Moments of the Reliability, *R* = *P*(*Y*<*X*), As a Random Variable

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ABSTRACT

The estimation of the reliability R = P (Y<X), when X and Y are independent and identically distributed random variables, has been extensively studied in the literature. Distribution functions such as Normal, Burr Type X, Exponential, Gamma, Weibull, Logistic, and Extreme Value have been considered. The expressions to estimate R have been expressed as function of positive parameters from the assumed distributions. No article yet has considered the expression for R as a random variable by itself. In this paper we will do so and consider some distributions that have been introduced in the literature to estimate it. The objective of this paper is to derive the probability density function (pdf) of R, under the assumption that the parameter of the distribution under investigation is itself a random variable with some specified pdf. Once the pdf of R is derived, its moments and other properties are investigated. Several distributions including the Exponential, Chi-Square, Gamma, Burr Type X and Weibull are assumed for this research and are investigated. Furthermore, two cases are considered for each distribution. It has been found that the moment generating function techniques for finding the moments do not work.

I. INTRODUCTION AND BACKGROUND

The product reliability seems, finally, becoming the top priority for the third millennium and a technically sophisticated customer. Manufacturers and all other producing entities are sharpening their tools to satisfy such customer. Estimation of the reliability has become a concern for many quality control professionals and statisticians. Let *Y* represent the random value of a stress that a device (or a component) will be subjected to in service and *X* represent the strength that varies from product to product in the population of devices. The device fails at the instant that the stress applied to it exceeds the strength and functions successfully whenever *Y* is less than *X*. Then, the reliability (or the measure of reliability) *R* is defined as *P*(*Y* < *X*), that is, the probability that a randomly selected device functions successfully. Algebraic forms of *R*, for different distributions, have been studied in the literature. Among those distributions considered are the normal, exponential, gamma, Weibull, Pareto, and recently extreme value family by Nadarajah (2003). Haghighi and Shayib (2009) have considered the logistic and extreme value distributions. In all cases, *X* and *Y* are assumed to be independent random variables. If $H_Y(y)$ and $f_X(x)$ are the cumulative distribution function (cdf) and the probability density function (pdf) of *Y* and *X*, respectively, then it is well known that

$$R = P(Y < X) = \int_{-\infty}^{\infty} H_Y(z)\, f_X(z)\, dz. \tag{1}$$

The estimation of *R* is a very common concern in statistical literature in both distribution-free and parametric cases. Enis and Geisser (1971) studied Bayesian approach to estimate *R*. Different distributions have been assumed for the random variables *X* and *Y*. Downton (1973) and Church and Harris (1970) have discussed the estimation of *R* in the normal and gamma cases, respectively. Studies of stress-strength model and its generalizations have been gathered in Kotz et al. (2003). Gupta and Lvin (2005) studied the monotonicity of the failure rate and mean residual life function of the generalized log-normal distribution.

Out of 12 different forms of cdf Burr (1942) introduced for modeling survival data, Burr Type X and Burr Type XII distributions have received an extensive attention. Awad and Gharraf (1986) used the Burr Type X model to simulate a comparison of three estimates for *R*, namely, the minimum variance unbiased, the maximum likelihood, and the Bayes estimators. They also studied the sensitivity of the Bayes estimator to the prior parameters. Ahmad et al. (1997) used the same assumptions except they further assumed that the scale parameters are known. Then, they used maximum likelihood (MLE), Bayes and empirical techniques to deal with the estimation of *R* in such a case. They compared the three methods of estimation using the Monte-Carlo simulation. Additionally, they presented a comparison among the three estimators and some characterization of the distribution. The first characteristic they studied was based on the recurrence relationship between two successively conditional moments of a certain function of the random variable, whereas the second was given by the conditional variance of the same function. Surles and Padgett (1998) introduced the scaled Burr Type X distribution and name it as Scaled Burr Type X or the Generalized Rayleigh Distribution (GRD). They considered the inference on *R* when *X* and *Y* are independently distributed Burr Type X random variables, discussed the existing and new results on the estimation of *R*, using MLE, and introduced a significance test for *R*. They also presented Bayesian inference on *R* when the parameters are assumed to have independent gamma distribution. They, further, offered an algorithm for finding the highest posterior density interval for *R* and using different methods of estimation, calculated the value of *R* under the assumed distributions.

Baklizi and Abu-Dayyeh (2006) have considered the problem of estimating $R=(Y<X)$ when *X* and *Y* have independent exponential distributions with parameters θ and λ, respectively. Also, Baklizi and El-Masri (2004) had considered the same problem as in Baklizi and Abu Dayyeh (2006), but with a common location parameter μ. Surles and Padgett (1998) considered the inference on *R* when *X* and *Y* are independent and identically distributed as Burr-type-*X* random variables with parameters θ and λ, respectively. In addition, Raqab and Kundu (2009) considered the scaled Burr Type X distribution for comparing different estimators of *R*. Moreover, the expression for *R* came out as is specified in

$$R=P(Y<X)=\frac{1}{1+\frac{\lambda}{\theta}}, \tag{2}$$

where the parameter λ is of the stress *Y* and θ is that of the strength *X*. It is to noted here that *R* increases as the ratio λ/θ decreases, and, thus, the parameterλ of the stress *Y* needs to be much smaller than θ of the strength *X*. Shayib (2005) had considered the effect of the sample size and the ratio between the parameters on the estimation of *R*, when *X* and *Y* shared the Burr Type X distribution. Haghighi sand Shayib (2008), considered the Weibull distribution when estimating *R*. A common application of the Weibull distribution is to model the lifetimes of components such as bearings, ceramics, capacitors, and dielectrics. Haghighi sand Shayib (2009) also considered the estimation of *R* in the case of the logistic distribution.

Generally, the *cdf* and *pdf* of a two-parameter logistic distribution are given, respectively, as:

$$F_X(x)=\frac{1}{1+e^{-\frac{x-\alpha}{\beta}}} \text{ and } f_X(x)=\frac{e^{-\frac{x-\alpha}{\beta}}}{\beta\left(1+e^{-\frac{x-\alpha}{\beta}}\right)^2},\quad -\infty<x,\,\alpha<\infty,\,\beta>0, \tag{3}$$

where α and β are the location and scale parameters, respectively. We consider the unknown-one-parameter logistic distributions for both *X* and *Y*, that is, we assume that the location parameter α is 0. Hence, *cdf* and *pdf* are, respectively, as follow:

$$H_Y(y)=\frac{1}{1+e^{-\frac{y}{\beta_1}}} \text{ and } h_Y(y)=\frac{e^{-\frac{y}{\beta_1}}}{\beta_1\left(1+e^{-\frac{y}{\beta_1}}\right)^2},\quad -\infty<y<\infty,\,\beta_1>0, \tag{4}$$

$$F_X(x)=\frac{1}{1+e^{-\frac{x}{\beta_2}}} \text{ and } f_X(x)=\frac{e^{-\frac{x}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x}{\beta_2}}\right)^2},\quad -\infty<x<\infty,\,\beta_2>0. \tag{5}$$

Haghighi sand Shayib (2009) proved the following Theorem:

Theorem: When X and Y are independent random variables with logistic probability distribution and the location parameter $\alpha = 0$, then

$$R(\beta_1,\, \beta_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{1 + \dfrac{m}{n+1}\dfrac{\beta_2}{\beta_1}}. \tag{6}$$

They also considered the estimation of R in the case of the extreme value distribution. They assumed that both X and Y are independent and each has an unknown-one-parameter extreme-value type 1 (or double exponential or Gumbel-type) distribution. Without loss of generality, it can be assumed that the location parameter is zero for each of X and Y. Hence, in this case, the *cdf* and *pdf* for Y and X are, respectively, as follows:

$$F_Y(y;\,\theta_1) = e^{-\frac{y}{\beta_1}} \text{ and } f(y;\,\theta_1) = \frac{1}{\theta_1} e^{-\frac{y}{\beta_1}}\, e^{-e^{\frac{-y}{\beta_1}}},\quad -\infty < y < \infty,\, \theta_1 > 0, \tag{7}$$

and

$$H_X(x;\,\theta_2) = e^{-e^{\frac{-x}{\theta_2}}} \text{ and } h(x;\,\theta_2) = \frac{1}{\theta_2} e^{-\frac{x}{\theta_2}}\, e^{-e^{\frac{-x}{\theta_2}}},\quad -\infty < x < \infty,\, \theta_2 > 0. \tag{8}$$

Moreover, Haghighi sand Shayib (2009) proved the following Theorem:

Theorem: Given (7) and (8), if

$$k\frac{\theta_2}{\theta_1} < 1,\;\; k = 0, 1, 2, \cdots,$$

then,

$$R(\theta_1, \theta_2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Gamma\left(k\frac{\theta_2}{\theta_1} + 1\right). \tag{9}$$

II. R AS A RANDOM VARIABLE

We have seen that R has been expressed as a ratio of one positive parameter over the sum of two positive parameters. This ratio has appeared under the assumption of exponential, Burr Type X, gamma, and Wiebull distributions. In the cases of logistic and extreme-value distributions, an infinite series presentation of R has been derived; see Haghighi sand Shayib (2009). Thus, in the remaining of the sequel we will assume

$$R = \frac{U}{U+V}, \tag{10}$$

where $U > 0$, $V > 0$ and that U and V are independent random variables.

We will consider the following distribution functions: the exponential, Chi-Square, gamma, Burr Type X, and Weibull. Without loss of generality and for simplifying the presentation, the one parameter in the considered distributions will be denoted by β. As a special case of the gamma distribution, we will consider the distributions that correspond to the exponential and Chi Square. This will be dealt with in section 3 for the exponential distribution as the underlying assumption for the variables that are involved in R. The Chi-Square distribution will be taken as the underlying one in section 4. In Section 5, we will investigate a special case of the gamma distribution. Conclusion remarks are summarized in the last section.

III. EXPONENTIAL DISTRIBUTIONS FOR *U* AND *V*

We will assume that U and V are independent and identically distributed random variables that share an exponential distribution, but with different parameters, namely,

$$U \sim Exp(\beta_1) \text{ and } V \sim Exp(\beta_2),$$

where β_1 and β_2 are positive. In this case the probability density functions are known as:

$$f_1(u) = \begin{cases} \frac{1}{\beta_1} e^{\frac{-u}{\beta_1}}, & u > 0 \\ 0, & u \le 0, \end{cases} \tag{11}$$

and

$$f_2(v) = \begin{cases} \frac{1}{\beta_2} e^{\frac{-v}{\beta_2}}, & v > 0 \\ 0, & v \le 0, \end{cases} \tag{12}$$

Since *U* and *V* are independent, their joint probability density function will be

$$g(u,v) = \begin{cases} \frac{1}{\beta_1\,\beta_2} e^{-\left(\frac{u}{\beta_1} + \frac{v}{\beta_2}\right)} & u, v > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

In this case, $g(u,v) > 0$, $\forall (u,v)$ such that $u > 0$ and $v > 0$. We are interested in *R* as a function of *U* and *V* in the form (10).

Let us define

$$R_1 = \frac{u}{u+v} = h_1(u,v) \text{ and } R_2 = u + v = h_2(u,v). \tag{14}$$

This choice of R_2 yields an inverse transformation of the following form

$$U = R_1 R_2 = h_1^{-1}(R_1, R_2) \text{ and } V = R_2(1 - R_1) = h_2^{-1}(R_1, R_2). \tag{15}$$

The Jacobian of this transformation will be

$$J = \det \begin{pmatrix} R_2 & R_1 \\ -R_2 & 1 - R_1 \end{pmatrix} = R_2. \tag{16}$$

Thus, the joint probability density function of R_1 and R_2 is given by

$$f(R_1, R_2) = \begin{cases} \frac{R_2}{\beta_1\,\beta_2} e^{-\left[\frac{\beta_2 R_1 R_2 + \beta_1 R_2 (1 - R_1)}{\beta_1\,\beta_2}\right]}, & R_1,\ R_2 > 0,\ R_2(1 - R_1) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

From (17), the marginal density functions for R_1 and R_2. For instance, for R_1 we have

$$h(R_1) = \frac{\beta_1 \beta_2}{\left[(\beta_2 - \beta_1) R_1 + \beta_1\right]^2}, \quad 0 < R_1 < 1,\ \beta_1,\ \beta_2 > 0. \tag{18}$$

Note that when the two parameters are equal $h(R_1) = 1$, and R_1 is uniformly distributed over (0, 1). This situation can arise when the stress and strength, that the component is exposed to, share the same density function that follow the exponential, Burr Type X, or the Weibull distribution. This in turn shows the result that was calculated by Shayib (2005) that the value of *R* is 1/2, and the first moment will not exceed that. Moreover, when the two parameters are equal, then R_2 will have a gamma density with parameters $\alpha = 2$ and β, the common parameter of the assumed distributions.

Now after establishing the probability density function for *R*, we will find its moments. Relation (18) is the density function of *R*. Thus, we have

$$E(R) = \mu = \int_0^1 r h(r)\, dr = \beta_1 \beta_2 \int_0^1 \frac{r}{[(\beta_2 - \beta_1) r + \beta_1]^2}\, dr$$

$$= \frac{\beta_1 \beta_2}{(\beta_2 - \beta_1)^2} \left[\ln \left(\frac{\beta_2}{\beta_1}\, e^{-\frac{\beta_2 - \beta_1}{\beta_2}} \right) \right], \qquad \frac{\beta_2}{\beta_1}\, e^{-\frac{\beta_2 - \beta_1}{\beta_2}} > 1. \tag{19}$$

From (19) we will have

$$\left[E(R)\right]^2 = \mu^2 = \frac{\beta_1^2 \beta_2^2}{(\beta_2 - \beta_1)^4} \left[\ln \left(\frac{\beta_2}{\beta_1}\, e^{-\frac{\beta_2 - \beta_1}{\beta_2}} \right) \right]^2 \tag{20}$$

and from (18) we have

$$E(R^2) = \int_0^1 r^2 h(r)\, dr = \beta_1 \beta_2 \int_0^1 \frac{r^2}{[(\beta_2 - \beta_1) r + \beta_1]^2}\, dr. \tag{21}$$

Thus, the variance of *R* can be found from (20) and (21).

Subsequently, other moments can be found. It is to be noted here that the Moment Generating Function technique to find the moments is a forbidden task.

IV. CHI-SQUARE DISTRIBUTION FOR BOTH *U* AND *V*

In this section, we will assume that *U* and *V* are iid random variables that share the Chi square distribution when the parameters are different, i.e., with degrees of freedoms d_1 and d_2, $d_1 \neq d_2$. It can be shown that the pdf of *R* will have a Beta density function with parameters and $d_2/2$,$d_1/2$, that is

$$g(r) = \begin{cases} r^{\left[\frac{d_1}{2} - 1\right]} (1 - r)^{\left[\frac{d_2}{2} - 1\right]} \dfrac{\Gamma\left(\dfrac{d_1}{2} + \dfrac{d_2}{2}\right)}{\Gamma\left(\dfrac{d_1}{2}\right)\Gamma\left(\dfrac{d_2}{2}\right)}, & 0 < r < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

From (22), the mean and the variance are given by

$$\mu = \frac{d_1}{d_1 + d_2} \tag{23}$$

and

$$\sigma^2 = \frac{2 d_1 d_2}{\left(d_1 + d_2 + 2\right)\left(d_1 + d_2\right)^2}, \tag{24}$$

respectively.

When the two Chi square distributions share the same degrees of freedom, then *R* will have a Beta distribution with parameters *d*/2 and *d*/2, with the following probability density function:

$$g(r) = \begin{cases} r^{\left[\frac{d}{2} - 1\right]} (1 - r)^{\left[\frac{d}{2} - 1\right]} \dfrac{\Gamma\left(\dfrac{d}{2} + \dfrac{d}{2}\right)}{\Gamma\left(\dfrac{d}{2}\right)\Gamma\left(\dfrac{d}{2}\right)}, & 0 < r < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

In this case, the mean and variance of *R* are well known and are given by

$$\mu = \frac{1}{2} \text{ and } \sigma^2 = \frac{1}{4d + 4},$$

see Hogg and Tanis (1988), for instance.

We note that the moment generating function for the Beta distribution cannot be expressed in a close form. Thus, to find other moments, we will appeal to integration methods.

V. GAMMA DISTRIBUTION FOR BOTH *U* AND *V*

Although the gamma distribution has two identifying parameters, by fixing one at a time, two more distributions will emerge. Using the notation that if the random variable *X* has a gamma distribution with parameters α and β, that is, $X \sim G(\alpha, \beta)$. It is well known that when $\alpha = 1$, G will be the exponential distribution with parameter β.

Now, in *G* we assume $\beta = 1$. Thus, if *U* and *V*, each has a gamma distribution with the same parameter, say α, we see that the pdf for both *U* and *V* will, respectively, be

$$f_1(u) = \begin{cases} \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u}, & 0 < u < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

and

$$f_2(v) = \begin{cases} \frac{1}{\Gamma(\alpha)} v^{\alpha-1} e^{-v}, & 0 < v < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Since *U* and *V* are independent, it is clear that their joint density function will be

$$h(u,v) = \begin{cases} \frac{1}{\left[\Gamma(\alpha)\right]^2} (uv)^{\alpha-1} e^{-(u+v)}, & 0 < u,\, v < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Thus, in this case the joint density functions of R_1 and R_2 is given by

$$h(r_1, r_2) = \begin{cases} \dfrac{r_1^{\alpha-1} r_2^{2\alpha-1} (1-r_1)^{\alpha-1}}{\left[\Gamma(\alpha)\right]^2} e^{-r_2}, & r_2 > 0,\, 0 < r_1 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

In accordance with Theorem 1, Section 2.4 of Hogg and Craig (1995), the random variables R_1 and R_2 are independent. The Marginal pdf of R_1 is given by

$$g(r_1) = \begin{cases} \dfrac{r_1^{\alpha-1} (1-r_1)^{\alpha-1} \Gamma(2\alpha)}{\left[\Gamma(\alpha)\right]^2}, & 0 < r_1 < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

This pdf is a Beta distribution with parameters α and α. Thus, the mean and variance of *R* are given, respectively, by

$$\mu = \frac{1}{2} \text{ and } \sigma^2 = \frac{1}{4\alpha + 4}, \tag{30}$$

When the parameters of the assumed gamma distributions are different, that is, with α_1 and α_2, the pdf of *R* will take the following form:

$$g(r_1) = \begin{cases} r_1^{\left[\frac{\alpha_1}{2}-1\right]} (1-r_1)^{\left[\frac{\alpha_2}{2}-1\right]} \dfrac{\Gamma\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right)}{\Gamma\left(\frac{\alpha_1}{2}\right)\Gamma\left(\frac{\alpha_2}{2}\right)}, & 0 < r_1 < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

That is, *R* will have a Beta distribution with parameters $\alpha_1/2$ and $\alpha_2/2$. Hence, the mean and the variance are given by

$$\mu = \frac{\alpha_1}{\alpha_1+\alpha_2} \text{ and } \sigma^2 = \frac{2\alpha_1\alpha_2}{(\alpha_1+\alpha_2+2)(\alpha_1+\alpha_2)^2}, \tag{32}$$

see Hogg and Tanis 1988.

Note that the moment generating function for the Beta distribution cannot be expressed in a close form for finding other moments.

VI. WEIBULL DISTRIBUTION FOR *U* AND *V*

In this section we consider the case that the distributions of *U* and *V* are independent Weibull random variables, sharing the same known shape parameter α, but unknown and different scale parameters, namely, $U \sim \text{Weibull}(\alpha,\beta_1)$ and $V \sim \text{Weibull}(\alpha,\beta_2)$. The Weibull distribution is commonly used for a model for life lengths and in the study of breaking strengths of materials because of the properties of its failure rate function. Resistors used in the construction of an aircraft guidance system have life time lengths that follow a Weibull distribution with $\alpha = 2$ and $\beta = 10$ (with measurements in thousands of hours).

In this paper we consider the Weibull distribution with two parameters, namely, α and β, where α is the shape parameter while $\beta^{1/\alpha}$ is the scale parameter of the distribution. The probability density function of the Weibull distribution that we will use based on this setup is

$$f(u) = \begin{cases} \dfrac{\alpha}{\beta_1} u^{\alpha-1} e^{-\frac{u^\alpha}{\beta_1}}, & u,\alpha,\beta_1 > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

$$f(v) = \begin{cases} \dfrac{\alpha}{\beta_2} v^{\alpha-1} e^{-\frac{v^\alpha}{\beta_2}}, & v,\alpha,\beta_2 > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

$$g(u,v) = \begin{cases} \alpha^2\,\beta_1\,\beta_2\,(uv)^{\alpha-1}\,e^{-(\beta_1 u+\beta_2 v)}, & \alpha,\,\beta_1,\,\beta_2,\,u,\,v>0, \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

Thus, from (14) – (16) we have:

$$g(r_1,r_2) = \begin{cases} \alpha^2\,\beta_1\,\beta_2\,r_1 r_2^{2\alpha-1}\,(1-r_1)^{\alpha-1} e^{-r_2^\alpha\left[\beta_1 r_1^\alpha+\beta_2(1-r_1)^{\alpha-1}\right]}, & \alpha,\,\beta_1,\,\beta_2,\,r_2>0,\ 0<r_1<1, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

When $\beta_1 = \beta_2 \equiv \beta$, (36) becomes

$$g(r_1,r_2) = \begin{cases} \alpha^2\,\beta^2\,r_1 r_2^{2\alpha-1}\,(1-r_1)^{\alpha-1} e^{-\beta r_2^\alpha\left[r_1^\alpha+(1-r_1)^{\alpha-1}\right]}, & \alpha,\,\beta,\,r_2>0,\ 0<r_1<1, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Hence,

$$h(r) = \frac{\alpha r^{\alpha-1}(1-r)^{\alpha-1}}{\left[r^\alpha+(1-r)^\alpha\right]^2}, \qquad 0<r<1. \tag{38}$$

Note:

(i) For $\alpha = 1$, certainly, *R* is uniformly distributed over the interval (0, 1). Hence, the mean and the variance of *R*, in this case, are, respectively, given by 1/2 and 1/12.

(ii) When $\alpha = 2$, we obtain

$$h(r) = \begin{cases} \frac{2r(1-r)}{\left[r^2+(1-r)^2\right]^2}, & 0 < r < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

It can be shown that *h*(*r*) given in (39) is a pdf for 0 < *r* < 1. Hence, the mean and the variance of *R* are given as $1/2$ and $(\pi - 3)/4$, respectively.

(iii) In general, for all values of α, we have the following Theorem:

Theorem: The function in (38), for any α, is a pdf over the range of *r*.

Proof:

It is clear that the function in (38) is positive for all assumed values of *r*. To prove that the integral of $h(r)$, over the range of *r* is 1, let $r = \sin^2 t,\ 0 \le t \le \pi/2$. Then, $dr = 2\sin t\ \cos t\, dt$. Hence, we have the following:

$$\begin{aligned} \int_0^{\pi/2} h(r)\, dr &= \int_0^{\pi/2} \frac{2\alpha(\sin t)^{2(\alpha-1)}\,(\cos t)^{2(\alpha-1)} \sin t\ \cos t}{(\sin^{2\alpha} t + \cos^{2\alpha} t)^2}\, dt \\ &= \int_0^{\pi/2} \frac{2\alpha(\sin t)^{2\alpha-1}\,(\cos t)^{2\alpha-1}}{(\sin^{2\alpha} t + \cos^{2\alpha} t)^2}\, dt \\ &= \int_0^{\pi/2} \frac{2\alpha(\sin t)^{2\alpha-1}\,(\cos t)^{2\alpha-1}}{\cos^{4\alpha} t(\tan^{2\alpha} t + 1)^2}\, dt \\ &= \int_0^{\pi/2} \frac{2\alpha(\sin t)^{2\alpha-1}}{\cos^{2\alpha+1} t(\tan^{2\alpha} t + 1)^2}\, dt \\ &= \int_0^{\pi/2} \frac{2\alpha(\tan t)^{2\alpha-1}}{(\tan^{2\alpha} t + 1)^2}\ \sec^2 t\, dt \\ &= -\left[\frac{1}{1 + \tan^{2\alpha} t}\right]_0^{\pi/2} \\ &= 1. \end{aligned}$$

VII. BURR TYPE X DISTRIBUTION FOR BOTH *U* AND *V*

We now assume that *U* and *V* are iid random variables that share a Burr Type X distribution with a common parameter β. Thus, the density functions of *U* and *V* are, respectively, as:

$$f_1(u) = \begin{cases} 2\beta u e^{-u^2}(1 - e^{-u^2})^{\beta-1}, & u,\ \beta > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{40}$$

and

$$f_2(v) = \begin{cases} 2\beta v e^{-v^2}(1 - e^{-v^2})^{\beta-1}, & v,\ \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

Hence, the joint density is,

$$g(u,v)=\begin{cases}4\beta^2uve^{-(u^2+v^2)}\left[(1-e^{-u^2})(1-e^{-v^2})\right]^{\beta-1}, & u,v,\beta>0,\\ 0, & \text{otherwise,}\end{cases} \tag{42}$$

Once again, note that from (14) and (15), there are many values of (*u*, *v*) that will give the same value for R_1.

From (16) we have the joint probability density functions of R_1 and R_2 as

$$g(r_1,r_2)=\begin{cases}4\beta^2r_1r_2^3(1-r_1)e^{-\left[r_1^2r_2^2+r_2^2(1-r_1)^2\right]}\left[\left(1-e^{-r_1^2r_2^2}\right)\left(1-e^{-r_2^2(1-r_1)^2}\right)\right]^{\beta-1}, & \\ & 0<r_1<1, r_2>0,\\ 0, & \text{otherwise.}\end{cases} \tag{43}$$

For $\beta=1$, (43) becomes

$$g(r_1,r_2)=\begin{cases}4r_1r_2^3(1-r_1)e^{-\left[r_1^2r_2^2+r_2^2(1-r_1)^2\right]}, & 0<r_1<1, r_2>0,\\ 0, & \text{otherwise.}\end{cases} \tag{44}$$

Thus,

$$h(r)=\begin{cases}\dfrac{2r(1-r)}{\left[r^2+(1-r)^2\right]^2}, & 0<r_1<1, r_2>0,\\ 0, & \text{otherwise.}\end{cases} \tag{45}$$

It is easy to show that (45) is a pdf when 0 < *r* <1. Hence, again *E*(*R*) = 1/2 and the variance of *R* is $(\pi-3)/4$.

When $\beta=2$, the function in (44), will take the form

$$g(r_1,r_2)=\begin{cases}16r_1r_2^3(1-r_1)e^{-r_2^2\left[r_1^2+(1-r_1)^2\right]}\left[1-e^{-r_1^2r_2^2}-e^{-r_2^2(1-r_1)^2}+e^{-r_2^2[r_1^2+(1-r_1)^2]}\right], & \\ & 0<r_1<1, r_2>0,\\ 0, & \text{otherwise.}\end{cases} \tag{46}$$

Letting $K=r_1^2+(1-r_1)^2$, then $g(r_1,r_2)$ can be expressed as

$$g(r_1,r_2)=\begin{cases}16r_1r_2^3(1-r_1)e^{-Kr_2^2}-16r_1r_2^3(1-r_1)e^{-r_2^2(K+r_1^2)} & \\ -16r_1r_2^3(1-r_1)e^{-r_2^2[K+(1-r_1)^2]}+16r_1r_2^3(1-r_1)e^{-2Kr_2^2}, & \\ & 0<r_1<1, r_2>0,\\ 0, & \text{otherwise.}\end{cases} \tag{47}$$

Thus, in this case,

$$h(r)=8r(1-r)\left[\frac{1}{K_1^2}+\frac{1}{(K_1+r^2)^2}+\frac{1}{\left[K_1+(1-r)^2\right]^2}+\frac{1}{4K_1^2}\right], \qquad 0<r<1, \tag{48}$$

where $K_1=r^2+(1-r)^2$.

Theorem: The function $h(r)$ in (48) is a pdf over the range of *R*.

Proof:

It is clear that for $0 < r < 1$, $h(r)$ given in (48) is positive. To show that the integral is 1 we proceed as follows. Let us write $h(r)$ as the following sum:

$$h(r) = g_1(r) + g_2(r) + g_3(r) + g_4(r),$$

where

$$g_1(r) = \frac{8r(1-r)}{K_1^2}, \; g_2(r) = \frac{8r(1-r)}{(K_1 + r_2^2)^2},$$
$$g_3(r) = \frac{8r(1-r)}{[K_1 + (1-r)^2]^2}, \text{ and } g_4(r) = \frac{2r(1-r)}{K_1^2}.$$

It is to be noted that $g_1(r) = 4h(r)$, where *h*(*r*) is as expressed in (45). Thus, the integral of $g_1(r)$ over the range of *r* has 4 as its value. Moreover, $g_4(r) = h(r)$, and, hence, the integral of $g_4(r)$ over the range of *r* is 1. Now for integrals of $g_2(r)$ and $g_3(r)$, let $x = r - \frac{1}{3}$. Thus, using this change of variable and the partial fractions technique, we will have the integrals of $g_2(r)$ and $g_3(r)$ equal to 2. This shows that the integral of *h*(*r*) is 1 and that completes the proof.

VIII. CONCLUSION

Different distributions were considered for the investigation *R* = *P*(*Y* < *X*), when R was taken as a random variable itself, and the parameters of the assumed distributions were taken as random variables themselves. The ratio of the parameters has a greater effect on the value of *R* than the sample size involved, especially when the parameters are equal. Even though the stress and the strength share the same distribution, the parameter of the stress has to be very small compared with that of the strength, and, thus, the ratio will contribute positively for the value of *R*. Two cases were investigated under the taken distributions when the two parameters are equal and when they are not. Regardless of the assumed distribution, the mean of *R* is 1/2 when the parameters are equal. The moment generating function technique, to find the moments did not work in the cases that were discussed in this paper. Additional work and investigation to find the moments of *R* is under way. It should be noted that when the parameters of *U* and *V* are equal, integrating their joint pdf is not easy; obviously, it will be more difficult when the parameters are unequal.

BIBLIOGRAPHY

[1] Ahmad, K. E., Fakhry, M. E. and Jaheen, Z. F. (1997). *Empirical Bayes' estimation of P(Y<X) and characterizations of Burr-Type X model*, Journal of Statistical Planning and Inference, 64, 297-308.

[2] Awad, A. M. and Gharraf, M. K. (1986). *Estimation of P(Y<X) in the Burr case: a Comparative Study*, Commun. Statist, Simula, 15, 389 – 403.

[3] Burr, I. W. (1942). Cumulative *frequency distribution*, Ann. Math. Statist, 13, 215 – 232.

[4] Church, J. D. and Harris, B., 1970, *The estimation of reliability from stress-strength Relationships*, Technometrics, 12, 49-54.

[5] Downton, F. (1973). *The estimation of Pr (Y<X) in the normal case*, Technometrics, 15, 551-558.

[6] Gupta and Lvin (2005). Mathematical and Computer Modeling (Elsevier), 42, 939 – 946.

[7] Haghighi, A. M. and Shayib, M. A. (2008). Shrinkage Estimators for Calculating Reliability, Weibull case, Journal of Applied Statistical Science, Vol. 17, Issue 2, pp. 219-233.

[8] Haghighi, A. M. and Shayib, M. A. (2009). Reliability Computation Using Logistic and extreme –Value Distributions, International Journal of statistics and economics, (Formerly Known by the "Bulletin of Statistics and economics"), Vol. 4 pp. 54-73.

[9] Hogg, R. V, and Tanis, E. A. (1988). Probability and Statistical Inference, 3rd Edition, MacMillan.

[10] Kotz, S., Lumelskii, Y., and Pensky, M. (2003). T*he Stress-Strength Model and its Generalizations*, World Science Press, Singapore.

[11] Nadarajah, S. (2003). *Reliability for Extreme Value Distributions*, Mathematical and Computer Modeling, 37, 915 – 922.

[12] Raqab, Mohammad Z. and Kundu, Debasis (2005). *Comparison of Different Estimators of P(Y<X) for a Scaled Burr Type X Distribution*, Communications in Statistics – Simulation and Computation (Taylor & Francis), 34, pp. 564 – 483

[13] Shayib, M. A. (2005). *Effects of Parameters and Sample Size on the Estimation of P(Y<X), A Simulation Study, Part I*, JSM Proceedings, 142 – 144.

[14] Surles, J. G. and Padgett, W.J. (1998). Inference for *P (Y < X)* in the Burr Type *X* Model, *Journal of Applied Statistical Science*, 7, 225-238.