Integral Solutions of the Homogeneous Cubic Equation

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ABSTRACT:

The cubic equation $x^3+y^3+xy(x+y)-z^3-w^3-zw(z+w)=(x+y+z+w)X^2$ *is analysed for its non-zero integral solutions. A few interesting relations between the solutions and special numbers are exhibited.*

Keywords: *Homogeneous equation with five unknowns, Integral solutions.*

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NOTATIONS:

Special Number	Notations	Definitions
Gnomonic number	G_n	$2n-1$
Pronic number	P_n	$n(n+1)$
Star number	S_n	$6n(n-1)+1$
Octahedral number	OH_n	$\frac{n(2n^2+1)}{3}$

I. I.INTRODUCTION

Integral solutions for the homogeneous or non-homogeneous Diophantine cubic equations is an interesting concept as it can be seen from[1-2]. In [3-13] a few special cases of cubic Diophantine equations with 4 unknowns are studied. In [14-15], the cubic equation with five unknowns is studied for its non-zero integral solutions. This communication concerns with a another interesting cubic equation with five unknowns given by $x^3+y^3+xy(x+y)-z^3-w^3-zw(z+w)=(x+y+z+w)X^2$ for determining its integral solutions. A few interesting relations between the solutions are presented.

II. METHOD OF ANALYSIS

The cubic Diophantine equation with five unknowns to be solved for getting non-zero integral solutions is

$$x^3+y^3+xy(x+y)-z^3-w^3-zw(z+w)=(x+y+z+w)X^2 \quad (1)$$

On substituting the linear transformations

$$x=u+v, y=u-v, z=u+p, w=u-p \quad (2)$$

in (1) leads to

$$v^2-p^2=X^2 \quad (3)$$

2.1Pattern 1:

Equation (3) can be written as,

$$v^2=p^2+X^2 \quad (4)$$

which is satisfied by

$$\left.\begin{array}{l} v = m^2 + n^2,\ p = 2mn,\ X = m^2 - n^2 \\ v = m^2 + n^2,\ p = m^2 - n^2,\ X = 2mn \end{array}\right\} m > n > o \qquad (5)$$

Substituting (5) in (2) , the two sets of solutions satisfying (1) are obtained as follows:

SET 1:

$$x(u,m,n) = u + m^2 + n^2$$
$$y(u,m,n) = u - m^2 - n^2$$
$$z(u,m,n) = u + 2mn$$
$$w(u,m,n) = u - 2mn$$
$$X(m,n) = m^2 - n^2$$

SET 2:

$$x(u,m,n) = u + m^2 + n^2$$
$$y(u,m,n) = u - m^2 - n^2$$
$$z(u,m,n) = u + m^2 - n^2$$
$$w(u,m,n) = u - m^2 + n^2$$
$$X(m,n) = 2mn$$

2.2Pattern 2:

Equation (3) can be written as

$$p^2 + X^2 = v^2 * 1 \qquad (6)$$

Assume $\quad v = a^2 + b^2$ (7)

Write 1 as

$$1 = \frac{(1+i)^{2n}(1-i)^{2n}}{2^{2n}} \qquad (8)$$

Substituting (7) and (8) in (6) and using the method of factorization, define

$$p + iX = (a+ib)^2 \frac{(1+i)^{2n}}{2^n} \qquad (9)$$

Equating real and imaginary parts of (9),we have

$$p = (a^2 - b^2)\cos\frac{n\pi}{2} - 2ab\sin\frac{n\pi}{2}$$

$$X = (a^2 - b^2)\sin\frac{n\pi}{2} + 2ab\cos\frac{n\pi}{2}$$

The corresponding integral values of x, y, z, w and X satisfying (1) are obtained as,

$$x(u,a.b) = u + a^2 + b^2$$
$$y(u,a,b) = u - a^2 - b^2$$
$$z(u,a,b) = u + (a^2 - b^2)\cos\frac{n\pi}{2} - 2ab\sin\frac{n\pi}{2}$$
$$w(u,a,b) = u - (a^2 - b^2)\cos\frac{n\pi}{2} + 2ab\sin\frac{n\pi}{2}$$
$$X(a,b) = (a^2 - b^2)\sin\frac{n\pi}{2} + 2ab\cos\frac{n\pi}{2}$$

Properties:

1.Each of the following expression is a nasty number

i. $3[x(u,a,b) - y(u,a,b)]^2 + (-1)^n[12x(u,a,b)^2 - 3(z(u,a,b) - w(u,a,b))^2]$

ii. $6[x(u,a,b)\times y(u,a,b) + (a^2 + b^2)^2]$

2. $(2x(u,a,b), z(u,a,b) - w(u,a,b), x(u,a,b) - y(u,a,b))$ forms a Pythagorean triple.

3.If a, b are taken as the generators of the Pythagorean triangle (α,β,γ) whose sides are $\alpha = a^2 - b^2; \beta = 2ab; \gamma = a^2 + b^2$ then the product $\left[X(a,b)\sin\frac{n\pi}{2} + \left(\frac{z(u,a,b) - w(u,a,b)}{2}\right)\cos\frac{n\pi}{2}\right]\left[X(a,b)\cos\frac{n\pi}{2} - \left(\frac{z(u,a,b) - w(u,a,b)}{2}\right)\sin\frac{n\pi}{2}\right]$ represents two times its area.

4. $x(u,a,b)y(u,a,b) - z(u,a,b)w(u,a,b) \equiv 0 \pmod 2$

5. $x(u,a,b) \pm y(u,a,b) \equiv 0 \pmod 2$

2.3Pattern 3:

In (6) 1 can be written as

$$1 = \frac{(p^2 + q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2} \quad ; \qquad p > q > 0$$

Proceeding as in Pattern II

$$p + iX = (a + ib)^2 \frac{(p^2 + q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2} \tag{10}$$

Equating real and imaginary parts,

$$p = (a^2 - b^2)\frac{(p^2 - q^2)}{(p^2 + q^2)} - 4ab\frac{pq}{(p^2 + q^2)} \tag{11}$$

$$X = (a^2 - b^2)\frac{2pq}{(p^2 + q^2)} + 2ab\frac{(p^2 - q^2)}{(p^2 + q^2)} \tag{12}$$

Since our aim is to find the integral solutions, substituting $a=(p^2+q^2)A, b=(p^2+q^2)B$ in (7),(11) and (12)

$$v=(p^2+q^2)^2(A^2+B^2) \tag{13}$$

$$p=(p^2+q^2)\left[(A^2-B^2)(p^2-q^2)-4ABpq\right] \tag{14}$$

$$X=2pq(p^2+q^2)(A^2-B^2)+2(p^2+q^2)AB(p^2-q^2) \tag{15}$$

Substituting (13), and (14) in (2) and using (15) we have the integral solutions of (1) as,

$$x(u,A,B)=u+(p^2+q^2)^2(A^2+B^2)$$
$$y(u,A,B)=u-(p^2+q^2)^2(A^2+B^2)$$
$$z(u,A,B)=u+(p^2+q^2)(A^2-B^2)(p^2-q^2)-(p^2+q^2)4ABpq$$
$$w(u,A,B)=u-(p^2+q^2)(A^2-B^2)(p^2-q^2)+(p^2+q^2)4ABpq$$
$$X(A,B)=2pq(p^2+q^2)(A^2-B^2)+2(p^2+q^2)AB(p^2-q^2)$$

Properties:

1. $4pq\left[z(u,p,q)-w(u,p,q)\right]-2(p^2+q^2)x(u,p,q)\equiv 0(\text{mod }8)$

2. $2G_p(p^4-q^4)+w(u,p,p-1)-z(u,p,p-1)\equiv 0(\text{mod}8)$

3. $pq\left[z(u,p,p-1)-w(u,p,p-1)\right]-(p^2-q^2)x(u,p,p-1)+2(p^2+q^2)^3P_{p-1}=0$

4. $2(p^2+q^2)^2S_p+3\left[y(u,p,p-1)-x(u,p,p-1)\right]\equiv 0(\text{mod}4)$

5. $3OH_{pq}+z(u,p,p)-w(u,p,p)-pq\equiv 0(\text{mod }2)$

6. $w(u,p,q)-z(u,p,q)+x(u,p,q)+10(p^2+q^2)q^2$ is a perfect square.

7. Each of the following expression is a nasty number

   i. $6u\left[x(u,A,B)+y(u,A,B)+z(u,A,B)+w(u,A,B)\right]$

   ii. $6\left[x(u,A,A)-y(u,A,A)\right]$

III. CONCLUSION

To conclude one may search for other patterns of solutions and their corresponding properties.

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