Observations on the Ternary Cubic Equation $x^2 - xy + y^2 = 7z^3$

S.Vidhyalakshmi[1], M.A.Gopalan[2], A.Kavitha[3]

[1] Professor, PG and Research Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamilnadu, India

[2] Professors, PG and Research Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamilnadu, India

[3] Lecturers, PG and Research Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamilnadu, India

ABSTRACT:

The non-homogeneous cubic equation with three unknowns represented by the diophantine equation $X^2 - XY + Y^2 = 7Z^3$ *is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.*

Keywords: *Integral solutions, non-homogeneous cubic equation with three unknowns.*

M.Sc 2000 mathematics subject classification: 11D25

Notations:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

S_n : Star number of rank n

$\Pr_n$: Pronic number of rank n

j_n : Jacobsthal lucas number of rank n

J_n : Jacobsthal number of rank n

CP_n^m : Centered Pyramidal number of rank n with size m.

I. INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-17] for cubic equations with three unknowns. This communication concerns with yet another interesting equation $X^2 - XY + Y^2 = 7Z^3$ representing non-homogeneous cubic with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

II. METHOD OF ANALYSIS

The ternary non-homogeneous cubic Diophantine equation to be solved for its distinct non-zero integral solution is $x^2 - xy + y^2 = 7z^3$ (1)

2.1Pattern:1

Introduction of the transformations

$$x = u + v, \qquad y = u - v \tag{2}$$

In (1) leads to

$$u^2 + 3v^2 = 7z^3 \tag{3}$$

Let $z = a^2 + 3b^2$ (4)

Write 7 as

$$7=\frac{(5*2^{n}+i2^{n}\sqrt{3})(5*2^{n}-i2^{n}\sqrt{3})}{2^{2n+2}} \quad (5)$$

Using (4) and (5) in (3) and applying the method of factorization, define

$$u+i\sqrt{3}v=\frac{(5*2^{n}+i2^{n}\sqrt{3})}{2^{n+1}}(a+i\sqrt{3}b)^{3}$$

Equating real and imaginary parts, we get

$$u=\frac{2^{n}}{2^{n+1}}[5(a^{3}-9ab^{2})-9(a^{2}b-b^{3})] \quad (6)$$

$$v=\frac{2^{n}}{2^{n+1}}[(a^{3}-9ab^{2})+15(a^{2}b-b^{3})] \quad (7)$$

Using (6) and (7) in (2), we have

$$\left.\begin{aligned} x(a,b)&=\frac{2^{n}}{2^{n+1}}[6(a^{3}-9ab^{2})+6(a^{2}b-b^{3})]\\ y(a,b)&=\frac{2^{n}}{2^{n+1}}[4(a^{3}-9ab^{2})-24(a^{2}b-b^{3})]\\ z(a,b)&=(a^{2}+3b^{2}) \end{aligned}\right\} \quad (8)$$

Where n= 0,1, 2……..

Properties:

(1) $x(2^{n},1)=3[j_{3n}-9j_{n}+3J_{2n}-(-1)^{3n}+9(-1)^{n}+(-1)^{2n}-1]$

(2) $y(2^{n},1)=3J_{3n+1}-9j_{n+1}-3j_{2n+2}+(-1)^{3n+1}+9(-1)^{n+1}+15$

(3) $z(1,2^{n})=3j_{2n}-2$

For illustration and clear understanding, substituting n=1 in (8), the corresponding non-zero distinct integral solutions to (1) are given by

$$x(a,b)=3(a^{3}-9ab^{2})+3(a^{2}b-b^{3})]$$

$$y(a,b)=2(a^{3}-9ab^{2})-12(a^{2}b-b^{3})]$$

$$z(a,b)=(a^{2}+3b^{2})$$

Properties:

(1) $x(1,n)=3-29t_{4,n}+4t_{3,n}-2CP_{n}^{9}$

(2) $y(1,n)=3CP_{n}^{4}-62t_{3,n}+31t_{4,n}-S_{n}+14$

(3) $x(1,n)+y(1,n)+z(1,n)=CP_{n}^{12}-16t_{3,n}-37t_{4,n}+6$

(4) $x(1,n)+y(1,n)-6P_{n}^{7}+6t_{6,n}\equiv 9(\bmod 49)$

(5) $y(n,1)-2P_{n}^{8}+t_{28,n}\equiv 12(\bmod 29)$

2.2Pattern:2

Introducing the linear transformations

$$u=\alpha+3T, \qquad v=\alpha-T \quad (9)$$

Substituting in (3) we get

$$4\alpha^{2}+12T^{2}=7z^{3} \quad (10)$$

Let $z=a^{2}+12b^{2}$ (11)

Write 7 as

$$7=\frac{(4+i\sqrt{12})(4-i\sqrt{12})}{4} \quad (12)$$

Using (11) and (12) in (10), we get

$$2\alpha + i\sqrt{12}T = \frac{4 + i\sqrt{12}}{2}(a + i\sqrt{12}b)^3$$

Equating real and imaginary parts, we obtained

$$2\alpha = 2(a^3 - 36ab^2) - 6(3a^2b - 12b^3)$$

$$2T = (a^3 - 36ab^2) + 4(3a^2b - 12b^3)$$

Hence the values of x and y satisfies (1) are given by

$$x(a,b) = 3(a^3 - 36ab^2) - 2(3a^2b - 12b^3)$$

$$y(a,b) = 2(a^3 - 36ab^2) + 8(3a^2b - 12b^3)$$

$$z(a,b) = (a^2 + 12b^2)$$

Properties:

(1) $x(1,n) = 3[6CP_n^{25} - 6CP_n^{24} - 70t_{3,n} + 33t_{4,n} + 8]$

(2) $z(1,n) = t_{24,n} + 20t_{3,n} - 9t_{4,n} + 1$

(3) $y(n,1) = 3CP_n^4 + 2t_{23,n} + 2t_{5,n} - 106t_{3,n} + 53t_{4,n} - 96$

(4) $x(n,1) + y(n,1) - 10P_n^5 - t_{28,n} \equiv -72 \pmod{168}$

(5) $y(n,n) - z(n,n) + 284P_n^5 - 43t_{8,n} \equiv 0 \pmod{86}$

2.3Pattern:3

Instead of (12) we write 7 as

$$7 = \frac{(10 + i\sqrt{12})(10 - i\sqrt{12})}{16}$$

Following the procedure as presented in pattern 2 the corresponding non-zero distinct integral solutions to (1) are obtained as

$$x(a,b) = 3(a^3 - 36ab^2) + 2(3a^2b - 12b^3)$$

$$y(a,b) = 3(a^3 - 36ab^2) + 10(3a^2b - 12b^3)$$

$$z(a,b) = (a^2 + 12b^2)$$

Properties:

(1) $y(n,1) = 2CP_n^3 + t_{24,n} + 12t_{21,n} - 20t_{3,n} + 10t_{4,n} - 120$

(2) $x(1,n) = 3[-3CP_n^{16} - CP_{6,n} - 33t_{4,n} + 2]$

(3) $x(n,1) - 6P_n^5 - t_{8,n} \equiv -24 \pmod{106}$

(4) $x(n,1) - z(n,1) - 6P_n^5 - t_{6,n} \equiv -36 \pmod{107}$

(5) $z(1,n) - x(1,n) - 48P_n^5 - 48t_{6,n} \equiv -2 \pmod{42}$

2.4Pattern:4

Introducing the linear transformations

$u = \alpha - 3T, \qquad v = \alpha + T$ (13)

Let $z = a^2 + 3b^2$ (14)

Write 7 as

$7 = (2 + i\sqrt{3})(2 - i\sqrt{3})$ (15)

Substituting (13), (14) and (15) in (3) and repeating the process as in pattern2, the non-zero distinct integral solutions to (1) are obtained as

$$x(a,b) = (a^3 - 9ab^2) - 15(a^2b - b^3)$$

$$y(a,b) = -2(a^3 - 9ab^2) - 12(a^2b - b^3)$$

$$z(a,b) = a^2 + 3b^2$$

Properties:

(1) $x(n,1) - y(n,1) = 3[2CP_n^3 - CP_{4,n} + t_{20,n} - 8t_{4,n} + 2]$

(2) $x(2^n,1) - y(2^n,1) = 3[j_{3n} - 27J_n - j_{2n} - (-1)^{3n} - 9(-1)^n + 2]$

(3) Each of the following represents a nasty number:

(a) $y(3a^2,1) - x(3a^2,1) - t_{14,3a^2} + 6P^5_{3a^2} + 3$

(b) $7\{42 - 2x(a,1) - y(a,1)\}$

(c) $6\{x(a,a) - z(a,a) + 16P^5_a\}$

(d) $2\{32P^5_a - y(a,a) - z(a,a)\}$

2.5Pattern:5

Instead of (15) we write 7 as

$$7 = \frac{(8 + i2\sqrt{12})(8 - i2\sqrt{12})}{16}$$

Following the procedure as presented in pattern4, the corresponding non-zero distinct integral solutions to (1) are

$$x(a,b) = (a^3 - 36ab^2) - 10(3a^2b - 12b^3)$$
$$y(a,b) = -2(a^3 - 36ab^2) - 8(3a^2b - 12b^3)$$
$$z(a,b) = a^2 + 12b^2$$

Properties:

(1) $x(1,n) - y(1,n) = 6CP^{24}_n - t_{24,n} - 2t_{6,n} - 93t_{4,n} + 3$

(2) $x(n,1) = 2CP^3_n - 2CP_{29,n} - 16t_{3,n} + 7t_{4,n} = 122$

(3) $y(n,1) = -2[2CP^3_n - 74t_{3,n} + 37t_{4,n}] - 8[t_{8,n} + 4t_{3,n} - 2t_{4,n} - 12]$

(4) Each of the following represents a nasty number:

(a) $7\{110P^5_n - x(n,n) - z(n,n)\}$

(b) $6\{x(n,n) - y(n,n) + z(n,n) + 174P^5_n\}$

2.6Pattern:6

(1) can be written as

$$(2x - y)^2 + 3y^2 = 28z^3 \quad (16)$$

One may write 28 as

$$28 = (5 + i\sqrt{3})(5 - i\sqrt{3}) \quad (17)$$

Substituting (17) and (4) in (16), employing the method of factorization, we have

$$(2x - y) + i\sqrt{3}y = (5 + i\sqrt{3})(a + i\sqrt{3}b)^3$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} x(a,b) &= 3(a^3 - 9ab^2) + 3(a^2b - b^3) \\ y(a,b) &= (a^3 - 9ab^2) + 15(a^2b - b^3) \end{aligned}\right\} \quad (18)$$

Thus (14) and (18) represents the non-zero distinct integral solutions to (1)

Properties:

(1) $x(n,1) = 2CP^9_n + t_{8,n} - 48t_{3,n} + 24t_{4,n} - 3$

(2) $y(1,n) = -6CP^{13}_n - 2CP^9_n + 2CP^3_n - t_{20,n} - 6t_{3,n} + 2t_{4,n} + 1$

(3) $y(1,n) + 18P^7_n \equiv 1 \pmod{9}$

(4) $x(n,1) - 6P^5_n + 3 \equiv 0 \pmod{27}$

(5) $6P^3_n - y(n,1) - 3t_{10,n} \equiv -15 \pmod{2}$

III. REMARKS

(1) Let (x_0, y_0, z_0) be the initial solution of (1)

Let $\left.\begin{aligned} x &= 7^3 x_0 + h \\ y_1 &= 7^3 y_0 + h \\ z_1 &= 7^2 z_o \end{aligned}\right\}$ (19)

be the first solution of (1).
Substituting (18) in (1), we get

$h = -7^3 (x_0 + y_0)$ (20)

Using (20) in (19) we obtain the general solution as follows:

EVEN ORDERED SOLUTION:

$x_{2n} = 7^{6n} x_0,$

$y_{2n} = 7^{6n} y_0,$

$z_{2n} = 7^{4n} z_0,$

where n=1, 2, 3, …..

ODD ORDERED SOLUTION:

$x_{2n-1} = -7^{3(2n-1)} y_0,$

$y_{2n-1} = -7^{3(2n-1)} x_0,$

$z_{2n-1} = -7^{2(2n-1)} z_0,$

where n=1, 2, 3, …..

Properties:

1. $48\left[\left(\frac{z_0}{x_0}\right)\sum_{n=0}^{N-1}\left(\frac{x_{2n}}{z_{2n}}\right)\right] + 1 \equiv 0 (\mathrm{mod}\ 7)$

2. $x_0^2 z_{2n}^3 = x_{2n}^2 z_0^3$

3. Each of the following is a nasty number:

(i). $6\left[\frac{x_{2n} y_{2n} z_{2n}}{x_0 y_0 z_0}\right]$, (ii). $6\left(\frac{z_{2n}}{z_{2n-1}}\right)$, (iii). $6\left(\frac{z_0 y_{2n}}{y_0 z_{2n}}\right)$, (iv). $6\left(\frac{z_0 x_{2n}}{x_0 z_{2n}}\right)$, (v). $6\left(\frac{z_{2n}}{z_{2n-1}}\right)$

4. Each of the following is a cubic integer:

(i). $\left(\frac{x_{2n}}{x_0}\right)\left(\frac{-y_0}{x_{2n-1}}\right)$, (ii). $\left(\frac{-x_{2n}}{y_{2n-1}}\right)$, (iii). $\left(\frac{-y_{2n}}{x_{2n-1}}\right)$, (iv). $\left(\frac{y_{2n}}{y_0}\right)\left(\frac{-x_0}{y_{2n-1}}\right)$

II. Employing the solutions(x, y, z) of (1), the following relations among the special polygonal and pyramidal numbers are obtained.

1. $$\left[\frac{3P_{x-2}^3}{t_{3,x-2}}\right]^2 - \left[\frac{3P_{x-2}^3}{t_{3,x-2}}\right]\left[\frac{P_y^5}{t_{3,y}}\right] + \left[\frac{P_y^5}{t_{3,y}}\right]^2 = 7\left[\frac{6P_{z-1}^4}{t_{3,2(z-1)}}\right]^3$$

2. $$\left[\frac{6P_{x-1}^4}{t_{3,2(x-1)}}\right]^2 - \left[\frac{6P_{x-1}^4}{t_{3,2(x-1)}}\right]\left[\frac{3P_{y-2}^3}{t_{3,y-2}}\right] + \left[\frac{3P_{y-2}^3}{t_{3,y-2}}\right]^2 = 7\left[\frac{P_z^5}{t_{3,z}}\right]^3$$

3. $$\left[\frac{2P_{x-1}^8}{t_{3,2x-3}}\right]^2 - \left[\frac{2P_{x-1}^8}{t_{3,2x-3}}\right]\left[\frac{6P_y^4}{t_{3,2y+1}}\right] + \left[\frac{6P_y^4}{t_{3,2y+1}}\right]^2 = 7\left[\frac{3P_z^3}{t_{3,z+1}}\right]^3$$

4. $$\left[\frac{6P_x^4}{t_{3,2x+1}}\right]^2 - \left[\frac{6P_x^4}{t_{3,2x+1}}\right]\left[\frac{6P_{y-1}^4}{t_{3,2(y-1)}}\right] + \left[\frac{6P_{y-1}^4}{t_{3,2(y-1)}}\right]^2 = 7\left[\frac{3P_{z-2}^3}{t_{3,z-2}}\right]^3$$

IV. CONCLUSION

To conclude, one may search for other pattern of solutions and their corresponding properties.

REFERENCES

[1]. Dickson. I. E. '"History of the Theory of Numbers'', Vol 2. Diophantine analysis, New York, Dover, 2005.

[2]. Mordell .L J., "Diophantine Equations'' Academic Press, New York,1969.

[3]. Carmichael. R.D. "The Theory of numbers and Diophantine Analysis'', New York, Dover, 1959.

[4]. Gopalan.M.A.,Anbuselvi.R. Integral solutions of Ternary Cubic Diophantine Equations $x^2 + y^2 + 4N = zxy$,Pure and Applied Mathematica Sciences, Vol.LXVII(NO.1-2):107-111,March(2008).

[5]. Gopalan.M.A., Manjusomnath and Vanith N., "On Ternary Cubic Diophantine Equations $x^2 + y^2 = 2z^3$ '', Advances in Theoretical and Applied Mathematics,Vol.1(No.3)Pg.227-231(2006).

[6]. Gopalan.M.A., Manjusomnath and Vanith N., "On Ternary Cubic Diophantine Equations $x^2 + y^2 = 2z^3$ '', Acta Ciencia Indica, Vol.XXXIIIM(No.3) Pg.705-707(2007).

[7]. Gopalan.M.A.,and Pandiselvi .V., "Integral solutions of Ternary Cubic Diophantine Equations $x^2 - xy + y^2 = (k^2 - 2kz + 4)z^3$ '',Pacific Asian journal of Mathematics, Vol.2.No.1-2,Pg.91-91,2008.

[8]. Gopalan.M.A., and KaligaRani.J., "Integral solutions of $x^2 - ay^2 = (a-1)z^3$ ($a > 1$ and a is square free) '', Impact J.Sci.Tec,Vol.2(4)Pg.201-204.2008

[9]. Gopalan.M.A.,Devibala.S., and Manjusomnath., "Integral solutions of $x^3 + x + y^3 + y = 4(z-2)(z+2)$ '', Impact.J.Sci.Tech.Vol.2(2)Pg.65-69,2008

[10]. Gopalan.M.A., Manjusomnath and Vanith N., "Ternary Cubic diophantine Equation $2^{2a-1}(x^2 + y^2) = z^3$ '',Acta Ciencia Indica,Vol.XXXIVM(No.3)Pg.135-137(2008).

[11]. Gopalan.M.A., and KaligaRani.J., "Integral solutions of $x^3 + y^3 + 8k(x+y) = (2k+1)z^3$ '',Bulletin of pure and Applied Sciences,Vol.29E(No.1)Pg.95-99(2010).

[12]. Gopalan.M.A., and Janaki.G., "Integral solutions of $x^2 - y^2 + xy = (m^2 - 5n^2)z^3$ '',Antartica J.Math.,7(1)Pg.63-67(2010).

[13]. Gopalan.M.A., and Shanmuganantham.P., " On the Equation $x^2 + xy - y^2 = (n^2 + 4n - 1)z^3$ '', Bulletin of pure and Applied Sciences, Vol.29E, Pg.231-235 Issue 2(2010).

[14]. Gopalan.M.A., and Vijayasankar.A., " Integral Solutions of Ternary Cubic Equation $x^2 + y^2 - xy + 2(x+y+2) = z^3$ '',Antartica J.Math.,Vol.7(N0.4). Pg.455-460(2010).

[15]. Gopalan.M.A., and KaligaRani.J., " Integral solutions of Ternary Cubic Equation $x^3 + y^3 + 4z^3 = 3xy(x+y)$ '', Antartica J.Math.,7(3)Pg.311-315(2010).

[16]. Gopalan.M.A.,and Pandiselvi .V., "On the Ternary Cubic Diophantine Equation $y^2 + gz^2 = (k^2 + g)z^3$ '',Impact.J.Sci.Tech.,Vol.4,No.4,Pg117-23.(2010).

[17]. Gopalan.M.A,Vidhyalakshmi.S.,Sumathi.G., "On the homogeneous cubic equation with threer unknowns $x^3 + y^3 = 14z^3 + 3(x+y)$ '',Discovery Science, Vol.2, No.4. Pg.37-39.,Oct 2012