"A Study of $\tilde{H}-$Function Transform And Its Inversion With Properties"

[1,]Dr. A.K. Ronghe, [2,] Mrs. Kulwant Kaur Ahluwalia
[1],(S.S.L. Jain, P.G. College, Vidisha, MP (India) Pin-464001)
[2,(]Mata Gujri Mahila Mahavidyalaya, Jabalpur, MP (India) Pin-482002)

Abstract: In this research paper we have defined $\tilde{H}-$function transform and developed its inversion formula, and it is further detected that particular cases of $\tilde{H}-$function transform comes out as Fox's H-function transform defined by Gupta and Mittal [6.7], G-function transform defined by Bhise [1] and other etc.

1. Introduction And Preliminaries:

The Mellin transform of the $\tilde{H}-$function is defined as follows:

$$\int_0^1 x^{\xi-1} H_{p,q}^{m,n}\left[ax\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]dx.d\xi$$

$$=a^{-\xi}\theta(-\xi) \qquad [1.1]$$

Where,

$$\theta(-\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j-A_j\xi)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j-B_j\xi)^{\beta_j}\}\prod_{j=n+1}^{p}\Gamma(a_j+A_j\xi)}, \qquad [1.2]$$

Provided $\min\limits_{1\le J\le m}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]<\mathrm{Re}(\xi)<\left\{\min\limits_{1\le J\le n}\left[\mathrm{Re}\left(\frac{1-a_j}{a_j}\right)\right]\right\}$ [1.3]

And other convergence conditions will be those of the $\tilde{H}-$ function associated in the definition of $\tilde{H}-$ function, see [(3), (8), (9)] etc. Integral involving product of Hyper geometric function and $\tilde{H}-$ function given as follows,

$$\int_0^\infty x^{-\alpha}(x-1)^{\beta-1}\,{}_2F_1\left[\begin{matrix}\gamma+\beta+\alpha,\lambda+\beta-\alpha\\\beta\end{matrix};(1-x)\right]\times$$

$$\tilde{H}_{p,q}^{m,n}\left[(ax)\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\Gamma(\beta)\tilde{H}_{p+3,q+3}^{m+1,n+2}\left[a\left|\begin{matrix}(\alpha-\beta;1,1),(\gamma+\beta+\lambda-\alpha;1,1)\\(\alpha-\beta;1),(b_j,\beta_j)_{1,m},(b_j,B_j,\beta_j)_{m+1,q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\alpha,1)\\(\gamma,1,1),(\lambda,1,1)\end{matrix}\right]$$

Provided $\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\lambda)>0$, [1.4]

and other convergence conditions stated in definition of $\tilde{H}-$function, see [(8) and (9)].

Proof: To establish (1.4), we first express. $\tilde{H}$ and ${}_2F_1$, occurring in the left hand side of (1.4) in term of Mellin Barnes contour integral with the help of definition of $\tilde{H}-$function, given by Inayat Hussain [8] and [9], series form of ${}_2F_1$ and using the property of Gamma-function, we arrive at the R.H.S. of (1.4) after a little simplification.

2. The $\tilde{H}-$Function Transform:

An integral transform of function $f(x)$ whose kernel is $\tilde{H}-$function defined by Inayat Hussain is called $\tilde{H}-$function transform, which is defined as follows:

$$\phi(\xi)=\int_0^\infty (\xi x)^\rho \tilde{H}_{p,q}^{m,n}\left[(\xi x)\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$\times f(x)dx,$$

Provided $\mathrm{Re}(\xi)>0, \mathrm{Re}(\rho+1)>0$, [2.1]

We may represent $\tilde{H}-$function transform as follows:

$$\bar{f}(x) \text{ or } \{\bar{f}(x);\xi\}$$ [2.2]

3. Special Cases:

(i) If $\rho=0,\alpha_j=\beta_j=1,$ in (2.1) we get Fox's H-function transform defined by Gupta and Mittal (6) in 1970 is as follows:

$$\phi(\xi)=\int_0^\infty H_{p,q}^{m,n}\left[(\xi x)\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]f(x)dx$$ [3.1]

(ii) If $\rho=0,\alpha_j=\beta_j=1,$, $a_j=c_j+d_j, b_j=d_j$, $m=m+1,$ $n=0, p=m, q=m+1$ in (2.1) we get G-function transform defined by Bhise [1] in 1959 as follows:

$$\phi(\xi)=\int_0^\infty G_{m,m+1}^{m+1,0}\left[(\xi x)\left|\begin{matrix}(c_j+d_j)_{1,m}\\(d_j)_{1,m},1\end{matrix}\right.\right]$$

Note: A number of other transform involving various special functions which are the special cases of $\tilde{H}-$function given by Saxena [11] can also be obtained from $\tilde{H}-$function transform, but we do not record them here, due to lack of space for that see [10].

4. Inversion formula for $\tilde{H}-$function transform: Multiplying on both sides of equation (2.1) by ξ^{-k} and integrating with respect to "ξ" between the limits $[0,\infty]$, we have,

$$\int_0^\infty \xi^{-k}\phi(\xi)d\xi=\int_0^\infty \xi^{-k}\left\{\int_0^\infty (\xi x)^\rho \times\right.$$

$$\left.\tilde{H}_{p,q}^{m,n}\left[(\xi x)\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_jB_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]f(x)dx\right\}d\xi,$$ [4.1]

Now changing the order of integration and using the definition of Mellin transform of H-function given in (1.1) we get,

$$\int_0^\infty \xi^{-k}\phi(\xi)d\xi=\int_0^\infty \theta[(k-1-\rho)]x^{k-1}f(x)dx,$$ [4.2]

Where,

$$\theta[(k-1-\rho)]=\frac{\prod_{J=1}^{m}\Gamma(b_j+\beta_j(1-k+\rho))\prod_{J=1}^{n}\Gamma(1-a_j+\alpha_j(1-k+\rho))^{\alpha_j}}{\prod_{J=m+1}^{q}\{\Gamma(1-b_j-\beta_j(1-k+\rho))\}^{\beta_j}\prod_{J=n+1}^{p}\Gamma(a_j+\alpha_j(1-k+\rho))}$$ [4.3]

Let $\int_0^\infty \xi^{-k}\phi(\xi)d\xi=F(x)$, [4.4]

Then, $F(x)=\int_0^\infty \theta[(k-1-\rho)]x^{k-1}f(x)dx,$ [4.5]

OR

$$\int_0^\infty x^{k-1} f(x)\,dx = \frac{F(x)}{\theta[k-1-\rho]}, \qquad [4.6]$$

Now using the definition of inverse Mellin transform we get,

$$\frac{f(x-0)+f(x+0)}{2} = \frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty} \frac{x^{-k}F(x)}{\theta[(k-1-\rho)]}dx \qquad [4.7]$$

Where, $F(x)$ is given by (4.5) and $\theta[(k-1-\rho)]$ is given by (4.2).

Provided $\min_{1\le J\le m}\left[\rho+\text{Re}\left(\frac{b_J}{\beta_J}\right)\right] < (1-c)$ $\left\{\min_{1\le J\le m}\left[\rho+\text{Re}\left(\frac{1-a_J}{\alpha_J}\right)\right]\right\}$ and other convergence conditions are stated in the definition of $\tilde{H}-$function.

4. Special Cases:

(i) If $\rho=0, \alpha_j=\beta_j=1$ in (4.7) we get inversion formula for Fox's H-function transform, which developed by Gupta and Mittal [7] as follows:

$$\frac{f(x-0)+f(x+0)}{2} = \frac{1}{2\pi\omega}\int_{c-\omega\theta}^{c+\omega\theta} \frac{x^{-k}F(x)}{\theta_1[(k-1)]}dx, \qquad [4.8]$$

Where,

$$\theta_1[(k-1)] = \frac{\prod_{J=1}^{m}\Gamma(b_j+\beta_j(1-k))\prod_{J=1}^{n}\Gamma(1-a_j+\alpha_j(1-k))^1}{\prod_{J=m+1}^{q}\{\Gamma(1-b_j+\beta_j(1-k))\}^1\prod_{J=n+1}^{p}\Gamma(a_j+\alpha_j(1-k))} \qquad [4.9]$$

(ii) $\alpha_j=\beta_j=$ unity, $a_j=c_j+d_j$, $b_j=d_j, m=m+1, n=0$ $\;p=m, q=m+1,$ in (4.7), we get an inversion formula for G-function transform which developed by Bhise [1] as follows:

$$\frac{f(x-0)+f(x+0)}{2} = \frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty} \frac{x^{-k}F(x)}{\theta_2[(k-1)]}dk,$$

Where,

$$\theta_2[(k-1)] = \frac{\prod_{J=1}^{m}\Gamma(d_j+(1-k))\prod_{J=1}^{n}\Gamma(\sigma_j+(1-k))^1}{\prod_{J=1}^{m}\Gamma[(c_j+d_j)+(1-k)]^1}$$

5. Properties of $\tilde{H}-$function transform:

Ist property:

If, $\tilde{H}\{f(x):\xi\}=\phi_1(\xi)$ and $\tilde{H}\{f_2(x):\xi\}=\phi_2(\xi)$ then

$$\tilde{H}\{C_1 f_2(x)\pm C_2 f_2(x):\xi\} = C_1\phi_1(\xi)\pm C_2\phi_2(\xi) \qquad [5.1]$$

Where C_1 and C_2 are arbitrary constant.

IInd Property:

If $\tilde{H}\{f(x):\xi\}=\phi(\xi)$ and $\tilde{H}\{(\lambda x):\xi\}=\phi_1(\xi)$ then $\phi_1(\xi)=\frac{1}{\lambda}\phi\left(\frac{s}{\lambda}\right)$ [5.2]

6. Property:

If $\tilde{H}\{f(x):\xi\}=\phi(\xi)$ and $\tilde{H}\{f(x^5):\xi\}=\phi_1(\xi)$, then $\phi_1(\xi)=\xi^{\sigma-1}\phi(\xi^\sigma)$, where $\text{Re}(\sigma)>0,$ [5.3]

IVth Property:

If $\tilde{H}\{f(x):\xi\}=\phi(\xi)$ and $\tilde{H}\left\{f\left(\frac{1}{2}\right):\xi\right\}=\phi_1(\xi)$ [5.4]

Proof of all properties given in [10, P, 110]

7. Acknowledgement:

The author are thankful to prof. R.D Agrawal (SATI Vidisha) for useful disscusions.

References

[1] Bhise, V.M., (1959); Inversion Formula For Generalized Laplace, Transform, Vikram, Univ. 3, (P- 57-63)

[2] Bhise. V.M. (1967);Certain Properties Of Meijer Laplace Transform, Comp. Math. 18, (P- 1.6).Bushman, R.G. And The H-Function Associated With A Certain Class Of

[3] Shrivastava, H.M. (1990); Feynmann Integrasl. J. Phys.; A Math, Gen (23), (P- 4707-4710).

[4] Fox, C. (1961); The G Function And H-Function As Symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98. (P- 395-429).

[5] Gupta, K.C. (1964);On The Inverse Meijer Transform Of G-Function. Collect Math 16. (P- 45-54).

[6] Gupta K.C. And The H-Function Transform, J. Austral Math. Soc. Mittal P.K. (1970); 11 (P- 142-148).

[7] Gupta K.C. And; The H-Function Transform II. J Austral. Math. Soc. Mittal P.K. (1971) 12. (P- 444-450).

[8] Inayat Hussain A.A. (1987); New Properties Of Hypergeometric Series Dvivable From Feynmann Integrals : I Transformation And Reduction Formulae J. Phys. A Math. Gen 20 (P- 4109-4117).

[9] Inayat-Hussain A.A. (1987); New Properties Of Hypergeometric Series Derivable From Feynmann Integral II Generalization Of The $\tilde{H}-$ Function J. Phys. A. Math. Gen (20), (P- 4119-4129).Ahluwalia K.K. (2013);

[10] Comparative Study Of The I-Function, The $\tilde{H}-$ Function, The Generalized Legendre's Associated Function And Their Applications, Ph.D. Thesis, Univ. Of B.U. Bhopal, India.

[11] Saxena, R.K. J. Ram And Unified Fractional Integrals Formulae For The

[12] And Kalla, S.L. (2002); Generalized $\tilde{H}-$ Function Rev. Acad. Canor Cienc. 14 (P- 97-101)