Efficient Triple Connected Domination Number of a Graph

G. Mahadevan[1] N. Ramesh[2] Selvam Avadayappan[3] T. Subramanian[4]

[1]Dept. of Mathematics, Anna University : Tirunelveli Region, Tirunelveli - 627 007
[2]Udaya School of Engineering, Kanyakumari - 629 204
[3]Dept.of Mathematics, VHNSN College, Virudhunagar - 626 001
[4]Research Scholar, Dept. of Mathematics, Anna University : Tirunelveli Region, Tirunelveli - 627 007

ABSTRACT

The concept of triple connected graphs with real life application was introduced in [16] by considering the existence of a path containing any three vertices of a graph G. In [3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. A subset S of V of a nontrivial connected graph G is said to be triple connected dominating set, if S is a dominating set and the induced sub graph <S> is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by γ_{tc}. A subset S of V of a nontrivial graph G is said to be an efficient dominating set, if every vertex is dominated exactly once. The minimum cardinality taken over all efficient dominating sets is called the efficient domination number and is denoted by γ_e. In this paper we introduce new domination parameter efficient triple connected domination number of a graph with real life application. A subset S of V of a nontrivial connected graph G is said to be an efficient triple connected dominating set, if S is an efficient dominating set and the induced subgraph <S> is triple connected. The minimum cardinality taken over all efficient triple connected dominating sets is called the efficient triple connected domination number and is denoted by γ_{etc}. We also determine this number for some standard graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters are also investigated.

KEY WORDS: *Domination Number, Triple connected graph, Triple connected domination number, Efficient triple connected domination number.*

AMS (2010): *05C69*

I. INTERDUCTION

REAL LIFE APPLICATION OF TRIPLE CONNECTED DOMINATION NUMBER[3] AND EFFICIENT TRIPLE CONNECTED DOMINATION NUMBER

The concept of triple connected graphs with real life application was introduced in [16] by considering the existence of a path containing any three vertices of a graph G. In [3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. In this paper we introduce new domination parameter efficient triple connected domination number of a graph with real life application.

Suppose we want to locate the three armed forces (Army, Air, Navy) along the border of a country. First we construct a graph whose nodes (Vertices) indicating the critical areas and the edges indicating the paths which are connecting the critical areas of the border. Check whether this graph is triple connected or not.

Case (i)

If the constructed graph is triple connected, then any of the three armed forces (Army, Air, Navy) can locate their strategy in any of the three nodes (i.e.,) the critical areas so that if anyone of the three armed force needs help from the other two forces then it can be call them immediately.

Case (ii)

If the constructed graph is not triple connected, then construct a triple connected dominating set (or efficient triple connected dominating set) in the graph and then the three armed forces (Army, Air, Navy) can locate their strategy in any of the three nodes in the triple connected dominating set (or efficient triple connected dominating set) of the graph so that the critical areas dominated by the triple connected dominating set can contact at least one force directly (exactly one in the case of efficient triple connected dominating set) and the other two forces can be called by that force if necessary.

Case (iii)

If the constructed graph is neither triple connected nor we can find a triple connected dominating set in the graph, then do some necessary corrections in the construction (either by increasing the edges or by increasing the nodes) and make the constructed graph is triple connected or there can find a triple connected dominating set (or efficient triple connected dominating set).

II. INTRODUCTION

By a **graph** we mean a finite, simple, connected and undirected graph G(V, E), where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has p vertices and q edges. **Degree** of a vertex v is denoted by d(v), the **maximum degree** of a graph G is denoted by Δ(G). We denote a **cycle** on p vertices by $C_{p,}$ a **path** on p vertices by P_p, and a **complete graph** on p vertices by K_p. A graph G is **connected** if any two vertices of G are connected by a path. A maximal connected subgraph of a graph G is called a **component** of G. The number of components of G is denoted by ω(G). The **complement** $\overline{G}$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A **tree** is a connected acyclic graph. A **bipartite graph** (or **bigraph**) is a graph whose vertex set can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A **complete bipartite graph** is a bipartite graph where every vertex of V_1 is adjacent to every vertex in V_2. The complete bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, is denoted by $K_{m,n}$. A **star**, denoted by $K_{1,p-1}$ is a tree with one root vertex and p – 1 pendant vertices. The **open neighbourhood** of a set S of vertices of a graph G, denoted by N(S) is the set of all vertices adjacent to some vertex in S and N(S) ∪ S is called the **closed neighbourhood** of S, denoted by N[S]. A **cut – vertex (cut edge)** of a graph G is a vertex (edge) whose removal increases the number of components. A **vertex cut**, or **separating set** of a connected graph G is a set of vertices whose removal results in a disconnected. The **connectivity** or **vertex connectivity** of a graph G, denoted by κ(G) (where G is not complete) is the size of a smallest vertex cut. A connected subgraph H of a connected graph G is called a **H -cut** if ω(G – H) ≥ 2. The **chromatic number** of a graph G, denoted by χ(G) is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colour. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. A **Nordhaus -Gaddum-type** result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [13].

A subset S of V is called a **dominating set** of G if every vertex in V − S is adjacent to at least one vertex in S. The **domination number** γ(G) of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a **connected dominating set** of G if the induced sub graph <S> is connected. The minimum cardinality taken over all connected dominating sets is the **connected domination number** and is denoted by γ_c.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [17, 18]. Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph J. et. al.,[16] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be **triple connected** if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In[3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. A subset S of V of a nontrivial graph G is said to be a **triple connected dominating set**, if S is a dominating set and the induced subgraph <S> is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the **triple connected domination number** of G and is denoted by $\gamma_{tc}(G)$. Any triple connected dominating set with γ_{tc} vertices is called a γ_{tc} -set of G. In[4, 5, 6, 7, 8, 9, 10, 11, 12] G. Mahadevan et. al., was introduced **complementary triple connected domination number, complementary perfect triple connected domination number**, **paired triple connected domination number, restrained triple connected domination number, triple connected two domination number, dom strong triple connected domination number, strong triple connected domination number, weak triple connected domination number, triple connected complementary tree domination number of a graph** and investigated new results on them.In this paper, we use this idea to develop the concept of **efficient triple connected dominating set** and **efficient triple connected domination number of a graph.**

Theorem 1.1 [16] A tree T is triple connected if and only if $T \cong P_p$; $p \geq 3$.

Theorem 1.2 [16] A connected graph G is not triple connected if and only if there exists a H -cut with $\omega(G - H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components C_1, C_2, and C_3 of G – H.

Notation 1.3 [16] Let *G* be a connected graph with *m* vertices $v_1, v_2, ..., v_m$. The graph obtained from *G* by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, ..., n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

III. EFFICIENT TRIPLE CONNECTED DOMINATION NUMBER OF A GRAPH

Definition 2.1 A subset *S* of *V* of a nontrivial graph *G* is said to be a *efficient triple connected dominating set*, if *S* is a efficient dominating set and the induced subgraph <*S*> is triple connected. The minimum cardinality taken over all efficient triple connected dominating sets is called the *efficient triple connected domination number* of *G* and is denoted by $\gamma_{etc}(G)$. Any efficient triple connected dominating set with γ_{etc} vertices is called a γ_{etc} -set of *G*.

Example 2.2 For the graph G_1 in figure 2.1, $S = \{v_1, v_2, v_7\}$ forms a γ_{etc} -set of G_1. Hence $\gamma_{etc}(G_1) = 3$.

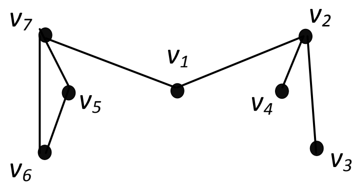


Figure 2.1 : Graph with γ_{etc} = 3.

G_1

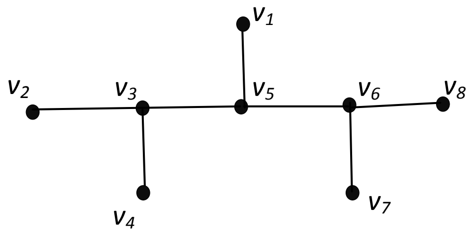
Observation 2.3 Efficient Triple connected dominating set (γ_{etc} -set or etcd set) does not exists for all graphs and if exists, then $\gamma_{etc}(G) \geq 3$.

Example 2.4 For K_5, there does not exists any efficient triple connected dominating set.

Remark 2.5 Throughout this paper we consider only connected graphs for which efficient triple connected dominating set exists.

Observation 2.6 The complement of the efficient triple connected dominating set need not be a efficient triple connected dominating set.

Example 2.7 For the graph G_2 in figure 2.2, $S = \{v_3, v_5, v_6\}$ forms a efficient triple connected dominating set of G_2. But the complement $V - S = \{v_1, v_2, v_4, v_7, v_8\}$ is not a efficient triple connected dominating set.



G_2

Figure 2.2: Graph in which V – S is not an etctd set

Observation 2.8 Every efficient triple connected dominating set is a dominating set but not conversely.

Example 2.9 For the graph G_3 in figure 2.3, $S = \{v_1, v_2, v_3\}$ forms the efficient triple connected dominating set as well as the dominating set of G_3. For the graph G_4 in figure 2.3, $S = \{v_7\}$ is a dominating set but not an efficient triple connected dominating set of G_3.

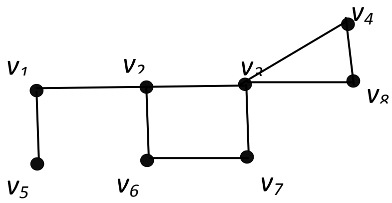
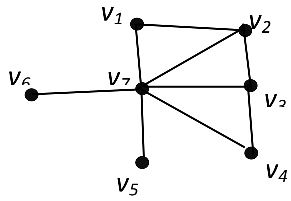


Figure 2.3



G₃ ***G₄***

Theorem 2.10 If the induced subgraph of each connected dominating set of *G* has more than two pendant vertices, then *G* does not contain an efficient triple connected dominating set.
Proof The proof follows from *theorem 1.2*.

Exact value for some standard graphs

1) For any cycle of order $p \geq 3$, $\gamma_{etc}(C_p) = \begin{cases} p & \text{if } p < 5 \\ p-2 & \text{if } p \geq 5. \end{cases}$
2) For any path of order $p \geq 3$, $\gamma_{etc}(P_p) = \begin{cases} p & \text{if } p = 3 \\ p-1 & \text{if } p = 4 \\ p-2 & \text{if } p \geq 5. \end{cases}$
3) For any complete graph of order $p \geq 3$, $\gamma_{etc}(K_p) = p$.
4) For any complete bipartite graph of order $p \geq 4$, $\gamma_{etc}(K_{m,n}) = p$, (where $m + n = p$).
5) For any star of order $p \geq 3$, $\gamma_{etc}(K_{1,n}) = 3$, (where $n+1 = p$).
6) For any wheel graph of order $p \geq 3$, $\gamma_{etc}(W_p) = p$.

Exact value for some special graphs

1) The **Möbius–Kantor graph** is a symmetric bipartite cubic graph with 16 vertices and 24 edges as shown in figure 2.4.

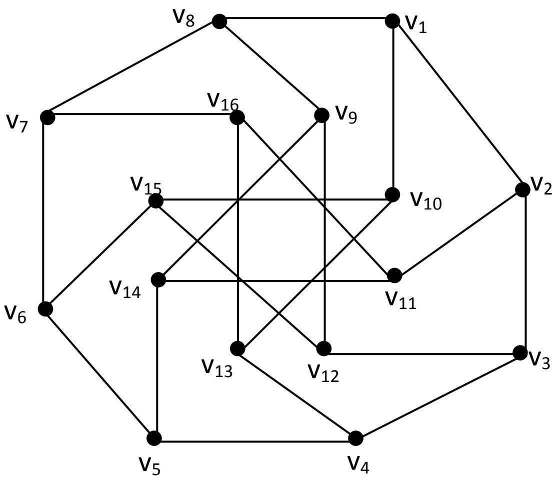


Figure 2.4

For the Mobius – Kantor graph G, $\gamma_{etc}(G) = 8$. Here S = {v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8} is an efficient triple connected dominating set.

2) The **Desargues graph** is a distance-transitive cubic graph with 20 vertices and 30 edges as shown in figure 2.5.

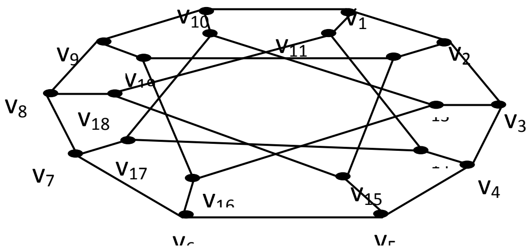


Figure 2.5

For any Desargues graph G, γ_{etc} (G) = 10. Here S = {v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}} is an efficient triple connected dominating set.

3) The **Chvátal graph** is an undirected graph with 12 vertices and 24 edges as shown in figure 2.6.

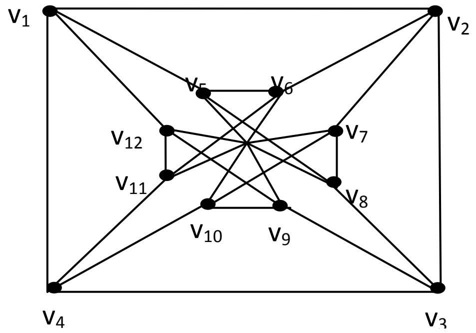


Figure 2.6

For the Chvátal graph G, γ_{etc}(G) = 4. Here S = {v_1, v_2, v_3, v_4} is an efficient triple connected dominating set.

4) The **Dürer graph** is an undirected cubic graph with 12 vertices and 18 edges as shown below in figure 2.7.

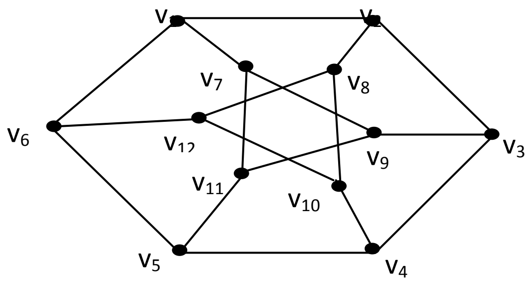


Figure 2.7

For the Dürer graph G, γ_{etc}(G) = 6. Here S = {v_1, v_2, v_3, v_4, v_5, v_6} is an efficient triple connected dominating set.

5) Any path with a pendant edge attached at each vertex as shown in figure 2.8 is called **Hoffman tree** and is denoted by P_n^+.

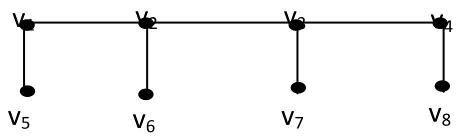


Figure 2.8

For the Hoffman tree G, γ_{etc}(G) = 4. Here S = {v_1, v_2, v_3, v_4} is an efficient triple connected dominating set.

Theorem 2.11 For any connected graph *G* with $p \geq 3$, we have $3 \leq \gamma_{etc}(G) \leq p$ and the bounds are sharp.

Proof The lower and upper bounds follows from *definition 2.1*. For C_5, the lower bound is attained and for $K_{3,5}$ the upper bound is attained.

Observation 2.12 For any connected graph G with 3 vertices, $\gamma_{etc}(G) = p$ if and only if $G \cong P_3, C_3$.

Observation 2.13 For any connected graph G with 4 vertices, $\gamma_{etc}(G) = p$ if and only if $G \cong K_4, W_4, C_4, K_4 - e$.

The Nordhaus – Gaddum type result is given below:

Theorem 2.15 Let G be a graph such that G and $\bar{G}$ have no isolates of order $p \geq 3$. Then

(i) $\gamma_{etc}(G) + \gamma_{etc}(\bar{G}) \leq 2p$

(ii) $\gamma_{etc}(G).\ \gamma_{etc}(\bar{G}) \leq p^2$.

Proof The bound directly follows from *theorem 2.11*. For path C_5, both the bounds are satisfied.

IV. RELATION WITH OTHER GRAPH THEORETICAL PARAMETERS

Theorem 3.1 For any connected graph G with $p \geq 3$ vertices, $\gamma_{etc}(G) + \kappa(G) \leq 2p - 1$ and the bound is sharp if and only if $G \cong K_p$.

Proof Let G be a connected graph with $p \geq 3$ vertices. We know that $\kappa(G) \leq p - 1$ and by *theorem 2.11*, $\gamma_{etc}(G) \leq p$. Hence $\gamma_{etc}(G) + \kappa(G) \leq 2p - 1$. Suppose G is isomorphic to K_p. Then clearly $\gamma_{etc}(G) + \kappa(G) = 2p - 1$. Conversely, Let $\gamma_{etc}(G) + \kappa(G) = 2p - 1$. This is possible only if $\gamma_{etc}(G) = p$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{etc}(G) = p$. Hence $G \cong K_p$.

Theorem 3.2 For any connected graph G with $p \geq 3$ vertices, $\gamma_{etc}(G) + \chi(G) \leq 2p$ and the bound is sharp if and only if $G \cong K_p$.

Proof Let G be a connected graph with $p \geq 3$ vertices. We know that $\chi(G) \leq p$ and by *theorem 2.11*, $\gamma_{etc}(G) \leq p$. Hence $\gamma_{etc}(G) + \chi(G) \leq 2p$. Suppose G is isomorphic to K_p. Then clearly $\gamma_{etc}(G) + \chi(G) = 2p$. Conversely, let $\gamma_{etc}(G) + \chi(G) = 2p$. This is possible only if $\gamma_{etc}(G) = p$ and $\chi(G) = p$. Since $\chi(G) = p$, G is isomorphic to K_p for which $\gamma_{etc}(G) = p$. Hence $G \cong K_p$.

Theorem 3.3 For any connected graph G with $p \geq 3$ vertices, $\gamma_{etc}(G) + \Delta(G) \leq 2p - 1$ and the bound is sharp.

Proof Let G be a connected graph with $p \geq 3$ vertices. We know that $\Delta(G) \leq p - 1$ and by *theorem 2.11*, $\gamma_{etc}(G) \leq p$. Hence $\gamma_{etc}(G) + \Delta(G) \leq 2p - 1$. For K_6, the bound is sharp.

REFERENCES

[1] E. J. Cokayne and, S. T. Hedetniemi, *Total domination in graphs*, Networks,Vol.10 (1980), 211–219.

[2] E. Sampathkumar and, H. B. Walikar, *The connected domination number of a graph*, *J.Math. Phys. Sci.*, 13 (6) (1979), 607–613.

[3] G. Mahadevan, A. Selvam, J. Paulraj Joseph and, T. Subramanian, *Triple connected domination number of a graph*, International Journal of Mathematical Combinatorics, Vol.3 (2012), 93-104.

[4] G. Mahadevan, A. Selvam, J. Paulraj Joseph, B. Ayisha and, T. Subramanian, *Complementary triple connected domination number of a graph*, Advances and Applications in Discrete Mathematics, (2012), Vol. 12(1) (2013), 39-54.

[5] G. Mahadevan, A. Selvam, A. Mydeen bibi and, T. Subramanian, *Complementary perfect triple connected domination number of a graph*, International Journal of Engineering Research and Application, Vol.2, Issue 5 (2012) , 260-265.

[6] G. Mahadevan, A. Selvam, A. Nagarajan, A. Rajeswari and, T. Subramanian, *Paired Triple connected domination number of a graph*, International Journal of Computational Engineering Research, *Vol. 2, Issue 5* (2012), *1333-1338.*

[7] G. Mahadevan, A. Selvam, B. Ayisha, and, T. Subramanian, *Triple connected two domination number of a graph*, International Journal of Computational Engineering Research Vol. 2, Issue 6 (2012),101-104.

[8] G. Mahadevan, A. Selvam, V. G. Bhagavathi Ammal and, T. Subramanian, *Restrained triple connected domination number of a graph*, International Journal of Engineering Research and Application, Vol. 2, Issue 6 (2012), 225-229.

[9] G. Mahadevan, A. Selvam, M. Hajmeeral and, T. Subramanian, *Dom strong triple connected domination number of a graph*, American Journal of Mathematics and Mathematical Sciences, Vol. 1, Issue 2 (2012), 29-37.

[10] G. Mahadevan, A. Selvam, V. G. Bhagavathi Ammal and, T. Subramanian, *Strong triple connected domination number of a graph*, International Journal of Computational Engineering Research, Vol. 3, Issue 1 (2013), 242-247.

[11] G. Mahadevan, A. Selvam, V. G. Bhagavathi Ammal and, T. Subramanian, *Weak triple connected domination number of a graph*, International Journal of Modern Engineering Research, Vol. 3, Issue 1 (2013), 342-345.

[12] G. Mahadevan, A. Selvam, N. Ramesh and, T. Subramanian, *Triple connected complementary tree domination number of a graph*, International Mathematical Forum, Vol. 8, No. 14 (2013), 659-670.

[13] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, 2008.

[14] J. Paulraj Joseph and, S. Arumugam, *Domination and connectivity in graphs*, International Journal of Management Systems, 8 (3) (1992), 233–236.

[15] J. Paulraj Joseph and, S. Arumugam, *Domination and coloring in graphs*, International Journal of Management Systems, 8 (1) (1997), 37–44.

[16] J. Paulraj Joseph, M. K. Angel Jebitha, P. Chithra Devi and, G. Sudhana, *Triple connected graphs*, Indian Journal of Mathematics and Mathematical Sciences,Vol. 8, No.1 (2012), 61-75.

[17] J. Paulraj Joseph and, G. Mahadevan, *On complementary perfect domination number of a graph*, Acta Ciencia Indica, Vol. XXXI M, No. 2. (2006), 847–853.

[18] T. W. Haynes, S. T. Hedetniemi and, P. J. Slater, *Domination in graphs*, Advanced Topics, Marcel Dekker, New York (1998).

[19] T. W. Haynes, S. T. Hedetniemi and, P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York (1998).