Integral solution of the biquadratic Equation

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ABSTRACT:

We obtain infinitely many non-zero integer quadruples (x, y, z, w) satisfying the biquadratic equation with four unknowns $x^4 - y^4 = (k^2+1)(z^2 - w^2)$. *Various interesting relations between the solutions, polygonal numbers, pyramidal numbers and centered pyramidal numbers are obtained.*

Keywords: *Biquadratic equations with four unknowns, integral solutions, special numbers, figurative numbers, centered pyramidal numbers*

MSC 2000 Mathematics subject classification: 11D25

Notations:

$Gno_n = 2n - 1$ - Gnomonic number.

$S_n = 6n(n-1)+1$ -Star number of rank n.

$J_n = \frac{1}{3}\left(2^n - (-1)^n\right)$- Jacobsthal number of rank n.

$j_n = 2^n + (-1)^n$ - Jacobsthal-Lucas number of rank n.

$KY_n = (2^n+1)^2 - 2$ - Keynea number.

$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ - Polygonal number of rank n with size m.

$P_n^m = \frac{1}{6}n(n+1)((m-2)n + 5 - m)$ -Pyramidal number of rank n with size m.

$OH_n = \frac{1}{3}(n(2n^2+1)$ - Octa hedral number of rank n.

$CP_n^6 = -n^3$ -Centered hexagonal pyramidal number of rank n.

$CP_n^{14} = \frac{n(7n^2-4)}{3}$ -Centered tetra decagonal pyramidal number of rank n.

$PT_n = \frac{n(n+1)(n+2)(n+3)}{4!}$ -Pentatope number of rank n.

$F_{4,n,4} = \frac{n(n+1)^2(n+2)}{12}$ - Four dimensional Figurative number of rank n whose generating polygon is a square

I. INTRODUCTION:

The biquadratic Diophantine (homogeneous or non-homogeneous) equation offer an unlimited field for research due to their varietyDickson.L.E [1],Mordell.L.J[2], Carmichael R.D [3].In particular,one may refer Gopalan M.A et.al[4-17] for non-homogeneous biquadratic equations, with three and four unknowns. This communication concerns with yet another interesting non-homogenous biquadratic equation with four unknowns given by $x^4 - y^4 = (k^2+1)(z^2 - w^2)$ .A few interesting relations between the solutions, special numbers, figurative numbers and centered pyramidal numbers are obtained.

II. METHOD OF ANALYSIS

The non-homogeneous biquadratic diophantine equation with four unknowns is

$$x^4 - y^4 = (k^2+1)(z^2 - w^2) \quad (1)$$

It is worth to note that (1) is satisfied by the following non-zero distinct integer quadruples

$((a^2+b^2)(1+k), (a^2+b^2)(1-k), (a^2+b^2)^2(2k+1), (a^2+b^2)^2(2k-1))$ and

$(10b^2(k-1), -10b^2(k+1), -100b^4(2k-1), -100b^4(2k+1))$.

However we have some more patterns of solutions to (1) which are illustrated below

To start with,

the substitution of the transformations

$$x = u+v, y = u-v, z = 2uv+\sigma^2, w = 2uv - \sigma^2 \quad (2)$$

in (1), leads to $u^2 + v^2 = (k^2+1)\sigma^2$ (3)

2.1 Pattern 1

Let $\sigma = a^2 + b^2$ (4)

Substituting (4) in (3) and using the method of factorization, define

$$u + iv = (k+i)(a+ib)^2 \quad (5)$$

Equating real and imaginary parts in (5), we have

$$\left.\begin{aligned} u &= k(a^2-b^2) - 2ab \\ v &= (a^2-b^2) + 2abk \end{aligned}\right\} \quad (6)$$

Substituting (4) , (6) in (2) and simplifying, the corresponding values of x,y,z and w are represented by

$$x(k,a,b) = k(a^2 - b^2 + 2ab) + (a^2 - b^2 - 2ab)$$

$$y(k,a,b) = k(a^2 - b^2 - 2ab) - (a^2 - b^2 + 2ab)$$

$$z(k,a,b) = 4k^2ab(a^2-b^2) + 2k(a^4 + b^4 - 6a^2b^2) - 4ab(a^2-b^2) + (a^2+b^2)^2$$

$$w(k,a,b) = 4k^2ab(a^2-b^2) + 2k(a^4 + b^4 - 6a^2b^2) - 4ab(a^2-b^2) - (a^2+b^2)^2$$

Properties

1. $4\left[z(k,a+1,a) - w(k,a+1,a) - 192 PT_a + 96T_a^4 + 24 PR_a\right]$ is a cubical integer.

2.Each of the following is a nasty number:

(i) 3(z(k,a,b)-w(k,a,b))

(ii) x(k,ka,a)

(iii) -6(x(k,a,a)+y(k,a,a)).

3. $z(k,a,b) + w(k,a,b) - 48(k^2-1)P_{a-1}^3 + k(32P_a^5 - 48F_{4,a,4} - 8t_{9,a} + 4) \equiv 0 \pmod{28}$

4. $x(k,2^n,1) - y(k,2^n,1) - 12kJ_n - KY_{2n} + j_{2n} \equiv o \pmod{k}$

5. $x(k,1,b)y(k,1,b) - 48kP_{b-1}^3 = (k^2-1)(24PT_b - 9OH_b - t_{19,b} + 18t_{6,b} - 36t_{4,b} + 1)$

2.2 Pattern2:

(3) is written as

$$u^2 + v^2 = (k^2+1)\sigma^2 * 1 \quad (7)$$

Write ‘1’ as

$$1 = \frac{(3+4i)(3-4i)}{25} \quad (8)$$

Using (4) and (8) in (7) and employing the method of factorization, define

$$u + iv = (k+i)\frac{(3+4i)}{5}(a+ib)^2 \quad (9)$$

Equating real and imaginary parts of (9) we get

$$\left.\begin{aligned} u &= \frac{1}{5}((3k-4)(a^2-b^2)-2ab(4k+3)) \\ v &= \frac{1}{5}(2ab(3k-4)+(4k+3)(a^2-b^2)) \end{aligned}\right\} \qquad (10)$$

Taking a=5A,b=5B in (10) and substituting the corresponding values of u,v in (2) the non-zero integral solutions of (1) are Given by

$$x(k,A,B) = 5(k(7A^2-7B^2-2AB)+(-A^2+B^2-14AB))$$

$$y(k,A,B) = 5(k(-A^2+B^2-14AB)+(-7A^2+7B^2+2AB))$$

$$\begin{aligned} z(k,A,B) = 50[&(2k^2(6(A^2-B^2)^2-24A^2B^2-7AB(A^2-B^2)) \\ &-k(7(A^2-B^2)^2-28A^2B^2+96AB(A^2-B^2))- \\ &2(6(A^2-B^2)^2-24A^2B^2-7AB(A^2-B^2))]+(25(A^2+B^2))^2 \end{aligned}$$

$$\begin{aligned} w(k,A,B) = 50[&(2k^2(6(A^2-B^2)^2-24A^2B^2-7AB(A^2-B^2)) \\ &-k(7(A^2-B^2)^2-28A^2B^2+96AB(A^2-B^2))- \\ &2(6(A^2-B^2)^2-24A^2B^2-7AB(A^2-B^2))]-(25(A^2+B^2))^2 \end{aligned}$$

Properties

1. $x(k,A,1)+y(k,A,1) = 5(k(S_A-20t_{3,A}+2t_{12,A}+4Gno_A-3)+(3CP_A^{14}+7CP_A^{6}-8PR_A+8))$

2. $A(y(k,A,A)-x(k,A,A))(3k-4)CP_A^{6}$

3. $x(k,k,k)+y(k,k,k)+160P_k^5 = 20t_{4,k}$

4. $2(z(k,A,A)-w(k,A,A))$ is a biquadratic integer.

III. REMARKS

It is worth mentioning here that, instead of (8), one may also consider 1 in general form, as

$$1 = \frac{(p^2-q^2+i2pq)(p^2-q^2-i2pq)}{(p^2+q^2)^2} \qquad \text{(or)}$$

$$1 = \frac{(2pq+i(p^2-q^2))(2pq-i(p^2-q^2))}{(p^2+q^2)^2}$$

Following a similar analysis as in Pattern 2,one can obtain the integral solutions to (1)

Further, instead of (2) one can also use the following transformations

$$(i)\, x = u+v,\, y = u-v,\, z = 2u+v\sigma^2,\, w = 2u-v\sigma^2$$

$$(ii)\, x = u+v,\, y = u-v,\, z = 2u\sigma^2+v,\, w = 2u\sigma^2-v$$

and obtain the corresponding integral solutions to (1)

IV. CONCLUSION:

One may search for other choices of solutions to the equation under consideration and their corresponding properties.

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