_i}{2} B^{[\alpha]} F_i \tag{8}$$

$$\left[\underbrace{\int_{t_i}^{t_{ij}} \cdots \int_{t_i}^{t_{ij}}}_{k-times} \tilde{f}(s)\, ds \right] = \left(\frac{h_i}{2} \right)^k \left(B^{[\alpha]} \right)^k F_i = \frac{\left(\frac{h_i}{2} \right)^k}{(k-1)!} B^{[\alpha]} K F_i \qquad (9)$$

$$\int_{t_i}^{t_{ij}} \tilde{f}(s,t)\, d s = \sum_{l=0}^{i-1} \frac{h_l}{2} \sum_{s=0}^{\nu} b_{\nu s}^{[\alpha]} \tilde{f}(t, t_{ls}) + \frac{h_i}{2} \sum_{s=0}^{\nu} b_{js}^{[\alpha]} \tilde{f}(t, t_{is}), \quad j = 0(1)\nu \qquad (10)$$

where

$$B^{[\alpha]} = [b_{js}^{[\alpha]}]_{j,s=0}^{\nu},\ b_{js}^{[\alpha]} = b_s^{[\alpha]}(c_j),\ F_i = [\tilde{f}(t_{i0}), \cdots, \tilde{f}(t_{i\nu})]^T$$

and $K = diag\ [(c_j - c_0)^{k-1}, \cdots, (c_j - c_\nu)^{k-1}]$.

We notice that the matrix B of El-Gendi's method [27], is the same matrix $B^{[\alpha]} = \left[2\ b_{js}^{[0]} \right]_{j,s=0}^{\nu}$. On the interval $(t_i, t_{i+1}]$, rewriting (1) in the form:

$$\tilde{y}(t) = \tilde{y}(t_i) \int_{t_i}^{t} \tilde{z}(s)\, d s,\ \tilde{z}(t) = f(t, \tilde{y}(.), \tilde{z}(.)) \qquad (11)$$

Suppose the approximations of $y(t)$ and $z(t)$ are given for $t \le t_i$. On $(t_i, t_{i+1}]$, we define the ν - stage method $[\nu - UM]$, as follows

$$\tilde{z}(t_{il}) = f(t_{il}, \tilde{y}(.), \tilde{z}(.)), \quad l = 1(1)\nu \qquad (12)$$

where

$$\tilde{y}(t) = \begin{cases} g(t) & if \quad t \le t_0 \\ \tilde{y}(t_{j-1}, \nu) + \frac{h_j}{2} b_0^{[\alpha]}(\theta)\, \tilde{z}(t_{j-1}, \nu) + \frac{h_j}{2} \sum_{l=1}^{\nu} b_l^{[\alpha]}(\theta)\, \tilde{z}(t_{jl}) & if \quad t = t_j + \theta h_j \in I_j \end{cases} \qquad (13)$$

$$\tilde{z}(t)=\begin{cases} g'(t) & if \quad t\le t_0 \\ \sum_{r=0}^{\nu} a_r\, C_r^{[\alpha]}\left(\frac{2(t-t_j)}{h_j}-1\right) & if \quad t=t_j+\theta h_j\in I_j \end{cases} \tag{14}$$

$$a_r=\frac{\pi}{\nu\,\lambda_r^{[\alpha]}}\sum_{l=0}^{\nu}{}'' \tilde{z}(t_{jl})\left(\sin\left(\frac{l\pi}{\nu}\right)\right)^{2\alpha} C_r^{[\alpha]}\left(-\cos\left(\frac{l\pi}{\nu}\right)\right) \tag{15}$$

2. 2 The Algorithm of ν -stage method

The described method represents a generalization of the methods given by Jackiewicz [7-8], for $\nu=1,2,3$ and $\alpha=0$. The algorithm of the ν -stage method is given below:-

STEP 1: Input $N, h_j, I_j,\ j=0(1)N-1$.

STEP 2: Put $i=0$

STEP 3: Compute $\tilde{y}(t_{ij})$ and $\tilde{z}(t_{ij})$ on $I_j,\ j=1(1)N-1$, by solving the system of ν -equation (12).

STEP 4: Store the computed values $\tilde{y}(t_{ij})$ and $\tilde{z}(t_{ij})$ on $I_j,\ j=1(1)N-1$.

STEP 5: If $i=N-1$ go to step 6, otherwise set $i=i+1$ and go to step 3.

STEP 6: Output the results $\tilde{y}(t_{ij})$ and $\tilde{z}(t_{ij})$ $,i=0(1)N-1$.

III. NUMERICAL EXAMPLES

In this section, we present the result of some computational experiments by applying our $\nu-UM$ method.

<u>Example 1</u>: (Jackiewicz [10])

$$y'(t)=e^{-y(t)}+\sin(y'(\beta(t)))-\sin\left(\frac{1}{3+\beta(t)}\right),\qquad t\in(0,10],$$
$$y(0)=\ln(3)\qquad,\qquad y'(0)=\frac{1}{3} \tag{16}$$

Here $\beta(t)=0.5\,t\,(1-\cos(2\pi t))$. The exact solution is $y(t)=\ln(3+t)$.

<u>Example 2</u>: (Jackiewicz [10])

$$y'(t) = \frac{\sin(t\ y'(t)) - \sin(e^{y(t)})}{16} + \frac{1}{t}, \qquad t \in (1,4],$$
$$y(1) = 0 \quad , \quad y'(1) = 1 \tag{17}$$

Here the exact solution is $y(t) = \ln(t)$.

	Table 1		Table 2	
	Example 1: $y(10) = 2.5649494$		**Example 2:** $y(4) = 1.386294361$	
h	$2-UM$	C_0C_1	$2-UM$	C_0C_1
2^{-1}	1.21E-4	-2.5E-4	-1.12E-4	-4.00E-1
2^{-2}	7.62E-6	-6.25E-2	-8.00E-6	-7.00E-2
2^{-3}	4.76E-7	6.25E-2	5.00E-7	-5.00E-3
2^{-4}	2.98E-8	7.42E-2	-3.13E-8	3.13E-3
2^{-5}	1.86E-9	2.93E-3	-1.95E-9	-3.13E-4
2^{-6}	1.16E-10	-7.32E-4	-1.22E-10	-3.13E-4
2^{-7}	1.02E-11	-7.93E-4	-7.63E-12	-8.79E-5
2^{-8}	-7.10E-12	-3.66E-4	-4.77E-13	-2.44E-5
2^{-9}	1.15E-11	-8.39E-5	-2.98E-14	-6.10E-6
2^{-10}	-4.88E-12	-5.72E-6	-4.66E-15	-1.53E-6

<u>Example 3</u>: (Jackiewicz [10])

$$y'(t) = e^{1-2t^2}\, y(t^2) \left[y'\left(\frac{t-1}{t+1}\right) \right]^{(1+t)}, \qquad t \in (0,1],$$
$$y(t) = e^{t}, \qquad t \in [-1, 0] \tag{18}$$

Here the exact solution is $y(t) = e^{t}$.

<u>Example 4</u>: (Jackiewicz [10])

$$y'(t) = 2\cos(2t) \left(y\left(\frac{t}{2}\right) \right)^{2\cos(t)} + \ln \left| \frac{y'\left(\frac{t}{2}\right)}{2\cos(t)} \right| - \sin(t), \quad t \in (0,1],$$
$$y(0) = 1, \qquad y'(0) = 2 \tag{19}$$

Here the exact solution is $y(t) = e^{\sin(2t)}$ **.**

	Table 3		Table 4	
	Example 3: $y(1) = 2.7182818$		**Example 4:** $y(1) = 2.4825777$	
h	$2-UM$	C_0C_1	$2-UM$	C_0C_1
2^{-1}	1.25E-3	2.75E-2	-5.94E-2	4.28E-1
2^{-2}	7.81E-5	-2.50E-3	-4.41E-3	5.44E-2
2^{-3}	4.88E-6	-2.03E-3	-3.08E-4	1.91E-2
2^{-4}	3.05E-7	-6.64E-4	-2.11E-5	5.59E-3
2^{-5}	2.86E-8	-1.86E-4	-1.42E-6	1.33E-3
2^{-6}	1.79E-9	-4.88E-5	-9.30E-8	3.13E-4
2^{-7}	7.45E-11	-1.22E-5	-5.96E-9	7.45E-5
2^{-8}	9.31E-12	-3.20E-6	-3.73E-10	1.82E-5
2^{-9}	7.28E-13	-8.01E-7	-2.33E-11	4.46E-6
2^{-10}	-2.73E-15	-2.00E-7	-1.40E-12	1.12E-6

In tables (1)-(4), we give **E** for **2-UM** method and **E** for the best method of Jackiewicz; $C_0 C_1$ method [10], where **E** denotes the global error at the end point t_N .

<u>**Example 5**</u>**: (Kappel-Kunish [15] and Jackiewicz [10])**

$$y'(t) = y(t) + y(t-1) - 0.25\, y'(t-1), \quad t \in (0,2],$$
$$y(t) = -t, \qquad y'(t) = -1, \qquad t \in [-1,0] \tag{20}$$

Here the exact solution is given by

$$y(t) = t + 0.25\, e^{t} - 0.25, \qquad t \in [0,1]$$

$$y(t) = -t + 0.25\, e^{t} + 0.5 + \frac{17 + 3t}{16} e^{(t-1)}, \qquad t \in [1, 2].$$

Here, the first derivative has a discontinuity at $t = 0$ and $t = 1$.

In example 5, we give **E** for the method **2-UM**, **5-UM**, **8-UM** and $C_0 C_1$ in Table (5), the ν **- UM** methods, for large ν , make a little improvement in the computed results, the reason is due to there exist discontinuity for the first derivative of $y(t)$ at $t = 0$ and $t = 1$. These results indicate that the ν **- UM** method is better than the one-step methods of Jackiewicz [10].

Table 5				
Example 5: $y(2) = 4.2547942$				
h	$2-UM$	$5-UM$	$8-UM$	C_0C_1
2^{-1}	1.6	2.03E-1	8.00E-2	6.53
2^{-2}	8.15E-1	1.00E-1	4.00E-2	4.22
2^{-3}	4.10E-1	4.95E-2	2.00E-2	2.35
2^{-4}	2.05E-1	2.48E-2	1.00E-2	1.22
2^{-5}	1.03E-1	1.23E-2	5.00E-3	6.14E-1
2^{-6}	5.13E-2	6.16E-3	2.50E-3	3.08E-1
2^{-7}	2.56E-2	3.08E-3	1.25E-3	1.54E-1
2^{-8}	1.28E-2	1.54E-3	6.25E-4	7.69E-2
2^{-9}	6.41E-3	7.70E-4	3.13E-4	3.84E-2
2^{-10}	3.20E-3	7.70E-4	1.56E-4	1.92E-2

Example 6: (Castleton-Grimm [4] and Jackiewicz [8])

$$y'(t) = \frac{-4\,t\,y^2(t)}{(\ln(\cos(2t)))^2+4} + \tan(2t) + \frac{\tan^{-1}(z(t))}{2}, \quad t \in (0, 0.75], \qquad (21)$$
$$y(0)=0, \qquad y'(0)=0$$

where $z(t) = y'\left(\dfrac{t\,y^2}{1+y^2(t)}\right)$. The theoretical solution is $y(t) = -0.5\ln(\cos(2t))$.

Example 7: (Castleton-Grimm [4] and Jackiewicz [8])

$$y'(t) = (1+u(t))\cos(t) + y(t)\,z(t) - \sin(t + t\sin^2(t)), \quad t \in (0,1], \qquad (22)$$
$$y(0)=0, \qquad y'(0)=1$$

where $u(t) = y(t\,y^2(t))$, $z(t) = y'(t\,y^2(t))$.

The theoretical solution is $y(t) = \sin(t)$.

Example 8: (Pouzet [16] and Jackiewicz [8])

$$y'(t) = e^{-t^2}(1+t^2) + t - y(t) + \int_0^t t^2 e^{-st}\, y(s)\, y'(s)\, ds, \quad y(0)=0, \; t \in (0,2] \qquad (23)$$

The exact solution is $y(t) = t$.

Table 6 : Example 6 (Castleton-Grimm [4] and Jackiewicz [8])					
t_n	$y(t_n)$	$y_n(2^{-4})$	$y_n(2^{-6})$	$y_n(2^{-8})$	$y_n(2^{-10})$
0.0625	0.003916465	0.003916470	0.003926723	0.003916465	0.003917107
		0.003916465	0.003916505	0.003916465	0.003916468
0.1250	0.015790526	0.015790547	0.015832887	0.015790526	0.015793178
		0.015790526	0.015790691	0.015790526	0.015790536
0.1875	0.036012506	0.036012544	0.036113197	0.036012505	0.036018812
		0.036012506	0.036012900	0.036012506	0.036012531
0.2500	0.065292120	0.065292132	0.065486194	0.065292114	0.065304276
		0.065292120	0.065292880	0.065292120	0.065292168
0.3125	0.104766351	0.104766181	0.105105260	0.104766333	0.104787582
		0.104766350	0.104767677	0.104766351	0.104766433
0.3750	0.156199949	0.156199327	0.156765982	0.156199923	0.156235405
		0.156199949	0.156202162	0.156199949	0.156200087
0.4375	0.222365362	0.222364313	0.223301956	0.222365342	0.222423942
		0.222365362	0.222369022	0.222365362	0.222365591
0.5000	0.307813235	0.307813411	0.309395557	0.307813221	0.307912401
		0.307813235	0.307819432	0.307813235	0.307813622
0.5625	0.420618861	0.420623077	0.423464892	0.420618861	0.420797810
		0.420618860	0.420630041	0.420618861	0.420619559
0.6250	0.577079895	0.577100894	0.582906497	0.577079967	0.577448783
		0.577079896	0.577102963	0.577079895	0.577081337
0.6875	0.818538930	0.818682074	0.834410205	0.818539584	0.819566071
		0.818538933	0.818603259	0.818538930	0.818542950
0.7500	1.324391827	1.328895912	1.430890306	1.324428196	1.332263471
		1.324391983	1.324892390	1.324391828	1.324423149

Table 7 :Example 7 (Castleton-Grimm [4] and Jackiewicz [8])					
$y(t_n = 0.25) = 0.247403959$, $y(t_n = 0.50) = 0.479425539$ $y(t_n = 0.75) = 0.681638760$, $y(t_n = 1.0) = 0.841470985$					
t_n	$y_n(2^{-2})$	$y_n(2^{-4})$	$y_n(2^{-6})$	$y_n(2^{-8})$	$y_n(2^{-10})$
0.25	0.247404002	0.245934360	0.247403959	0.247306957	0.247403959
	0.247397577	0.247403959	0.247403556	0.247403959	0.247403934
0.50	0.479423742	0.475355054	0.479425535	0.479158061	0.479425539
	0.479408626	0.479425539	0.479424471	0.479425539	0.479425472
0.75	0.681635238	0.672679530	0.681638752	0.681051318	0.681638760
	0.681602967	0.681638760	0.681636491	0.681638760	0.681638618
1.00	0.841459232	0.825030283	0.841470963	0.840379343	0.841470984
	0.841404641	0.841470985	0.841466775	0.841470985	0.841470722

Table 8 :Example 8 (Pouzet [16] and Jackiewicz [8])					
t_n	$y(t_n)$	$y_n(0.1)$	$y_n(0.05)$	$y_n(0.025)$	$y_n(0.0125)$
0.1	0.100000000	0.100000000	0.100000000	0.100000000	0.100000000
		0.099999999	0.099999999	0.100000000	0.100000000
0.2	0.200000000	0.200000000	0.200000000	0.200000000	0.200000000
		0.199999855	0.199999972	0.199999993	0.199999998
0.3	0.300000000	0.300000000	0.300000000	0.300000000	0.300000000
		0.299999115	0.299999806	0.299999958	0.299999988
0.4	0.400000000	0.399999999	0.400000000	0.400000000	0.400000000
		0.399996708	0.399999241	0.399999815	0.399999954
0.5	0.500000000	0.499999998	0.500000000	0.500000000	0.500000000
		0.499990890	0.499997844	0.499999469	0.499999868
0.6	0.600000000	0.599999994	0.600000000	0.600000000	0.600000000
		0.599979283	0.599995019	0.599998768	0.599999693
0.7	0.700000000	0.699999989	0.699999999	0.700000000	0.700000000
		0.699959044	0.699990053	0.699997533	0.699999384
0.8	0.800000000	0.799999981	0.799999999	0.800000000	0.800000000
		0.799927095	0.799982170	0.799995569	0.799998894
0.9	0.900000000	0.899999953	0.899999998	0.900000000	0.900000000
		0.899880395	0.899970604	0.899992684	0.899998173
1.0	1.000000000	0.999999953	0.999999997	1.000000000	1.000000000
		0.999816214	0.999954664	0.999988706	0.999997179
1.1	1.100000000	1.099999933	1.099999996	1.100000000	1.100000000
		1.099732385	1.099933803	1.099983497	1.099995877
1.2	1.200000000	1.199999907	1.199999994	1.200000000	1.200000000
		1.199627509	1.199907665	1.199976968	1.199994245
1.3	1.300000000	1.299999876	1.299999992	1.300000000	1.300000000
		1.299501081	1.299876121	1.299969086	1.299992275
1.4	1.400000000	1.399999839	1.399999990	1.399999999	1.400000000
		1.399353516	1.399839272	1.399959876	1.399989973
1.5	1.500000000	1.499999794	1.499999987	1.499999999	1.500000000
		1.499186056	1.499797427	1.499949416	1.499987358
1.6	1.600000000	1.599999740	1.599999984	1.599999999	1.600000000
		1.599000583	1.599751058	1.599937824	1.599984460
1.7	1.700000000	1.699999675	1.699999980	1.699999999	1.700000000
		1.698799343	1.699700726	1.699925339	1.699981314
1.8	1.800000000	1.799999596	1.799999975	1.799999998	1.800000000
		1.798584660	1.799647014	1.799911809	1.799977956
1.9	1.900000000	1.899999502	1.899999969	1.899999998	1.900000000
		1.898358666	1.899590455	1.899897665	1.899974420
2.0	2.000000000	1.999999389	1.999999962	1.999999998	2.000000000
		1.998123103	1.999531488	1.999882918	1.999970733

In Tables (6)-(8), we give the exact solution $y(t_n)$ in the second column, and $y_n(h)$ which denotes the computed value $y_h(t_n)$ at every $t = t_n$, the computed value of $y_n(h)$ are represented in two lines, the first line for **2-UM** and the second line for the method of Jackiewicz [8]. The results given in Tables (6) and (7) are much better than those obtained by Castleton-Grimm [4] and also, those obtained by Jackiewicz [8]. The results given in Table (8) are much better than those obtained by Jackiewicz [8].

When solving the nonlinear equations, the computations are terminated when two successive approximations differed by less than $10\ h_3$.

IV. CONCLUSIONS

In this paper we construct a method based on the Ultraspherical approximation. This method can be applied to solve different types of NFDEs. The experimental comparison, presented in this paper, shows that this method is more efficient than the previously introduced methods. In addition, the ν **- UM** method can be easily implemented on computer compared with the Lagrange multipliers and their integrals which given by Jackiewicz [10].

References

[1] Z. Kamont, M. Kwapisz, On the Cauchy problem for differential delay equations in a Banach space, Math. Nachr, 74:
173- 190, 1976.

[2] A. Bellen, Constrained mesh methods for functional differential equations, ISNM, 74:52-70, 1970.

[3] H. Brunner,The numerical solution of the neutral Volterra integro-differential equations with delay arguments, Annals of Numerical Mathematics, 1: 309-322, 1994.

[4] R. N. Castleton, A first order method for differential equations of neutral type, Math. Comput. 27: 571-577, 1973.

[5] C. W. Cryer, Numerical methods for functional differential equations In: Delay and functional differential equations and their applications (K. Schmitt, ed.), 17-101. New York: Academic Press, 1972.

[6] U. Hornung, Euler-Verfahren fur neutral functional differential glcichungen, Numer. Math., 24:233-240, 1975.

[7] Z. Jackiewicz, One-step methods for the numerical solution of Volterra functional differential equations of neutral type, Applicable anal., 12: 1-11, 1981.

[8] Z. Jackiewicz, The numerical solution of Volterra functional differential equations of neutral type}, SIAM J. Numer. Anal.18: 615-626, 1981.

[9] Z. Jackiewicz, Adams methods for neutral functional differential equations, Numer. Math. 39: 221-230, 1982.

[10] Z. Jackiewicz, One-step methods of any order for neutral functional differential equations, SIAM J. Numer. Anal. 21: 486-511, 1981.

[11] Z. Jackiewicz, Quasilinear multistep methods and variable-step predictor-corrector methods for neutral functional differential equations, SIAM J. Numer. Anal., 23: 423-452, 1986.

[12] Z. Jackiewicz, One-step methods for neutral delay differential equations with state dependent delays, Numerical Analysis Technical Report 651, 05-2. University of Arkansas, Fayetteville, 1985.

[13] Z. Jackiewicz, The numerical solution of functional differential equations, a survey, Mat. Stos. , 33: 57-78, 1991.

[14] Z. Jackiewicz, The numerical integration of NFDEs by fully implicit One-step methods}, ZAMM, 75:207-221, 1995.

[15] F. Kappel, K. Kunisch, Spline approximations for neutral functional differential equations, SIAM J. Numer. Anal., 18:1058-1080,1981.

[16] P. Pouzet, M'ethode d'intgration num'erique des e'quations inte'grales et inte'gro-diffe'rentielles du type de Volterra de seconde espe'ce. Formules de Runge-Kutta, Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-differential Equations, (Rome, 1960), Birkha"user Verlag, Basel, 362-368,1960.

[17] M. Zennaro, Natural continuous extensions of Runge-Kutta methods, Math. Comput. 46 (1986) 119-133.

[18] M. Zennaro, Delay differential equations theory and numerics, In: Theory and Numerics of Ordinary and Partial Differential Equations, Clarendon Press, Oxford, 291-333,1995.

[19] T. S. Zverkina, A modification of finite difference methods for integrating ordinary differential equations with nonsmooth solutions, (in Russian). Z. Vycisl Mat. i Mat. Fiz.[Suppl.] ,4 : 149-160, 1994.

[20] W. S. Wang, S. Fu Li, K. Su, Nonlinear stability of Runge-Kutta methods for neutral delay differential equations}, J. Comput. Appl. Math. , 214: 175-185, 2008.

[21] Y. Qu, M. Y. Li, J. Wei, Bufurcation analysis in a neutral differential equation, J. Math. Anal. Appl., 378: 387-402, 2011.

[22] J. J. Zhao, Y. Xu, M. Z. Liu, Stability analysis of numerical methods for linear neutral Volterra delay-integro-differential system, J. Appl. Math. Comput, 167:1062-1079, 2005.

[23] Z. Jackiewicz, E. Lo, Numerical solution of neutral functional differential equations by Adams methods in divided difference form, J. Comput. Appl. Math. , 189: 592-605, 2006.

[24] W. S. Wang, S. Fu Li, K. Su, Nonlinear stability of general linear methods for neutral delay differential equations, J. Comput. Appl. Math. , 214: 592-601, 2009.

[25] A. Bellen, N. Guglielmi, M. Zennaro, Numerical stability of nonlinear delay differential equations of neutral type, J. Comput. Appl. Math., 125 : 251-263,2000.

[26] M. A. Hussien, Chebyshev solution for neutral delay differential equations, Inter. J. Computer Math., 73: 253-260, 1999.

[27] S. E. El-Gendi, Chebyshev solution of differential, integral and integro-differential equations, Comput. J., 12: 282-287, 1969.

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