On The Stability and Accuracy of Some Runge-Kutta Methods of Solving Second Order Ordinary Differential Equations

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ABSTRACT.
This paper seeks numerical solutions to second order differential equations of the form $y'' = f(x, y, y')$ with initial value, $y(x_0) = y_0$, $y'(x_0) = y'_0$ using different Runge-Kutta methods of order two. Two cases of Explicit Runge-Kutta method were derived and their stability was determined, this is then implemented. The results were compared with the Euler's method for accuracy.

KEYWORDS. *Euler's Method, Ordinary Differential Equation, Runge-Kutta method, Stability and Taylor series*

I. INTRODUCTION.

Solutions to Differential equation have over the years been a focus to Applied Mathematics. The question then becomes how to find the solutions to those equations. The range of Differential Equations that can be solved by straight forward analytical method is relatively restricted [12]. Even solution in series may not always be satisfactory, either because of the slow convergence of the resulting series or because of the involved manipulation in repeated stages of differentiation [9].Runge-Kutta methods are among the most popular ODE solvers. They were first studied by Carle Runge and Martin Kutta around 1900. Modern developments are mostly due to John Butcher in the 1960s [4]. The Runge-Kutta method is not restricted to solving only first-order differential equations but Runge-Kutta methods are also used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations [14]. The higher order equations can be solved by considering an equivalent system of first order equations. However, it is also possible to develop direct single steps methods to solve higher order equation.

Definition: We define an explicit Runge-Kutta method with n slopes by the following equations [10].

$$y'' = f(x, y, y') \qquad x \in [x_0, b] \tag{1}$$

$$y_{n+1} = y_n + hy'_n + \sum_{i=1}^{n} r_i k_i \tag{2}$$

$$k_j = \frac{h^2}{2!} f\left(x_n + q_2 h, y_n + q_2 hy'_n + h^2 \sum_{i=1}^{j} q_{ij}, y'_n + \frac{c_{21}}{h} k_1\right) \tag{3}$$

Where a_{ij} ,$1 \le i, j \le n, r_1, r_2 .... \; r_n$ are arbitrary.

II. DERIVATION OF SECOND ORDER RUNGE-KUTTA METHOD.

To derive the Runge-Kutta methods for second order ordinary differential equation of the form $y'' = f(x, y, y')$ with the initial condition, $y(x_0) = y_0$, $y'(x_0) = y'_0$.

Let us define

$$k_1 = \frac{h^2}{2!} f(x_n, y_n, y'_n) \tag{2.1}$$

$$k_2 = \frac{h^2}{2!} f\left(x_n + q_2 h, y_n + q_2 h y'_n + q_{21} k_1, y'_n + \frac{c_{21}}{h} k_1 \right) \tag{2.2}$$

$$y_{n+1} = y_n + h y'_n + r_1 k_1 + r_2 k_2 \tag{2.3}$$

$$y'_{n+1} = y'_n + \frac{1}{h}(r'_1 k_1 + r'_2 k_2) \tag{2.4}$$

Where $q_2, q_{21}, b_{21}, r_1, r_2, r'_1$, and r'_2 are arbitrary constants to be determined. Using Taylor series expansion form equation (2.3) and (2.4) gives.

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \frac{h^4}{4!} y^{iv}_n + \ldots \tag{2.5}$$

$$y'_{n+1} = y'_n + h y''_n + \frac{h^2}{2!} y'''_n + \frac{h^3}{3!} y^{iv}{}_n + \ldots \tag{2.6}$$

$$y''_n = f(x_n, y(x_n), y'(x_n)) \tag{2.7}$$

$$y'''_n = \left(f_x + y' f_y + f f_{y'} \right)_n \tag{2.8}$$

$$y^{iv}_n = [f_{xx} + y'^2_n f_{yy} + f^2 f_{y'y'} + 2 y' f f_{xy} + 2 y' f f_{yy'} + 2 f f_{xy'} + f_{y'} \left(f_x + y' f_y + f f_{y'} \right) + f f_y]_n \tag{2.9}$$

We re-write equations (2.7), (2.8) and (2.9) as

$$y''_n = f_n$$

$$y'''_n = Df_n$$

$$y^{iv}_n = D^2 f_n + f_{y'} Df_n + f_n f_y$$

Where $D = \frac{\partial}{\partial x} + y'_n \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y'}$

So, equations (2.5) and (2.6) becomes

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} f_n + \frac{h^3}{3!} Df_n + \frac{h^4}{4!}\left(D^2 f_n + f_{y'} Df_n + f_n f_y\right) + \ldots \qquad (2.10)$$

$$y'_{n+1} = y'_n + hf_n + \frac{h^2}{2!} Df_n + \frac{h^3}{3!}\left(D^2 f_n + f_{y'} Df_n + f_n f_y\right) + \ldots \qquad (2.11)$$

Simplifying k_2, we have

$$\frac{2k_2}{h^2} = f_n + h\left(q_2 f_x + q_2 y'_n f_y + \frac{1}{2} c_{21} f_n f_{y'}\right) +$$
$$\frac{h^2}{2!}\left(q_2^2 f_{xx} + q_2^2 y_n'^2 f_{yy} + \frac{1}{4} c_{21}^2 f_{y''y'} + 2q_2^2 y'_n f_{xy} + q_2 c_{21} f_n f_y\right) + O\left(h^3\right)$$

$$k_2 = \frac{h^2}{2} f_n + \frac{h^3}{2} q_2 Df_n + \frac{h^4}{4}\left(q_2^2 Df_n + q_{21} f_n f_{y'}\right) + O\left(h^3\right) \qquad (2.12)$$

Where we have used $q_2 = \frac{1}{2} c_{21}$ $\qquad (2.13)$

The substitution of k_1 and k_2 in equation (2.3) and (2.4) yield

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}\left(r_1 + r_2\right) f_n + \frac{h^3}{2} q_2 r_2 Df_n + \frac{h^4}{4}\left(r_2 q_2^2 D^2 f_n + r_2 q_{21} f_n f_y\right) + O\left(h^5\right) \qquad (2.14)$$

$$y'_{n+1} = y'_n + \frac{h}{2}\left(r'_1 + r'_2\right) f_n + \frac{h^2}{2} q_2 r'_2 Df_n + \frac{h^3}{4}\left(r'_2 q_2^2 D^2 f_n + r'_2 q_{21} f_n f_y\right) + O\left(h^4\right) \qquad (2.15)$$

We compare the coefficient of equations (2.14) and (2.15) with equations (2.10) and (2.11) .to obtain.

$$r_1 + r_2 = 1, \quad r'_1 + r'_2 = 2, \quad q_2 r_2 = \frac{1}{3}, \quad q_2 r'_2 = 1 \qquad (2.16)$$

The coefficient of h^4 in y_{n+1} and h^3 in y'_{n+1} of equations (2.14) and (2.15) will not match with the corresponding coefficients in equation (2.10) and (2.11) for any choice of q_2, q_{21}, r_2 and r'_2. Thus the local truncation error is $O\left(h^4\right)$ in y and $O\left(h^2\right)$ in y'. A solution of equation (2.12) and (2.16) may be considered for different values of r_1. Here we consider two cases of r_1.

Case I: $r_1 = \frac{1}{2}$

Then, from equation (2.16), we get

$$r_2 = \frac{1}{2}, \quad q_2 = \frac{2}{3} = q_{21}, \quad b_{21} = \frac{4}{3}, \quad r'_1 = \frac{1}{2}, \quad r'_2 = \frac{3}{2}$$

If the function f is independent of y' then we can construct a Runge-Kutta method in which the local truncation error in y and y' is $O\left(h^4\right)$. Here we obtain,

$$r_1 + r_2 = 1, \quad r_1' + r_2' = 2, \quad q_2 r_2 = \frac{1}{3}, \quad q_2 r_2' = 1, \quad q_2^2 r_2' = \frac{2}{3}, \quad q_{21} r_2' = \frac{2}{3}$$

with the solution

$$r_1 = \frac{1}{2}, \quad q_2 = \frac{2}{3}, \quad q_1 = \frac{1}{2}, \quad r_1' = \frac{1}{2}, \quad r_2' = \frac{3}{2}$$

Thus, the Runge-kutta method for the second order initial value problem

$y'' = f(x, y, y')$ with initial value, $y(x_0) = y_0$, $y'(x_0) = y_0'$

Thus from equation (2.1), (2.2) , (2.3) and (2.4) we have

$$k_1 = \frac{h^2}{2!} f(x_n, y_n, y_n'), \quad k_2 = \frac{h^2}{2!} f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hy_n' + \frac{4}{9}k_1, y_n' + \frac{4}{3h}k_1\right) \qquad (2.17)$$

$$y_{n+1} = y_n + hy_n' + \frac{1}{2}(k_1 + k_2), \quad y_{n+1}' = y_n' + \frac{1}{2h}(k_1 + 3k_2) \qquad (2.18)$$

Case II: $r_1 = 0$

From equation (2.16), we have

$$r_2 = 1, q_2 = \frac{1}{3}, r_2' = 1, r_1' = 1, b_{21} = \frac{2}{3}, q_{21} = \frac{1}{9}$$

Therefore,

$$k_1 = \frac{h^2}{2!} f(x_n, y_n, y_n'), \quad k_2 = \frac{h^2}{2!} f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hy_n' + \frac{1}{9}k_1, y_n' + \frac{2}{3h}k_1\right) \qquad (2.19)$$

$$y_{n+1} = y_n + hy_n' + k_2, \quad y_{n+1}' = y_n' + \frac{1}{h}(k_1 + k_2) \qquad (2.20)$$

The two methods derived are explicit second order Runge-Kutta.

III. STABILITY ANALYSIS.

While numerically solving an initial value problem for ordinary differential equations, an error is introduced at each integration step due to the inaccuracy of the formula. Even when the local error at each step is small, the total error may become large due to accumulation and amplification of these local errors. This growth phenomenon is called numerical instability [11].

We shall discuss the stability of the Runge-Kutta method in (2.18) and (2.20)

Let us consider the differential equation

$$y'' = \alpha y \qquad (3.1)$$

Subject to initial condition

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad x \in [x_0, b]$$

Where α is a real number

We shall consider the case $\alpha = -k^2$ and $\alpha = k^2$ while $\alpha = 0$ give a trival solution. Recall that the second order ordinary differential equation is of the form $y'' = f(x, y, y')$

Then, $k_1 = \frac{h^2}{2} y''$ from equation (3.1)

Substitute this (3.1) in equation (2.17), we have

$$k_1 = \frac{h^2}{2}\alpha y_n, \quad k_2 = \left(\frac{h^2}{2}\alpha + \frac{h^4}{9}\alpha^2\right) y_n + \frac{h^3}{3}\alpha y'_n \qquad (3.2)$$

Substituting the equation (3.2) into equation (2.18), we have $y_{n+1} = \left(1 + \frac{h^2}{2}\alpha + \frac{h^4}{18}\alpha^2\right) y_n + \left(h + \frac{h^3}{6}\alpha\right) y'_n$,

$$y'_{n+1} = \left(\alpha h + \frac{h^3}{6}\alpha^2\right) y_n + \left(1 + \frac{h^2}{2}\alpha\right) y'_n \qquad (3.3)$$

Let us now consider the case when $\alpha = -k^2$. The solution in this case is oscillating.

We, therefore consider the eigenvalues of the matrix $\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ which is given by

$$\lambda = \frac{(q_{11} + q_{22}) \pm \left[(q_{11} - q_{22})^2 + 4 q_{12} q_{21}\right]^{1/2}}{2}$$

If $\lambda = \lambda_1, \lambda_2$, then

$$\lambda_1, \lambda_2 = \frac{(q_{11} + q_{22}) \pm \left[(q_{11} - q_{22})^2 + 4 q_{12} q_{21}\right]^{1/2}}{2} \qquad (3.4)$$

To determine the value of q_{11}, q_{12}, q_{21}, and q_{22}, we consider

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} \qquad (3.5)$$

We can re-write equation (3.5) as

$$y_{n+1} = q_{11} y_n + q_{12} y'_n$$

$$y'_{n+1} = q_{21} y_n + q_{22} y'_n \qquad (3.6)$$

when $\alpha = -k^2$ we have

$$q_{11} = 1 - \frac{h^2}{2}k^2 + \frac{h^4}{18}k^4, q_{12} = h - \frac{h^3}{6}k^2, q_{21} = -hk^2 + \frac{h^3}{6}k^4, q_{22} = 1 - \frac{h^2}{2}k^2 \qquad (3.7)$$

Substituting equation (3.7) in (3.4), we have

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\left(2 - h^2k^2 + \frac{h^4k^4}{18}\right) \pm \left[\left(\frac{hk}{18}\right)^2 \left(h^6k^6 - 36h^4k^4 + 432h^2k^2 - 1296\right)\right]^{1/2}\right]$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\left(2 - h^2k^2 + \frac{h^4k^4}{18}\right) \pm \left[\left(\frac{hk}{18}\right)^2 \left(h^2k^2 - 4.44044737\right)\left(h^4k^4 - 2\eta h^2k^2 + \eta^2 + \varphi^2\right)\right]^{1/2}\right]$$

Where $\eta = 15.779763$ and $\varphi = 6.5467418$

Computing λ_1 and λ_2 as functions of h^2k^2, we find that the roots have unit modulus $0 \le h^2k^2 \le 4.44$. Thus, the stability interval of the Runge-Kutta method (2.18) is $0 \le h^2k^2 \le 4.44$. which also show that is of order 2.

For $\alpha = k^2$, the solutions of (3.1) are exponential in nature and can be written in the matrix form as

$$\begin{bmatrix} y(x_n) \\ y'(x_n) \end{bmatrix} = e^{(x-x_0)b}\begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}$$

And $b = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix}$ for point $x = x_0 + nh = x_n$, this solution becomes

$$\begin{bmatrix} y(x_n) \\ y'(x_n) \end{bmatrix} = \begin{bmatrix} 1 + \frac{h^2k^2}{2} + \frac{h^4k^4}{24} + \ldots.. & h + \frac{h^3k^2}{6} + \ldots\ldots \\ hk^2 + \frac{h^3k^4}{6} + \ldots\ldots\ldots & 1 + \frac{h^2k^2}{2} + \frac{h^4k^4}{24} + \ldots.. \end{bmatrix}\begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}$$

And then,

$$\lim_{h \to 0} \frac{1}{h^4}\left[e^{bh} - \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\right] = \begin{bmatrix} -\frac{1}{72}k^4 & 0 \\ 0 & \frac{1}{24}k^4 \end{bmatrix}$$

The relative error for the method under discussion in case of a large number of integration interval (large x, small h) are to be considered.

The maximum eigenvalue of the matrix.

$$\begin{bmatrix} 1+\frac{h^2k^2}{2}+\frac{h^4k^4}{18} & h+\frac{h^3k^2}{6} \\ hk^2+\frac{h^3k^4}{6} & 1+\frac{h^2k^2}{2} \end{bmatrix}$$

Is obtained as

$$\lambda = \frac{1}{2}\left[(q_{11}+q_{22})+\left[(q_{11}-q_{22})^2+4q_{12}q_{21}\right]^{\frac{1}{2}}\right]$$

$$\lambda = 1+hk+\frac{1}{2}h^2k^2+\frac{1}{6}h^3k^3+\frac{1}{36}h^4k^4+\ldots$$

And therefore the relative error is given by

$$F_\infty \approx \frac{hk-\log\lambda}{h} = \frac{\log(e^{hk})-\log\lambda}{h} = \frac{1}{h}\log\left[1+\frac{e^{hk}-\lambda}{\lambda}\right] \approx \frac{1}{72}h^3k^3$$

We shall now consider the stability of the Runge-Kutta method (2.20) using differential equation (3.1). Equation (2.19) becomes

$$k_1 = \frac{h^2}{2}\alpha y_n, \qquad k_2 = \left(\frac{h^2}{2}\alpha+\frac{h^4}{36}\alpha^2\right)y_n+\frac{h^3}{6}\alpha y'_n \qquad (3.8)$$

Substituting the equation (3.9) into equation (2.21), we have

$$y_{n+1} = \left(1+\frac{h^2}{2}\alpha+\frac{h^4}{36}\alpha^2\right)y_n+\left(h+\frac{h^3}{6}\alpha\right)y'_n, \quad y'_{n+1} = \left(\alpha h+\frac{h^3}{36}\alpha^2\right)y_n+\left(1+\frac{h^2}{6}\alpha\right)y'_n \qquad (3.9)$$

We now consider the case when $\alpha = -k^2$. The eigenvalues is given by

$$\lambda_1,\lambda_2 = \frac{1}{2}\left[\left(2-\frac{h^2k^2}{3}+\frac{h^4k^4}{36}\right)\pm\left[\left(\frac{hk}{36}\right)^2\left(h^6k^6-48h^4k^4+1152h^2k^2-5184\right)\right]^{\frac{1}{2}}\right]$$

$$\lambda_1,\lambda_2 = \frac{1}{2}\left[\left(2-\frac{h^2k^2}{3}+\frac{h^4k^4}{36}\right)\pm\left[\left(\frac{hk}{36}\right)^2\left(h^2k^2-5.688510232\right)\left(h^4k^4-2\eta h^2k^2+\eta^2+\varphi^2\right)\right]^{\frac{1}{2}}\right]$$

Where $\eta = 21.15574488$ and $\varphi = 26.23461232$

Computing λ_1 and λ_2 as functions of h^2k^2, the roots have unit modulus $0 \le h^2k^2 \le 5.69$ Thus, the stability interval of the Runge-Kutta method (2.21) is $0 \le h^2k^2 \le 5.69$ and is of order 2.

For $\alpha = k^2$, the solutions of (3.1) are exponential in nature as above in case I.

The maximum eigenvalue of the matrix.

$$\begin{bmatrix} 1+\frac{h^2k^2}{2}+\frac{h^4k^4}{36} & h+\frac{h^3k^2}{6} \\ hk^2+\frac{h^3k^4}{36} & 1+\frac{h^2k^2}{6} \end{bmatrix}$$

Is obtained as

$$\lambda = \frac{1}{2}\left[(q_{11}+q_{22})+\left[(q_{11}-q_{22})^2+4q_{12}q_{21}\right]^{\frac{1}{2}}\right]$$

$$\lambda = 1+hk+\frac{1}{3}h^2k^2+\frac{1}{6}h^3k^3+\frac{1}{72}h^4k^4+\ldots$$

The relative error is given by

$$F_\infty \approx \frac{hk-\log\lambda}{h} = \frac{\log(e^{hk})-\log\lambda}{h} = \frac{1}{h}\log\left[1+\frac{e^{hk}-\lambda}{\lambda}\right] \approx \frac{1}{108}h^3k^3$$

IV. IMPLEMENTATION OF THE METHOD.

Consider the initial value problem [13]

$y''-y'=x$ Subjected to initial condition $y(0)=1,\ y'(0)=2,\ h=0.1$ (4.1)

The complementary solution is $y_c = A+Be^x$ and the particular solution gives $y_p = -\frac{1}{2}x^2 - x$ Using the initial condition, we have $A=-2$ and $B=3$

The general solution become

$$y = -2+3e^x-\frac{1}{2}x^2-x$$

Numerical solutions are preferred to the derived cases I and case II of explicit second order Runge-Kutta method of the IVP in (4.1), obtaining numerical solutions for values of x up to and including $x=1$ with a step size of 0.1 as found in Table 1 and Table 2.

Table1: Solutions of case I second ordered Runge-Kutta methods with $h=0.1$

X	Numerical solution y(x)	Analytical solution Y(x)	Absolute Error
0	1.000000000	1.000000000	0.000000000
0.1	1.210500000	1.210512754	1.288×10^{-5}
0.2	1.444127500	1.444208274	8.08×10^{-5}
0.3	1.704360888	1.704576424	2.16×10^{-4}
0.4	1.995043782	1.995474094	4.30×10^{-4}
0.5	2.320423378	2.321163813	7.40×10^{-4}
0.6	2.685192833	2.686356400	1.16×10^{-3}
0.7	3.094538080	3.096258121	1.72×10^{-3}
0.8	3.554189578	3.556622784	2.43×10^{-3}

0.9	**4.070479483**	**4.073809330**	$\mathbf{3.33} \times 10^{-3}$
1.0	**4.650404829**	**4.654845484**	$\mathbf{4.44} \times 10^{-3}$

Figure 1:

Table2: Solutions of case II second ordered Runge-Kutta methods with $h = 0.1$

X	Numerical solution y(x)	Analytical solution Y(x)	Absolute Error
0	**1.000000000**	**1.000000000**	**0.000000000**
0.1	**1.210500000**	**1.210512754**	$\mathbf{1.28} \times 10^{-5}$
0.2	**1.443075833**	**1.444208274**	$\mathbf{1.13} \times 10^{-3}$
0.3	**1.700988542**	**1.704576424**	$\mathbf{3.59} \times 10^{-3}$
0.4	**1.987830710**	**1.995474094**	$\mathbf{7.64} \times 10^{-3}$
0.5	**2.307560166**	**2.321163813**	$\mathbf{1.36} \times 10^{-2}$
0.6	**2.664537116**	**2.686356400**	$\mathbf{2.18} \times 10^{-2}$
0.7	**3.063565056**	**3.096258121**	$\mathbf{3.27} \times 10^{-2}$
0.8	**3.509935837**	**3.556622784**	$\mathbf{4.67} \times 10^{-2}$
0.9	**4.009479314**	**4.073809330**	$\mathbf{6.43} \times 10^{-2}$
1.0	**4.568618045**	**4.654845484**	$\mathbf{8.62} \times 10^{-2}$

Figure 2:

Euler's method

Euler's method is one of many methods for generating numerical solutions to differential equations. Besides, most of the other methods that might be discussed are refinements of Euler's method. This method is implemented and compared its accuracy and the error with the method in section 4.

The Euler's method generalized in the form [11].

$$y_{n+1} = y_n + hf\left(x_n, y_n, y'_n\right) \qquad (5.1)$$

$$y'_{n+1} = y'_n + hf\left(x_n, y_n, y'_n\right)$$

(5.2) Consider the initial value problem in equation (4.1)

$$y'' - y' = x$$

Subjected to initial condition

$$y(0) = 1, y'(0) = 2, h = 0.1$$

Table3: Solutions of Euler's methods with $h = 0.1$

X	Numerical solution y(x)	Analytical solution Y(x)	Absolute Error
0	**1.000000000**	**1.000000000**	**0.000000000**
0.1	**1.200000000**	**1.210512754**	$\mathbf{1.05} \times 10^{-2}$
0.2	**1.430000000**	**1.444208274**	$\mathbf{1.42} \times 10^{-2}$
0.3	**1.693000000**	**1.704576424**	$\mathbf{1.16} \times 10^{-2}$
0.4	**1.992300000**	**1.995474094**	$\mathbf{7.25} \times 10^{-3}$
0.5	**2.331530000**	**2.321163813**	$\mathbf{1.04} \times 10^{-2}$
0.6	**2.714680000**	**2.686356400**	$\mathbf{2.83} \times 10^{-2}$
0.7	**3.146145000**	**3.096258121**	$\mathbf{4.99} \times 10^{-2}$
0.8	**3.630756500**	**3.556622784**	$\mathbf{7.41} \times 10^{-2}$
0.9	**4.173829150**	**4.073809330**	$\mathbf{1.00} \times 10^{-1}$

1.0	**4.781209065**	**4.654845484**	1.26×10^{-1}

Figure 3:

V. CONCLUSION.

Investigation carried out on some explicit second ordered Runge-Kutta method in this paper has shown that the stability interval of the Runge-Kutta method in case I is $0 \le h^2 k^2 \le 4.44$ and the stability interval of the Runge-Kutta method in case II is $0 \le h^2 k^2 \le 5.69$. It is clear that case I is more stable than case II of the derived Runge-Kutta methods. The two methods are shown to be accurate, efficient and general in application for sufficiently solution of $y(x)$. The result obtained in the present work demonstrate the effectiveness and superiority for the solution of second order ordinary differential equation which gave a very high accuracy when compared with exact solution.

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