Second Derivative Free Modification with a Parameter For Chebyshev's Method

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ABSTRACT:
Chebyshev's method has not got much attention in recent years compared to Newton's method. Some of the papers which appeared recently discuss modification of Chebyshev's method free from second order derivative. In this paper, we have modified Chebyshev's method by suitably approximating second order derivative using Taylor's Series. The proposed method requires evaluation of only three functions and still maintains cubic convergence. For a particular choice of a parameter in the method, we get fourth order convergence. Examples are provided to show the efficiency of the method with classical Chebyshev's method and few other cubic convergent methods.

KEYWORDS: *Chebyshev's Method, Iterative Method, Non-linear equation, Second derivative free method, Cubic convergence.*

I. INTRODUCTION

One of the most important and challenging problem in applied mathematics and engineering is to find an approximate solution of the nonlinear equation

$$f(x) = 0 \qquad (1)$$

where $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function. Newton's method is one of the famous iterative methods to solve (1). It is well known that it has quadratic convergence. The iterative formula of Newton's method is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots \qquad (2)$$

By improving Newton's method, Chebyshev [8] derived a cubic convergent iterative method which requires computing of second derivative of the function (1). Without evaluating second derivative of f(x), cubic convergence was first established by the Arithmetic mean Newton's method [1, 16]. In [1, 4, 11, 14] third order accuracy is proved by using midpoint Newton's method. Modifications in the Newton's method using harmonic mean were suggested in [1, 7, 11, 14]. Nedzhibov [13] gave several classes of iterative methods using different quadrature rules for solving nonlinear equations. Different variants based on Simpson formula on Newton's theorem were proposed in [1, 2, 5, 18]. Zhou [17] gave a class of Newton's methods based on power means on the trapezoid formula. Tibor et al [12] gave third order accurate Newton's modification by using geometric mean. Jisheng et al [10] proposed a uniparametric Chebyshev type method free from second derivative. Herceg et al [6] presented a method for constructing new third order methods for solving (1) in which Halley's and Chebyshev's methods are special cases. Esmaeili et al [3] gave a uniparametric modification of Chebyshev's method. Different variants of Chebyshev's method with optimal order of convergence were proved in [15]. Jayakumar et al [9] modified Newton's method using the harmonic mean instead of arithmetic mean on the Simpson's formula. The methods presented in all the above papers require only the first derivative of f(x) and establishes third order convergence.

In this paper, we propose a modification in the Chebyshev's method [8] similar to the one given in [10] which is free from second order derivative of f(x) by considering a suitable approximation of f ″(x) using Taylor's Series. The proposed new method has the advantage of evaluating only the first derivative of f(x), less number of iterations and third order accuracy. With an additional condition on the parameter k, fourth order accuracy is obtained. In Section 2, we present some definitions related to our study. In Section 3, some known third order variants of Newton's method are discussed. Section 4 presents the new method and its analysis of convergence. Finally, Section 5 gives numerical results and discussion.

II. DEFINITIONS

Definition 2.1 [16]: Let $\alpha \in R$, $x_n \in R$, n = 0,1,2, … Then the sequence $\{x_n\}$ is said to converge to α if $\lim_{n\to\infty} |x_n - \alpha| = 0$. If, in addition, there exist a constant $c \geq 0$, an integer $n_0 \geq 0$ and p ≥ 0 such that for all $n > 0$, $|x_{n+1} - \alpha| \leq c\,|x_n - \alpha|^p$,

then $\{x_n\}$ is said to converge to α with order at least p. If p = 2 or 3, the convergence is said to be quadratic or cubic.

Definition 2.2 [8]: The efficiency index of an iterative method in the sense of Traub is defined by the equation $E^* = p^{1/n}$, where p is the order of the method and n is the total number of function and derivative evaluations at each step of the iteration.

Definition 2.3 [16]: Let α be a root of the function (1) and suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α. Then, the computational order of convergence (COC) ρ can be approximated using the formula $\rho \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}$.

III. KNOWN THIRD ORDER METHODS

Let α be a simple root of a sufficiently differentiable function $f(x)$. Consider the numerical solution of $f(x) = 0$. From Newton's theorem, we have

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \tag{3}$$

Arithmetic Mean Newton's method (AN): By using the trapezoidal rule in (3), we obtain

$$\int_{x_n}^{x} f'(t)\,dt \approx \frac{(x-x_n)}{2m}\left[f'(x_n) + 2\sum_{i=1}^{m-1} f'\left(x_n - \frac{if(x_n)}{mf'(x_n)}\right) + f'(x)\right] \tag{4}$$

From equations (3) and (4), a new approximation x_{n+1} for x is obtained and for m = 1, we get the AN [1, 16]

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, n = 0, 1, 2 \ldots \tag{5}$$

Since equation (5) is implicit, we can overcome this implicit nature by the use of Newton's iterative step (2).

Harmonic Mean Newton's method (HN): If we use the harmonic mean instead of arithmetic mean in (5), we obtain harmonic mean Newton's method [1, 7, 11, 14]:

$$x_{n+1} = x_n - \frac{f(x_n)\left(f'(x_n)+f'(y_n)\right)}{2f'(x_n)f'(y_n)},\ n = 0, 1, 2 \ldots, \tag{6}$$

where y_n is calculated from (2).

Midpoint Newton's method (MN): If the integral in (3) is approximated using the midpoint integration rule instead of trapezoidal rule, we get the midpoint Newton's method [1, 4, 11, 14]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + y_n}{2})},\ n = 0,1,2,\ldots, \tag{7}$$

where y_n is calculated from (2).

IV. MODIFICATION OF CHEBYSHEV'S METHOD AND ITS ANALYSIS OF CONVERGENCE

The classical Chebyshev's method (CCM) is given by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\left[\frac{f(x_n)\,f''(x_n)}{f'(x_n)^2}\right]\right)\frac{f(x_n)}{f'(x_n)}. \tag{8}$$

Use Taylor's series to approximate $f''(x_n)$ in (8) with $\frac{f'(y_n)-f'(x_n)}{y_n - x_n}$, where y_n is given by

$$y_n = x_n - k\frac{f(x_n)}{f'(x_n)},\ k \neq 0,\ n = 0,1,2,\ldots \tag{9}$$

Thus equation (8) is converted into

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]\left[\frac{1}{f'(x_n)}\right]\left[\frac{f'(y_n)-f'(x_n)}{y_n - x_n}\right]\right)\frac{f(x_n)}{f'(x_n)}. \qquad (10)$$

We can modify equation (10) by using (9) as below

$$x_{n+1} = x_n - \left(1 - \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]\left[\frac{1}{f'(x_n)}\right]\left[\frac{f'(y_n) - f'(x_n)}{-k\frac{f(x_n)}{f'(x_n)}}\right]\right)\frac{f(x_n)}{f'(x_n)},$$

where the parameter $k \neq 0$. Finally, we simplify the above equation to obtain the new method

$$x_{n+1} = x_n + \left(-1 + \frac{1}{2k}\left[\frac{f'(y_n) - f'(x_n)}{f'(x_n)}\right]\right)\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \ . \qquad (11)$$

Theorem 4.1 Assume that the function $f: D \subset \mathbb{R} \to \mathbb{R}$ has a simple root $\alpha \epsilon D$, where D is an open interval. If $f(x)$ is sufficiently differentiable in the interval D, then the methods defined by (11) converge cubically to α in a neighbourhood of α. for all values of the parameter $k \neq 0$. Moreover, if $f''(x_n) = 0$ and k = 2/3 then the method has fourth order convergence.

Proof. Let α be a simple zero of function $f(x) = 0$. (That is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$).

Let $e_n = x_n - \alpha$ and $d_n = y_n - x_n$, where y_n is defined by (9). By expanding $f(\alpha)$ by Taylor's series about x_n, we obtain

$$f(\alpha) = f(x_n) - e_n f'(x_n) + \frac{1}{2}f''(x_n)e_n^2 - \frac{1}{6}f'''(x_n)e_n^3 + O(e_n^4). \qquad (12)$$

Taking into account $f(\alpha) = 0$, we have from (12)

$$f(x_n) = e_n f'(x_n) - \frac{1}{2}f''(x_n)e_n^2 + \frac{1}{6}f'''(x_n)e_n^3 + O(e_n^4)\ . \qquad (13)$$

Let us denote $C_j = \dfrac{f^{(j)}(x_n)}{n!\, f'(x_n)}$, $j = 2, 3, \ldots$. Dividing equation (13) by $f'(x_n)$, we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2 e_n^2 + C_3 e_n^3 - C_4 e_n^4 + O(e_n^5). \qquad (14)$$

From (9) and (14), we have

$$d_n = y_n - x_n = -k\frac{f(x_n)}{f'(x_n)} = -ke_n + kC_2e_n^2 - kC_3e_n^3 + kC_4e_n^4 + O(e_n^5). \qquad (15)$$

Using Taylor's expansion for $f'(y_n)$ about x_n, we get

$$f'(y_n) = f'(x_n) + f''(x_n)d_n + \frac{f^{(3)}(x_n)}{2}d_n^2 + \frac{f^{(4)}(x_n)}{6}d_n^3 + \frac{f^{(5)}(x_n)}{24}d_n^4 + O(d_n^5). \qquad (16)$$

From (15) and (16), we get

$$f'(y_n) - f'(x_n) = f''(x_n)\, k\,[-e_n + C_2e_n^2 - C_3e_n^3 + C_4e_n^4] + \frac{f^{(3)}(x_n)}{2}k^2[e_n^2 - 2C_2e_n^3 + (C_2^2 + 2C_3)e_n^4$$
$$-2(C_4 + C_2C_3)e_n^5] + \frac{f^{(4)}(x_n)}{6}k^3[-e_n^3 + 3(C_2e_n^4 - C_2^2e_n^5 + C_3e_n^5)] + \frac{f^{(5)}(x_n)}{24}k^4[e_n^4 - 4C_2e_n^5] + O(e_n^6).$$
$$(17)$$

Dividing (17) by $f'(x)$ and the result is multiplied by $\dfrac{1}{2k}$ to get

$$\frac{1}{2k}\left(\frac{f'(y_n)-f'(x_n)}{f'(x_n)}\right) = -e_n C_2 + e_n^2\left(C_2^2+\frac{3}{2}kC_3\right) - e_n^3[C_2C_3(1+3k)+2k^2C_4] + e_n^4\left[3k\left(\frac{1}{2}C_2^2C_3+2C_3^2\right)+ C_2C_4(1+6\mathrm{k}^2)+\frac{5}{2}k^3C_5\right]+O(e_n^5) \quad (18)$$

From (14) and (18), we have

$$\left[\frac{1}{2k}\left(\frac{f'(y_n)-f'(x_n)}{f'(x_n)}\right)\right]\frac{f(x_n)}{f'(x_n)} = -e_n^2C_2 + e_n^3\left(2C_2^2+\frac{3k}{2}C_3\right) - e_n^4\left(C_2C_3\left(2+\frac{9}{2}k\right)+C_2^3+2k^2C_4\right)+O(e_n^5).$$

Again from (14) and the above equation, we get

$$-\frac{f(x_n)}{f'(x_n)}+\frac{1}{2k}\left[\left(\frac{f'(y_n)-f'(x_n)}{f'(x_n)}\right)\right]\frac{f(x_n)}{f'(x_n)} = -e_n + e_n^3\left(2C_2^2+\left(\frac{3k}{2}-1\right)C_3\right) - e_n^4\left(C_2C_3\left(2+\frac{9}{2}k\right)+C_2^3+C_4(2k^2-1)\right)+O(e_n^5) \quad (19)$$

Using the relation $e_n = x_n - \alpha$ in (11), we have

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} + \frac{1}{2k}\left[\left(\frac{f'(y_n)-f'(x_n)}{f'(x_n)}\right)\right]\frac{f(x_n)}{f'(x_n)}. \quad (20)$$

Finally, using (19) on (20), we obtain

$$e_{n+1} = e_n^3\left[2C_2^2 + C_3\left(\frac{3k}{2}-1\right)\right] - e_n^4\left[C_2^3 + C_2C_3\left(2+\frac{9k}{2}\right)+C_4(2k^2-1)\right]+O(e_n^5)\,. \quad (21)$$

From equation (21), we conclude that the methods defined by (11) are cubically convergent for all the values of $k \neq 0$. Also, taking $f''(x_n)=0$ (i.e. $C_2 = 0$) and $k = 2/3$ in equation (21), we get fourth order convergence.

V. NUMERICAL EXAMPLES AND DISCUSSION

In this section, we present the results of numerical calculations to compare the efficiency of the present method (MCM). We compare MCM with Newton's method (NM), classical Chebyshev method (CCM) and other cubic convergent Newton-type methods given in section III. Number of iterations and Computational Order of Convergence (COC) for different functions is given in **Table I.** Numerical computations are worked out in the MATLAB software with double precision accuracy and the results are presented below.
Stopping criterion for the iterations has been taken as $|x_{n+1}-x_n| + |f(x_{n+1})| < \varepsilon$, where $\varepsilon = 10^{-14}$. The following table gives functions that are taken as examples:

$f(x)$	Root (α)	$f(x)$	Root (α)
$f_1(x) = arctan(x)$	0	$f_4(x) = ln(x^3+x+1)$	0
$f_2(x) = e^{-x}sinx + \log(1+x^2)$	0	$f_5(x) = x^2 sin(x) - cos(x)$	6.3083089552381
$f_3(x) = x^3 - 9x^2 + 28x - 30$	3	$f_6(x) = (x+2)e^{-x}+x$	−1.6878939988

Discussion: In this work, a single parameter family of Chebyshev-type method free from second order derivatives for solving non-linear equation is presented. Third-order convergence is established for the proposed method for all values of the parameter $k \neq 0$. Also, we get fourth order convergence when $k = 2/3$ and $C_2 = 0$ which is displayed for the examples $f_1(x)$, $f_2(x)$ and $f_3(x)$. The efficiency index for MCM is $E^* = 1.4422$ and for the value of $k = 2/3$ and $C_2 = 0$, $E^* = 1.5874$ which are very good compared to Newton's method which has the efficiency $E^* = 1.4142$. This shows that MCM is preferable to Newton's method, especially when the computational cost of the first derivative is not more than that of the function itself. These proposed methods are also superior as there is no evaluation of second derivative compared to its classical predecessor, Chebyshev's method. From numerical examples, we have shown that these new methods have very good practical utility.

Table I

$f(x)$	x_0	Number of iterations							COC						
		NM	AN	MN	CCM	MCM k=1	MCM k=1/2	MCM k=2/3	NM	AN	MN	CCM	MCM k =1	MCM k =1/2	MCM k =2/3
$f_1(x)$	1.3	8	6	5	8	6	NC	NC	3.0	2.9	3.0	3.0	2.9	NC	NC
	1	6	5	5	6	5	6	5	2.9	2.9	3.0	3.0	2.9	3.3	4.9
	0.5	5	4	4	5	4	4	4	2.9	2.9	3.1	3.0	2.8	3.5	4.9
	−1	6	5	5	6	5	6	5	2.9	2.9	3	3.0	2.9	3.3	4.9
$f_2(x)$	1.3	5	4	5	5	5	5	4	3.0	2.2	2.9	2.56	2.9	2.8	4.0
	1	5	4	4	5	4	4	4	3.0	2.6	2.8	2.79	2.5	3.0	3.6
	0.5	4	4	4	4	4	4	4	2.4	2.6	2.7	2.78	2.6	2.7	3.8
	−1	6	5	5	5	5	5	5	3.0	3.1	3.0	3.07	3.1	2.8	4.4
$f_3(x)$	2	6	5	5	5	5	5	5	2.9	2.9	2.7	2.52	3.1	3.0	4.8
	2.5	7	6	5	6	6	6	7	2.9	2.9	2.9	2.97	3.0	2.8	4.8
	3.5	8	6	6	7	7	7	6	3.0	2.9	2.9	1.51	2.6	1.5	4.8
	1	9	7	7	7	7	7	7	2.9	2.9	2.9	2.78	3.0	2.9	4.8
$f_4(x)$	1.3	6	4	5	5	4	5	4	2.0	1.8	3.1	2.9	1.9	2.9	2.2
	1	6	4	4	5	5	4	4	2.0	2.0	2.6	3.0	2.9	3.5	2.3
	0.5	6	4	4	4	4	4	4	2.0	2.5	3.1	2.5	2.4	2.7	2.4
	−1	8	7	6	7	7	7	7	2.0	3.0	3.0	2.98	3.0	3.0	2.9
$f_5(x)$	4	6	4	4	5	4	5	4	1.9	2.4	2.5	2.9	2.2	2.9	2.3
	1	5	4	4	4	4	4	4	1.9	2.9	2.9	2.9	2.9	2.9	2.9
	0.5	7	5	5	7	9	8	7	2.0	3.0	3.0	3.0	3.1	2.9	2.8
	−4	6	4	5	4	5	5	5	1.9	2.1	2.9	2.05	2.9	2.9	2.9
$f_6(x)$	−3	8	6	6	8	6	6	6	1.9	2.8	2.9	1.99	2.9	2.9	2.9
	−2	6	4	4	6	5	5	5	1.9	2.8	2.9	1.99	2.9	2.9	2.9
	−1	9	7	6	13	15	13	D	2.0	3.0	3.0	1.99	2.9	2.9	D

NC – Not Convergent, D – Divergence, x_0– Initial point and COC – Computational Order of Convergence

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