

Algorithm on conditional probability on random graph using Markov Chain.

Habiba El-Zohny⁽¹⁾, Hend El- Morsy⁽²⁾

^{1,2}, Mathematics Department Faculty of science Alazhar University Cairo - Egypt

ABSTRACT:

In this paper we will illustrate the conditional probability on random graph. We will compute steady state which can defined as limits of stationary matrix S . The steady state will be computed for some examples. After that we will illustrate algorithm of random walk in weighted and un weighted graph.

Definitions:

Markov chain, Markov model: Is a mathematical model that makes it possible to study complex systems by establishing a state of the system and then effecting a transition to a new state, such a transition being dependent only on the values of the current state, and not dependent on the previous history of the system up to that point [1].

Probability matrix P: Conditional probability concerning a discrete Markov chain giving the probabilities of change from one state to another [4].

A stationary probability vector S: Is defined as a vector that does not change under application of the transition matrix; that is, it is defined as a left eigenvector of the probability matrix, associated with eigenvalue 1: $SP = S$ [2].

Random Graph: Is a graph in which properties such as the number of graph vertices, graph edges, and connections between them are determined in some random way [3].

Main Results:

Algorithm which calculate the steady state probability of random graph:

Let P is a probability matrix (For which each row sums to 1), an let S is stationary matrix, then the distribution over states can be written as general with relation: $S^{n+1} = S^n P$.

Where (n) describe time period of the system.

Then if at time n the system is in state 2, then 3 times periods later at time $n+3$ the distribution is $S^{n+3} = S^{n+2} P = (S^{n+1} P) P = S^{n+1} P^2$ and so on.

The Algorithm:

- [1] Create associated probability matrix for a given graph.
- [2] Get stationary matrix S .
- [3] Compute $S^0 P = S^1$ which describe the initial distribution of graph.
- [4] Compute $S^{n+1} P = S^n$ until you get steady state probability which defined as :
 - a. $\gamma = \lim_{n \rightarrow \infty} (S)^n$.

Example 1: The graph shown in Fig.(1) represent the probability connects vertices with each other. According to figure, vertex A has probability of 0.9 to connect with A and 0.075 to connect with B, 0.25 to connect with C.

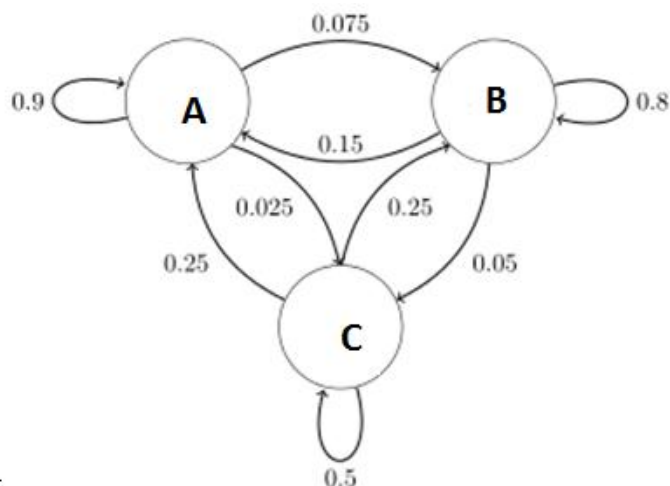


Fig.(1)

We will compute the steady state for this distribution as follows:

[1] Create associated probability matrix $P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$.

[2] Consider the stationary matrix $S = (0 \ 1 \ 0)$.

[3] Calculate :

$$S^0 P = (0 \ 1 \ 0) \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = (0.15 \ 0.8 \ 0.05)$$

[4] $S^1 P = (0.15 \ 0.8 \ 0.05) \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = (0.2675 \ 0.6637 \ 0.0687)$

[5] $S^2 P = (0.2675 \ 0.6637 \ 0.0687) \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = (0.3575 \ 0.56825 \ 0.0742)$

[6] $S^3 P = (0.3575 \ 0.5682 \ 0.0742) \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = (0.4255 \ 0.4999 \ 0.0744)$

Compute steady state distribution as follows:

$\gamma P = \gamma I = \gamma$, then

$\gamma (P - I) = 0$

$$(\gamma_1 \ \gamma_2 \ \gamma_3) \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0 \ 0 \ 0)$$

$$(\gamma_1 \ \gamma_2 \ \gamma_3) \begin{bmatrix} -0.1 & 0.075 & 0.025 \\ 0.15 & -0.2 & 0.05 \\ 0.25 & 0.25 & -0.5 \end{bmatrix} = (0 \ 0 \ 0)$$

Create a system of equation then solve it:

$\gamma_1 + 0.15 \gamma_2 + 0.025 \gamma_3 = 0$

$0.075 \gamma_1 - 0.2 \gamma_2 + 0.025 \gamma_3 = 0$

$0.025 \gamma_1 + 0.5 \gamma_2 - 0.5 \gamma_3 = 0$

And $\gamma_1 + \gamma_2 + \gamma_3 = 1$

This imply that:

$(\gamma_1 \ \gamma_2 \ \gamma_3) = (0.625 \ 0.3125 \ 0.0625)$

This mean that the steady state probabilities indicated that 62.5% of vertices connected with A , 31.5% connect with B and 6.25% connect with C .

Example 2: A Simple Markov Weather Model:

The probabilities of weather conditions, given the weather on the preceding day, can be represented by a transition matrix: $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$

The matrix P represents the weather model in which a sunny day is 90% likely to be followed by another sunny day, and a rainy day is 50% likely to be followed by another rainy day. The columns can be labelled "sunny" and "rainy" respectively, and the rows can be labelled in the same order. $(P)_{ij}$ is the probability that, if a given day is of type i , it will be followed by a day of type j . Notice that the rows of P sum to 1: this is because P is a stochastic matrix.

Predicting the weather

The weather on day 0 is known to be sunny. This is represented by a vector in which the "sunny" entry is 100%, and the "rainy" entry is 0%: $S^0 = (1 \ 0)$

The weather on day 1 can be predicted by:

$$S^1 = S^0 P = (1 \ 0) \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (0.9 \ 0.1).$$

Thus, there is an 90% chance that day 1 will also be sunny. The weather on day 2 can be predicted in the same way: $S^2 = S^1 P = S^0 P^2 = (0.9 \ 0.1) \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (0.86 \ 0.14).$

And so on.

Steady state of the weather

In this example, predictions for the weather on more distant days are increasingly inaccurate and tend towards a steady state vector. This vector represents the probabilities of sunny and rainy weather on all days, and is independent of the initial weather.

$\gamma P = \gamma I = \gamma$, then

$$\gamma (P - I) = 0$$

$$(\gamma_1 \ \gamma_2) \left(\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = (0 \ 0) .$$

$$(\gamma_1 \ \gamma_2) \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{bmatrix} = (0 \ 0).$$

$$-0.1 \gamma_1 + 0.5 \gamma_2 = 0$$

$\gamma_1 + \gamma_2 = 1$ by solving these equations we have:

$$(\gamma_1 \ \gamma_2) = (0.833 \ 0.167) . \text{ In conclusion, in the long term, 83\% of days are sunny.}$$

Example 3: Consider a company has two brands A,B .10 % market share using brand A and 40% use the other brand. the transition matrix given by: $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$ If $S^0 = (0.1 \ 0.4).$

$$S^1 = S^0 P = (0.1 \ 0.4) \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = (0.62 \ 0.38).$$

$$S^2 = (0.62 \ 0.38) \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = (0.724 \ 0.276).$$

$$S^3 = (0.724 \ 0.276) \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = (0.744 \ 0.255).$$

And so on ,then the steady state will be: $(\gamma_1 \ \gamma_2) \left(\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = (0 \ 0) .$

$$(\gamma_1 \ \gamma_2) \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & -0.6 \end{bmatrix} = (0 \ 0).$$

$$-0.2 \gamma_1 + 0.6 \gamma_2 = 0$$

$\gamma_1 + \gamma_2 = 1$ by solving these equations we have:

$$(\gamma_1 \ \gamma_2) = (0.75 \ 0.25) . \text{ this means that 75\% use brand A and 25\% use brand B.}$$

Algorithm which calculate random graph:

We will illustrate the algorithm for two kinds of graph , weighted and un weighted graph.

For un weighted graph:

Input: An undirected graph $G(V,E)$ with $|V| > 1, |E| > 0$.

Steps:

- [1] Look for random pair of nodes x, y that are nonadjacent .
- [2] If no such pair of nodes x, y exists , then output G' .
Else
- [3] Add element $\{ x, y \}$ to set E' .
- [4] Return to step 1 .
End while.
- [5] Output G' .
End algorithm.

We find that output graph G' is graph with the same number of connected component as G , and has maximum possible density as a clique.

Example 1:By using the previous algorithm find the output random graph shown in Fig. (4)

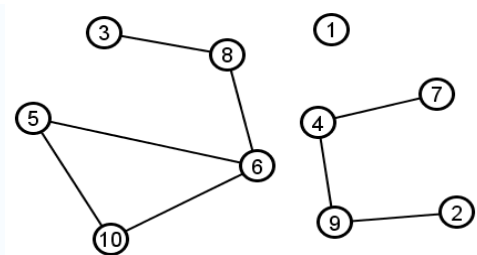


Fig.(4)

Input: Undirected graph with 10 vertices , and has 3 component , $|E| = 8$ edges.

Steps:

- [1] Start randomly with two nonadjacent nodes $\{ 8, 10 \}$
- [2] Add element $\{ 8, 10 \}$ to set E' .
- [3] Return to step 1 with another pair of nodes.
- [4] Output G' .
End algorithm.

The result graph G' shown in Fig.(5).

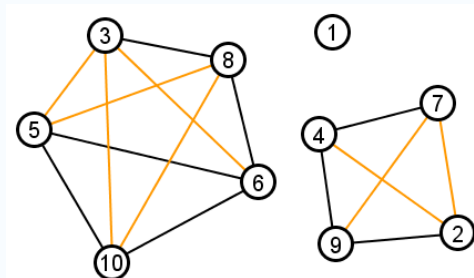


Fig.(5)

We find that the output graph is regular and complete.

Lemma 1: The output graph generating by algorithm of random graph will be regular and complete.

Example 2:For another graph shown in Fig.(6) with 10 vertices but 1 component.

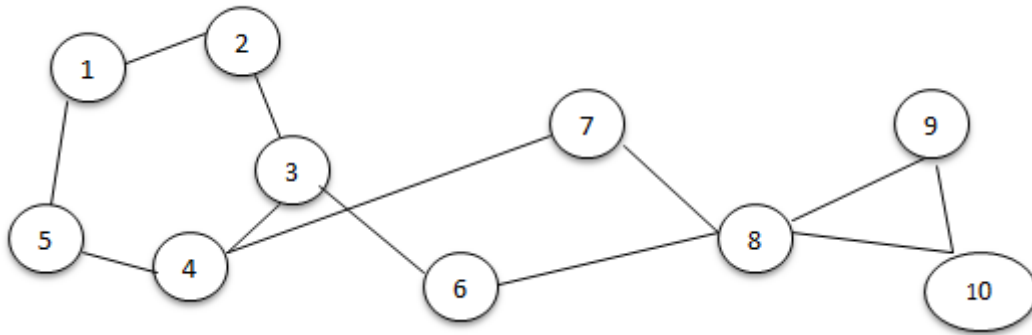


Fig.(6)

Input: Undirected graph with 10 vertices and 1 component.

Steps:

- [1] Start randomly with nonadjacent vertices s.t {1,6}.
- [2] Add element {1,6} to set E' .
- [3] Return to step 1.
- [4] Output G' .

End algorithm. Output graph shown in Fig.(7).

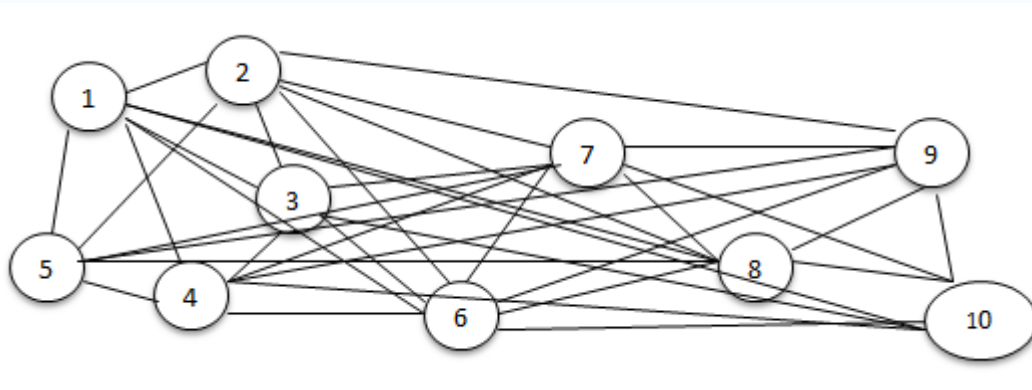


Fig.(7)

2b. For weighted graph:

Input: An undirected weighted graph $G(V, E)$ with $|V| > 1$, $|E| > 0$.

Steps:

1. Create the associated matrix .
2. Normalize this matrix.
3. Get the first transition probability matrix P .
4. Expand by taking P^{th} power of P .
5. Get stationary matrix S .
6. Compute $S^0 P = S^1$ which describe the initial distribution of graph.
7. Compute $S^{n+1} P = S^n$ until you get steady state probability which defined as :

$$\gamma = \lim_{n \rightarrow \infty} (S)^n$$

Notes: 1. When normalize matrix in this algorithm, column sums to one.

2. Each power of P consider as possibility distribution of random graph.

Example 3: Calculate random graph for weighted graph shown in Fig.(8).

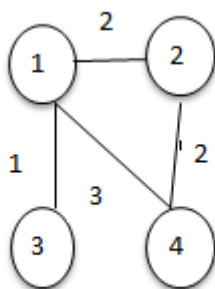


Fig.(8)

1. Associated matrix is:
$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

2. Normalize the matrix :
$$\begin{bmatrix} 0 & 1/2 & 1 & 3/5 \\ 1/3 & 0 & 0 & 2/5 \\ 1/6 & 0 & 0 & 0 \end{bmatrix}.$$

3.
$$P^2 = P \cdot P = \begin{bmatrix} 19/30 & 3/10 & 0 & 1/5 \\ 1/5 & 11/30 & 1/3 & 1/5 \\ 0 & 1/12 & 1/6 & 1/10 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/5 & 5/12 & 19/30 & 25/50 \\ 5/18 & 1/5 & 1/5 & 4/15 \\ 11/45 & 1/20 & 0 & 1/30 \end{bmatrix}$$
 And so on.

4. Get steady state :
$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \left(\begin{bmatrix} 0 & 1/2 & 1 & 3/5 \\ 1/3 & 0 & 0 & 2/5 \\ 1/6 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \begin{bmatrix} -1 & 1/2 & 1 & 3/5 \\ 1/3 & -1 & 0 & 2/5 \\ 1/6 & 0 & -1 & 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then $\gamma_1 =$ $\gamma_3 =$

$\gamma_2 =$ $\gamma_4 =$

Note: We normalize the matrix by dividing the column (Which expressed a vertex V) by sums of weights of all edges coming out of this vertex.

As example, vertex 1 adjacent with 3 edges of weights 2,3,1 then we divide column1 by (2+3+1) and so on.

Example 4: For graph shown in Fig.(9) calculate two possibility distributions for random graph.

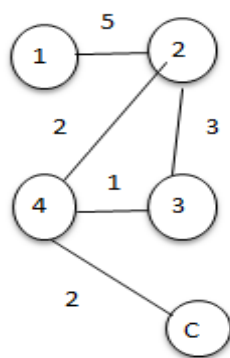


Fig.(9)

4. Associated matrix :
$$\begin{bmatrix} 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 3 & 2 & 0 \\ 0 & 3 & 0 & 1 & 0 \end{bmatrix}$$

5. Normalize matrix :
$$\begin{bmatrix} 0 & 5/10 & 0 & 0 & 0 \\ 1 & 0 & 3/4 & 2/5 & 0 \\ 0 & 3/10 & 0 & 1/5 & 0 \end{bmatrix} = P.$$

6.
$$P^2 = P \cdot P = \begin{bmatrix} 5/10 & 0 & 15/40 & 10/50 & 0 \\ 0 & 161/200 & 2/20 & 3/20 & 2/5 \\ 3/10 & 2/50 & 11/40 & 6/50 & 1/5 \end{bmatrix}$$

Application: Steady state for random bipartite graph:

Definition of random bipartite graph: Is a bipartite graph which two sets can be randomly joined with each other, such that the first set can be considered as begin then the other set consider as begin.

In random bipartite graph, we will compute steady state two times by changing the beginning set.

Example 4: Consider a bipartite graph as shown in Fig.(2) which consist of two sets : { 1,2,3 } and { A,B,C }, compute the steady state for this graph when set { 1,2,3 } is beginning set, then when { A,B,C } is beginning set.

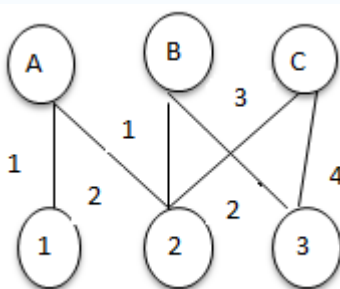


Fig.(2)

First, the incident matrix of this graph is:
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

Normalize the matrix:
$$\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & 1/4 & 2/6 \\ 0 & 3/4 & 4/6 \end{bmatrix}, S^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Then: } S^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & 1/4 & 2/6 \\ 0 & 3/4 & 4/6 \end{bmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 3/4 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 \\ 1/4 \\ 3/4 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & 1/4 & 2/6 \\ 0 & 3/4 & 4/6 \end{bmatrix} = \begin{pmatrix} 0 \\ 5/16 \\ 11/16 \end{pmatrix} \text{ and so on.}$$

Then the steady state can obtained as follows:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \left(\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & 1/4 & 1/3 \\ 0 & 3/4 & 2/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \begin{bmatrix} -2/3 & 0 & 0 \\ 2/3 & -3/4 & 1/3 \\ 0 & 3/4 & -1/3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2/3 \gamma_1 = 0 \quad \gamma_1 = 0,$$

$$2/3 \gamma_1 - 3/4 \gamma_2 + 1/3 \gamma_3 = 0, \quad \gamma_1 + \gamma_2 + \gamma_3 = 1$$

By solving these equations we find that:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4/13 \\ 9/13 \end{pmatrix}$$

We can compute steady state for the same graph by considered the set { 1,2,3} is the beginning set as follows:

$$\text{the incident matrix of this graph will be: } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\text{Normalize the matrix: } \begin{bmatrix} 1 & 2/5 & 0 \\ 0 & 1/5 & 3/7 \\ 0 & 2/5 & 4/7 \end{bmatrix}, S^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Then: } S^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 2/5 & 0 \\ 0 & 1/5 & 3/7 \\ 0 & 2/5 & 4/7 \end{bmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 3/4 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 \\ 1/4 \\ 3/4 \end{pmatrix} \begin{bmatrix} 1 & 2/5 & 0 \\ 0 & 1/5 & 3/7 \\ 0 & 2/5 & 4/7 \end{bmatrix} = \begin{pmatrix} 2/5 \\ 1/5 \\ 2/5 \end{pmatrix} \text{ and so on.}$$

Then the steady state can obtained as follows:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \left(\begin{bmatrix} 1 & 2/5 & 0 \\ 0 & 1/5 & 3/7 \\ 0 & 2/5 & 4/7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \begin{bmatrix} 0 & 2/5 & 0 \\ 0 & -4/5 & 3/7 \\ 0 & 2/5 & -3/7 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

After Creating a system of equation and solve it we find that:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

by comparing the two states we find that γ_1 at first has highest steady state and lowest steady state when we consider set {1,2,3} is the beginning set, the same note for γ_2, γ_3 .

Example 2:for bipartite graph shown in Fig.(3), compute the steady state for each vertex.

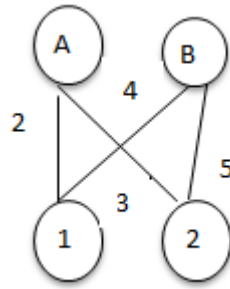


Fig.(3)

First , the incident matrix of this graph is : $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Normalize the matrix : $\begin{bmatrix} 2/6 & 3/8 \\ 4/6 & 5/8 \end{bmatrix}$

Then the steady state can obtained as follows:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \left(\begin{bmatrix} 1/3 & 3/8 \\ 2/3 & 5/8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \begin{bmatrix} -2/3 & 3/8 \\ 2/3 & -3/8 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since $\gamma_1 + \gamma_2 = 1$, $-2/3 \gamma_1 + \frac{3}{8\gamma_2} = 0$ then

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 9/25 \\ 16/25 \end{pmatrix}$$

We can compute steady state for the same graph by considered the set { 1,2} is the beginning set as follows:

the incident matrix of this graph will be: $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

Normalize the matrix: $\begin{bmatrix} 2/5 & 4/9 \\ 3/5 & 5/9 \end{bmatrix}$

$$\text{then : } \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \begin{bmatrix} -3/5 & 4/9 \\ 3/5 & -4/9 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3/5 \gamma_1 + 4/9 \gamma_2 = 0$$

$$\gamma_1 + \gamma_2 = 1$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 28/75 \\ 47/75 \end{pmatrix}. \text{ Which change of the first state.}$$

REFERENCES:

- [1]Saunders Comprehensive Veterinary Dictionary , 3 ed , Elsevier, Inc , (2007).
- [2]G. Latouche, V. Ramaswami. Introduction to Matrix Analytic Methods in Stochastic Modeling, 1st edition. Chapter 2: PH Distributions; ASA SIAM, 1999.
- [3]Bollobás, B. Random Graphs. London: Academic Press, 1985.
- [4]<http://www.answers.com/topic/transition-probability#ixzz2VQy2siJm> .