Fractional Derivative Associated With the Generalized M-Series and Multivariable Polynomials

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ABSTRACT

The aim of present paper is to derive a fractional derivative of the multivariable H-function of Srivastava and Panda [9], associated with a general class of multivariable polynomials of Srivastava [6] and the generalized Lauricella functions of Srivastava and Daoust [11] the generalized M-series. Certain special cases have also been discussed. The results derived here are of a very general nature and hence encompass several cases of interest hitherto scattered in the literature.

I. INTRODUCTION

In this paper the H-function of several complex variables introduced and studied by Srivastava and Panda [9] is an extension of the multivariable G-function and includes Fox's H-function, Meijer's G-function of one and two variables, the generalized Lauricella functions of Srivastava and Daoust [11], Appell functions etc. In this note we derive a fractional derivative of H-function of several complex variables of Srivastava and Panda [9], associated with a general polynomials (multivariable) of Srivastava [6] and the generalized Lauricella functions of Srivastavaand Daoust [11].Generalized M-series extension of the both Mittag-Laffler function and generalized hypergeometric functions.

II. DEFINITIONS AND NOTATIONS

By Oldham and Spanner [4] and Srivastava and Goyal [7] the fractional derivative of a function f(t) of complex order γ

$$ {}_{a}D_{t}^{\gamma}\{f(t)\} = \begin{cases} \dfrac{1}{\Gamma(-\gamma)}\displaystyle\int_{0}^{t} (t-x)^{-\gamma-1} f(x)\, dx, & \mathrm{Re}(\gamma) < 0 \\[2ex] \dfrac{d^{m}}{dt^{m}}\, {}_{a}D_{t}^{\gamma-m}\{f(t)\} & 0 \le \mathrm{Re}(\gamma) < m \end{cases} \qquad \ldots(2.1) $$

Where m is positive integer.

The multivariable H-function is defined by Srivastava and Panda [9] in the following manner

$$ H[z_1,\ldots, z_r] = H^{0,\lambda : (u',v');\ldots;(u^{(r)},v^{(r)})}_{A, C : [B',D'];\ldots;(B^{(r)},D^{(r)})} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[b^{(r)}:\phi^{(r)}] \\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}] \end{matrix} \right. \right] $$

$$ = \frac{1}{(2\pi i)^{r}} \int_{L_1} \ldots \int_{L_r} \psi_1(\xi_1,\ldots,\xi_r)\ldots\phi_1(\xi_1)\ldots\phi_r(\xi_r)\, z_1^{\xi_1}\ldots z_r^{\xi_r}\, d\xi_1\ldots d\xi_r , \qquad \ldots(2.2) $$

where $(i = \sqrt{-1})$.

The general class of multivariable polynomials defined by Srivastava [6] defined as

$$ S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}[x_1\ldots x_s] = \sum_{k_1=0}^{(q_1/p_1)} \ldots \sum_{k_s=0}^{(q_s/p_s)} \frac{(-q_1)_{p_1k_1}}{k_1!} \ldots \frac{(-q_s)_{p_sk_s}}{k_s!} $$

$$\times A\,[q_1, k_1;\dots; q_s, k_s]\, x_1^{k_1} \dots x_s^{k_s} \qquad \dots(2.3)$$

where $q_j = 0,1,2,\dots;\ p_j \neq 0\ (j = 1,\dots, s)$ are non-zero arbitrary positive integer the coefficients $A\,[q_1, k_1;\dots; q_s k_s]$ being arbitrary constants, real or complex.

The following known result of Srivastava and Panda [10]

Lemma. If ($\lambda \geq 0$), $0< x < 1$, Re $(1+p) > 0$, Re$(q) > -1$, $\lambda_i > 0$ and $\Delta_i > 0$ or $\Delta_i = 0$ and $|z_i| < \sigma$, $i = 1,2,\dots,r$ then

$$x^{\lambda} F\begin{pmatrix} z_1 x^{\lambda_1} \\ \vdots \\ z_r x^{\lambda_r} \end{pmatrix} = \sum_{M=0}^{\infty} \frac{(1+p+q+2M)\ (-\lambda)_M\ (1+p)_{\lambda}}{M\,!\,(1+p+q+M)_{\lambda+1}}$$

$$.\ F_M[z_1,\dots, z_r]\ {}_2F_1\left[\begin{matrix} -M,\ 1+p+q+M\ ; \\ 1+p \qquad ; \end{matrix}\ x\right] \qquad \dots(2.4)$$

where

$$F_M[z_1,\dots, z_r] = F^{E+2:U';\dots;U^{(r)}}_{p+2:V';\dots;V^{(r)}}\left[\begin{matrix} [(e):\eta';\dots;\eta^{(r)}][1+p+\lambda:\lambda_1,\dots,\lambda_r], \\ [(g):\xi';\dots;\xi^{(r)}][2+p+q+M+\lambda:\lambda_1,\dots,\lambda_r] \end{matrix}\right.$$

$$\left.\begin{matrix} [\lambda+1;\dots;\lambda_r]:[(w'):x'];\dots;[(w^{(r)}:x^{(r)}]; \\ [\lambda-M+1;\lambda_1,\dots,\lambda_r]:[(v'):t'];\dots;[(v^{(r)}):t^{(r)}]; \end{matrix}\ z_1,\dots, z_r\right] \qquad \dots(2.5)$$

where $M \geq 0$,

In this paper, we also use short notations as given

$$F^{E:U',\dots,U^{(r)}}_{P:V',\dots,V^{(r)}}\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_t \end{pmatrix} \equiv F\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix} \qquad \dots(2.6)$$

denote the generalized Lauricella function of several complex variable.

The special case of the fractional derivative of Oldham and Spanier [4] is

$$D_t^{\gamma}(t^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\gamma+1)}\, t^{\mu-\gamma} \qquad \text{Re}(\mu) > -1 \qquad \dots(2.7)$$

The generalized M-series is the extension of the both Mittag-Leffler function and generalized hypergeometric function.

It represent as following

$$M^{\lambda,\mu}_{p,q}(c_1,\dots, c_p; d_1,\dots, d_q; z) = M^{\lambda,\mu}_{p,q}(z)$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_p)_k}{(d_1)_k \dots (d_q)_k} \frac{z^k}{\Gamma(\lambda k+\mu)} z,\ \lambda, \mu \in c,\quad \text{Re}(\lambda) > 0 \qquad \dots(2.8)$$

III. THE MAIN RESULT

Our main result of this paper is the fractional derivative formula involving the Lauricella functions, generalized polynomials and the multivariable H-function and generalized M-series as given

$$D_{\ell}^{\gamma}\left\{(\ell - x)^{\sigma}\eta^{\sigma}(y-\ell)^{\sigma+\rho} F\begin{pmatrix} z_1\{\eta(y\ell)\}^{\sigma_1} \\ \vdots \\ z_r\{\eta(y-\ell)\}^{\sigma_r}\end{pmatrix} S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\begin{pmatrix}(\ell-x)^{a_1}(y-\ell)^{b_1} \\ \vdots \\ (\ell-x)^{a_s}(y-\ell)^{b_s}\end{pmatrix}\right.$$

$$\left.\times M_{\ell,m}^{\lambda,\mu}\{(\ell-x)^{\lambda_1}(y-\ell)^{\lambda_2}\} H\begin{bmatrix} w_1\{\ell(\ell-x)\}^{\sigma_1}\{\ell(y-\ell)\}^{\rho_1} \\ \vdots \\ w_r\{\ell(\ell-x)\}^{\sigma_r}\{\ell(y-\ell)\}^{\rho_r}\end{bmatrix}\right\}$$

$$= \sum_{\alpha,\beta=0}^{\infty}\sum_{k,M=0}^{\infty}\sum_{k_1=0}^{(N_1/M_1)}\cdots\sum_{k_s=0}^{(N_s/M_s)}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}A[N_1,k_1,\dots,N_s,k_s]$$

$$\Delta\, H_{A+3,C+3:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda+3\;:(u',v');\dots;(u^{(r)},v^{(r)})}\left[\begin{matrix} w_1(-x)^{\sigma_1}y^{\rho_1}\ell^{\rho_1+\sigma_1} \\ \vdots \\ w_r(-x)^{\sigma_r}y^{\rho_r}(\ell)^{\rho_1+\sigma_r}\end{matrix}\right|\begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1,\dots,\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i=1}^{s}a_ik_i-\lambda_1k:\sigma_1,\dots,\sigma_r\right)\end{matrix}$$

$$\begin{matrix}\left(-\sigma-\sum_{i=1}^{s}a_1k_1-\lambda_1k:\sigma_1,\dots,\sigma_r\right),\left(-\rho-k-\sum_{i=1}^{s}b_ik_i-\lambda_2k:\rho_1,\dots,\rho_r\right),[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}] \\ \left(\beta-\rho-k-\sum_{i=1}^{s}b_ik_i-\lambda_2k:\rho_1,\dots,\rho_r\right),(\gamma-\alpha-\beta:\rho_1+\sigma_1,\dots,\rho_r+\sigma_r):[(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'],\dots,[d^{(r)}:\delta^{(r)}]\end{matrix}\right] \qquad \dots(3.1)$$

where

$$\Delta = \frac{(-1)^{\beta}(1+\rho+q+2M)\;(1+p+q+M)_k\,(-M)_k\,(-\sigma)_M\,(1+p)_{\sigma}}{k!\,M!\,(1+p+q+M)_{\sigma+1}\,\Gamma(\lambda k+\mu)(1+p)_k\,\Gamma\alpha+1\Gamma\beta+1}$$

$$.\ \eta^{k}(-x)^{\sigma-\alpha+\lambda_1k+\sum_{i=1}^{s}a_i,k_i}(y)^{\rho+\lambda_2k-\beta+\sum_{i=1}^{s}b_ik_i}t^{\alpha+\beta-\gamma}$$

$$.\ F_M(z_1,\dots,z_r)\frac{(c_1)_R\dots(c_\ell)_R}{(d_1)_R\dots(d_m)_R}\quad \sigma_i>0, s_i>0,\ i=1,2,\dots,\ r$$

and

$$\mathrm{Re}(\sigma)+\sum_{i=1}^{r}\sigma_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

$$\mathrm{Re}\ (\rho)+\sum_{i=1}^{r}\rho_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

Proof. In order to prove (3.1) express the Lauricella function by (2.4) and the multivariable H-function in terms of Mellin-Barnes type of contour integrals by (2.2) and generalized polynomials given by (2.3) respectively and generalized M-series (2.8) and collecting the power of $(\ell - x)$ and $(y-\ell)$. Finally making use of the result (2.7), we get (3.1).

IV. PARTICULAR CASES

With $\lambda = A = C = 0$, the multivariable H-function breaks into product of Fox's H-function and consequently there holds the following result

$$D_{\ell}^{\gamma}\left\{(\ell-x)^{\sigma}\eta^{\sigma}(y-\ell)^{\sigma+\rho}F\begin{pmatrix} z_1\{\eta(y-\ell)\}^{\sigma_1} \\ \vdots \\ z_r\{\eta(y-\ell)\}^{\sigma_r}\end{pmatrix} S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{bmatrix}(\ell-x)^{a_1}(y-\ell)^{b_1} \\ \vdots \\ (\ell-x)^{a_s}(y-\ell)^{b_s}\end{bmatrix}\right.$$

$$\left.\times M_{\ell,m}^{\lambda,\mu}\{(\ell-x)^{\lambda_1}(y-\ell)^{\lambda_2}\}\prod_{i=1}^{r} H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}\left[w_i\{\ell(\ell-x)\}^{\sigma_i}\{\ell(y-\ell)\}^{\rho_i}\middle|\begin{matrix}[b^{(i)}:\phi^{(i)}] \\ [d^{(i)}:\delta^{(i)}]\end{matrix}\right]\right\}$$

$$=\sum_{\alpha,\beta=0}^{\infty}\sum_{k,M=0}^{\infty}\sum_{k_1=0}^{(N_1/M_1)}\cdots\sum_{k_s=0}^{(N_s/M_s)}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}A[N_1,k_1,\ldots,N_s,k_s]$$

$$\Delta\, H_{3,3:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,3:(u',v');\ldots;(u^{(r)},v^{(r)})}\left[\begin{matrix} w_1(-x)^{\sigma_1}y^{\rho_1}\ell^{\rho_1+\sigma_1} \\ \vdots \\ w_r(-x)\}^{\sigma_r}y^{\rho_r}\ell^{\rho_r+\sigma_r}\end{matrix}\middle|\begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i=1}^{s}a_ik_i-\lambda_1k:\sigma_1,\ldots,\sigma_r\right)\end{matrix}\right.$$

$$\left.\begin{matrix}\left(-\sigma-\sum_{i=1}^{s}a_1k_1-\lambda_1k:\sigma_1,\ldots,\sigma_r\right),\left(-\rho-k-\sum_{i=1}^{s}b_ik_i-\lambda_2k:\rho_1,\ldots,\rho_r\right),[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}] \\ \left(\beta-\rho-k-\sum_{i=1}^{s}b_ik_i-\lambda_2k:\rho_1,\ldots,\rho_r\right),(\gamma-\alpha-\beta:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r):[(d'):\delta'],\ldots,[d^{(r)}:\delta^{(r)}]\end{matrix}\right] \quad \ldots(4.1)$$

valid under the conditions surrounding (3.1).

II. If $\phi^{(i)} = \delta^{(i)} = 1$, $(i = 1,2,\ldots)$ equation (4.1) reduces to

$$D_{\ell}^{\gamma}\left\{(\ell-x)^{\sigma}\eta^{\sigma}(y-\ell)^{\sigma+\rho}F\begin{pmatrix} z_1\{\eta(y-\ell)\}^{\sigma_1} \\ \vdots \\ z_r\{\eta(y-\ell)\}^{\sigma_r}\end{pmatrix} S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{bmatrix}(\ell-x)^{a_1}(y-\ell)^{b_1} \\ \vdots \\ (\ell-x)^{a_s}(y-\ell)^{b_s}\end{bmatrix}\right.$$

$$\left.\times M_{\ell,m}^{\lambda,\mu}\{(\ell-x)^{\lambda_1}(y-\ell)^{\lambda_2}\}\prod_{i=1}^{r} G_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}\left[w_i\{\ell(\ell-x)\}^{\sigma_i}\{\ell(y-\ell)\}^{\rho_i}\middle|\begin{matrix}[b^{(i)}] \\ [d^{(i)}]\end{matrix}\right]\right\}$$

$$=\sum_{\alpha,\beta=0}^{\infty}\sum_{k,M=0}^{\infty}\sum_{k_1=0}^{(N_1/M_1)}\cdots\sum_{k_s=0}^{(N_s/M_s)}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}A[N_1,k_1,\ldots,N_s,k_s]$$

$$\Delta\, H_{3,3:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,3:(u',v');\ldots;(u^{(r)},v^{(r)})}\left[\begin{matrix} w_1(-x)^{\sigma_1}y^{\rho_1}\ell^{\rho_1+\sigma_1} \\ \vdots \\ w_r(-x)\}^{\sigma_r}(y)^{\rho_r}\ell^{\rho_r+\sigma_r}\end{matrix}\middle|\begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r), \\ \left(\alpha-\sigma-\sum_{i=1}^{s}a_ik_i-\lambda_1k:\sigma_1,\ldots,\sigma_r\right)\end{matrix}\right.$$

$$\left.\begin{matrix}\left(-\sigma-\sum_{i=1}^{s} a_1 k_1-\lambda_1 k:\sigma_1,\ldots,\sigma_r\right),\left(-\rho-k-\sum_{i=1}^{s} b_i k_i-\lambda_2 k:\rho_1,\ldots,\rho_r\right),[(b');\ldots;[(b^{(r)})] \\ \left(\beta-\rho-k-\sum_{i=1}^{s} b_i k_i-\lambda_2 k:\rho_1,\ldots,\rho_r\right),(\gamma-\alpha-\beta:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r):[(d'),\ldots,[(d^{(r)})]\end{matrix}\right] \quad \ldots(4.2)$$

valid under the conditions as obtainable from (3.1).

III. Let $N_i = 0$ ($i = 1,\ldots,s$), the result in (3.1) reduces to the known result given by Sharma and Singh [ ], after a little simplification.

IV. Replacing $N_1,\ldots,N_s$ by N in (3.1) we have a known result recently obtained by Chaurasia and Singh [ ].

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