New Travelling Waves Solutions for Solving Burger's Equations by Tan-Cot function method

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Abstract:
In this paper, we used the proposed Tan-Cot function method for establishing a traveling wave solution to Burger's equations. The method is used to obtain new solitary wave solutions for various types of nonlinear partial differential equations such as, one-dimensional Burgers, KDV-Burgers, coupled Burgers, and the generalized time delayed Burgers' equations. Proposed method has been successfully implemented to establish new solitary wave solutions for Burgers nonlinear PDEs.

Keywords:*Nonlinear PDEs, Exact Solutions, tan-Cot function method, one-dimensional Burgers, KDV-Burgers, coupled Burgers, and the generalized time delayed.*

I. INTRODUCTION

Many phenomena in many branches of sciences such as physical, chemical, economical and biological processes are described by Burgers equations which provide the simplest nonlinear model of turbulence . The existence of relaxation time or delay time is an important feature in reaction diffusion and convection diffusion systems. The approximate theory of flow through a shock wave traveling is applied in viscous fluid [1]. Fletcher using the Hopf-Cole transformation [2] gave an analytic solution for the system of two dimensional Burgers' equations.Several numerical methods such as algorithms based on and implicit finite-difference scheme [3].Soliman [4] used the similarity reductions for the partial differential equations to develop a scheme for solving the Burgers' equation. High order accurate schemes for solving the two-dimensional Burgers' equations have been used [5].The variational iteration method was used to solve the one-dimensional Burgers and coupled Burgers' equations [6]. Anwar et al [1] used the Tanh method for the multiple exact complex solutions of some different kinds of nonlinear partial differential equations, and new complex solutions for nonlinear equations were obtained.

This paper is to extend the Tan- Cot function method to solve four different types of nonlinear differential equations such as the Burgers, KdV-Burgers, coupled Burgers and the generalized time delayed Burgers' equations given respectively by [1,7,8]

$$u_t + \alpha\, u\, u_x - \omega\, u_{xx} = 0, \tag{1}$$
$$u_t + \alpha\, u\, u_x - \omega\, u_{xx} + \mu\, u_{xxx} = 0, \tag{2}$$
$$u_t - u_{xx} + 2\, u\, u_x + \alpha(u\, v)_x = 0\,, \tag{3}$$
$$v_t - v_{xx} + 2\, v\, v_x + \beta(u\, v)_x = 0\,, \tag{4}$$
$$\tau\, u_{tt} + u_t + p\, u^3 u_x - u_{xx} = 0 \tag{5}$$

Where ;α , β, ω, and μ are arbitrary constants.

II. TAN-COT FUNCTION METHOD

Consider the nonlinear partial differential equation in the form

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \dots\dots\dots) = 0 \tag{6}$$

where u(x, y, t) is a traveling wave solution of the nonlinear partial differential equation Eq. (6). We use the transformation,

$$u(x, y, t) = f(\xi) \tag{7}$$

Where

$$\xi = (kx + y + \lambda t) \tag{8}$$

where$k$, and λ are real constants. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = \lambda \frac{d}{d\xi}(.), \ \frac{\partial}{\partial x}(.) = k \frac{d}{d\xi}(.), \frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \tag{9}$$

Using Eq. (8) to transfer the nonlinear partial differential equation Eq. (6) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots \dots \dots \dots .) = 0 \tag{10}$$

The order of the ordinary differential equation Eq.(10) can be reduced by integrating the equation providing that all the terms contain derivatives. The solutions of many nonlinear equations can be expressed in the form [9]:

$$f(\xi) = \alpha \tan^{\beta}(\mu\xi) \ , \qquad |\xi| \le \frac{\pi}{2\mu}$$

or in the form (11)

$$f(\xi) = \alpha \cot^{\beta}(\mu\xi) \ , \qquad |\xi| \le \frac{\pi}{2\mu}$$

Where α ,μ, and β are parameters to be determined, μ is the wave number. We use

$$\begin{aligned} f(\xi) &= \alpha \tan^{\beta}(\mu\xi) \\ f'(\xi) &= \alpha\,\beta\,\mu\ [\tan^{\beta-1}(\mu\xi) + \tan^{\beta+1}(\mu\xi)] \\ f''(\xi) &= \alpha\,\beta\mu^2[(\beta-1)\ \tan^{\beta-2}(\mu\xi) + 2\beta\ \tan^{\beta}(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)] \end{aligned} \tag{12}$$

and their derivative. Or use

$$\begin{aligned} f(\xi) &= \alpha \cot^{\beta}(\mu\xi) \\ f'(\xi) &= -\alpha\,\beta\,\mu\ [\cot^{\beta-1}(\mu\xi) + \cot^{\beta+1}(\mu\xi)] \\ f''(\xi) &= \alpha\,\beta\mu^2[\ (\beta-1)\ \cot^{\beta-2}(\mu\xi) + 2\beta\cot^{\beta}(\mu\xi) + (\beta+1)\ \cot^{\beta+2}(\mu\xi)] \end{aligned} \tag{13}$$

and so on. We substitute Eq.(12) or Eq.(13) into the reduced equation (10), balance the terms of the tan functions when Eq.(12) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all the terms with the same power in $\tan^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown'sα , μ and β and solve the subsequent system.

III. APPLICATIONS

the Tan- Cot function method is implemented to solve four different types of nonlinear differential equations such as the Burgers, KdV-Burgers, coupled Burgers and the generalized time delayed Burgers' equations.

3.1 One-dimensional Burgers' equation

Consider the one-dimensional Burgers' equation which has the form[1];

$$u_t + \delta\, u\, u_x - \omega\, u_{xx} = 0 \tag{14}$$

Where α and ω are arbitrary constants. In order to solve Eq. (14) by the Tan method, we use the wave transformation $u(x,t) = U(\xi)$, with wave variable $\xi = (x + \lambda\, t)$, Eq. (14) takes the form of an ordinary differential equation.

$$\lambda U' + \delta\, U\, U' - \omega\, U'' = 0 \tag{15}$$

Integrating Eq. (15) once with respect to ξ and setting the constant of integration to be zero, we obtain:

$$\lambda U + \frac{1}{2}\delta\, U^2 - \omega\, U' = 0 \tag{16}$$

Seeking the solution in eq.(11)

$$\lambda \tan^{\beta}(\mu\xi) + \frac{1}{2}\,\delta\,\alpha\,\tan^{2\beta}(\mu\xi) - \omega\,\beta\,\mu\,[\tan^{\beta-1}(\mu\xi) + \tan^{\beta+1}(\mu\xi)] = 0 \qquad (17)$$

Equating the exponents and the coefficients of each pair of the tan functions we find the following algebraic system:

$$2\beta = \beta + 1 \;\; \rightarrow \beta = 1 \qquad (18)$$

Substituting Eq. (18) into Eq. (17) to get:

$$\alpha = \frac{2\omega\,\mu}{\delta} \qquad (19)$$

Then by substituting Eq.(19) into Eq.(12), the solution of equation (14) can be written in the form

$$u(x,t) = \frac{2\omega\,\mu}{\delta}\,\tan\,\left(\mu(x+\lambda t)\right) \qquad (20)$$

For $\mu = \lambda = 1, \omega = 0.5\,,\; \delta = 0.1$Eq.(20) becomes

$$u(x,t) = 10\,\tan\,(x+t) \qquad (21)$$

3.2 KdV-Burgers' equation

Another important example is the KdV-Burgers' equation [1], which can be written as

$$u_t + \delta\,u\,u_x - \omega\,u_{xx} + \rho\,u_{xxx} = 0 \qquad (22)$$

Where $\alpha\,,\omega,$ and μ are arbitrary constants. In order to solve Eq. (22) by the tan method, we use the wave transformation $u(x,t) = U(\xi)$, with wave variable $\xi = (x + \lambda\,t)$, : Eq. (22) takes the form of an ordinary differential equation

$$\lambda U' + \frac{1}{2}\delta\,(U^2)' - \omega\,U'' + \rho\,U''' = 0 \qquad (23)$$

Integrating Eq. (23) once with respect to ξ and setting the constant of integration to be zero, we obtain:

$$\lambda U + \frac{1}{2}\delta\;U^2 - \omega\,U' + \rho\,U'' = 0 \qquad (24)$$

Seeking the method in (12)

$$\lambda\,\tan^{\beta}(\mu\xi) + \frac{1}{2}\delta\;\alpha\,\tan^{2\beta}(\mu\xi) - \omega\,\beta\,\mu\,[\tan^{\beta-1}(\mu\xi) + \tan^{\beta+1}(\mu\xi)] + \rho\;\beta\mu^2[(\beta-1)\,\tan^{\beta-2}(\mu\xi) + 2\beta\,\tan^{\beta}\,(\mu\xi) + (\beta+1)\,\tan^{\beta+2}(\mu\xi\,)] = 0$$

(25)

Equating the exponents and the coefficients of each pair of the tan function, we obtain

$2\beta = \beta + 1$, so that $\beta = 1$ (26)

Substitute (26) into (25) give the following system of equations

$$\lambda + 2\rho\mu^2 = 0$$
$$\frac{1}{2}\delta\;\alpha\;\; - \omega\,\mu\; = 0 \qquad (27)$$

Then by solving system (27) to get:

$$\lambda = -2\,\rho\;\mu^2\,,\quad \alpha = 2\,\frac{\omega\,\mu}{\delta} \qquad (28)$$

Then by substituting Eq.(28) into Eq.(12), the exact soliton solution of the system of equation (22)can be written in the form

$$u(x,t) = 2\,\frac{\omega\,\mu}{\delta}\;\tan(\mu\,(x - 2\,\rho\,\mu^2 t)\,) \tag{29}$$

For $\mu = \rho = 1$, $\omega = 0.5$, $\delta = 0.1$ Eq.(22) becomes

$$u(x,t) = 10\;\tan(x - 2\,t) \tag{30}$$

3.3 Coupled Burgers' equations

The third instructive example is the homogeneous form of a coupled Burgers' equations [1]. We will consider the following system of equations

$$u_t - u_{xx} + 2\,u\,u_x + \delta(u\,v\,)_x = 0 \tag{31}$$

$$v_t - v_{xx} + 2\,v\,v_x + \gamma(u\,v\,)_x = 0 \tag{32}$$

In order to solve Eqs. (31,32) by the tan method. We use the wave transformations u(x; t) = U(ξ) and v(x; t) = V(ξ) with wave variable $\xi = (x + \lambda t)$: Eqs. (31,32) take the form of ordinary differential equations

$$\lambda\,U' - U'' + 2\,UU' + \delta\,(UV)' = 0 \tag{33}$$

$$\lambda V' - V'' + 2VV' + \gamma\,(UV)' = 0 \tag{34}$$

Integrating Eqs. (33,34) once with respect to ξ and setting the constant of integration to zero, we obtain

$$\lambda U - U' + U^2 + \delta\,UV = 0 \tag{35}$$

$$\lambda\,V - V' + V^2 + \gamma UV = 0 \tag{36}$$

Let :

$$U = \alpha_1\,\tan^{\beta_1}(\mu\xi) \tag{37}$$

$$V = \alpha_2\,\tan^{\beta_2}(\mu\xi) \tag{38}$$

$$U' = \alpha_1\beta_1\,\mu\;[\tan^{\beta_1 - 1}(\mu\xi) + \tan^{\beta_1 + 1}(\mu\xi)] \tag{39}$$

$$V' = \alpha_2\beta_2\,\mu\;[\tan^{\beta_2 - 1}(\mu\xi) + \tan^{\beta_2 + 1}(\mu\xi)] \tag{40}$$

Substitute (37-38) and their derivatives then (35,36) become

$$\lambda\tan^{\beta_1}(\mu\xi) - \beta_1\,\mu\;[\tan^{\beta_1 - 1}(\mu\xi) + \tan^{\beta_1 + 1}(\mu\xi)] + \alpha_1\,\tan^{2\beta_1}(\mu\xi) + \delta\,\alpha_2\,\tan^{\beta_1+\beta_2}(\mu\xi) = 0 \tag{41}$$

and

$$\lambda\tan^{\beta_2}(\mu\xi) - \beta_2\,\mu\;[\tan^{\beta_2 - 1}(\mu\xi) + \tan^{\beta_2 + 1}(\mu\xi)] + \alpha_2\,\tan^{2\beta_2}(\mu\xi) + \gamma\,\alpha_1\tan^{\beta_1+\beta_2}(\mu\xi) = 0 \tag{42}$$

Equating the exponents and the coefficients of each pair of the tan function, we obtain

$\beta_1 + 1 = \beta_1 + \beta_2$, so that$\beta_2 = 1$

$\beta_1 + \beta_2 = 2\beta_2$, so that $\beta_1 = 1$

Substitute $\beta 1 = 1$, and $\beta 2 = 1$ into (41), and (42) give the following system of equations

$$-\mu + \alpha_1 + \delta\,\alpha_2 = 0$$

$$-\;\mu\; + \alpha_2 + \gamma\,\alpha_1 = 0 \tag{43}$$

Solving system (43) then:

$$\alpha_1 = \frac{[1-\delta]}{[1-\delta\gamma]}\,\mu\;,\ \alpha_2 = \frac{[1-\gamma]}{[1-\delta\gamma]}\,\mu \tag{44}$$

Then by substituting Eq.(44) into Eqs.(37), (38), the exact solution of the system of equations (31) and (32) can be written in the form

$$u(x,t) = \frac{[1-\delta]}{[1-\delta\gamma]}\ \mu \tan\ (\mu(x + \lambda t)) \quad (45)$$

$$v(x,t) = \frac{[1-\gamma]}{[1-\delta\gamma]}\mu \tan\ (\mu(x + \lambda t)) \quad (46)$$

For $\mu = \lambda = 1,\ \delta = \gamma = 0.5$

$$u(x,t) = \frac{2}{3}\tan(x + t) \quad (47)$$

$$v(x,t) = \frac{2}{3}\tan(x + t) \quad (48)$$

3.4 Generalized time-delayed Burgers equation

The time-delayed Burgers equation [1,8] is given by

$$\tau u_{tt} + u_t + p\, u^s u_x - u_{xx} = 0 \quad (49)$$

we use the wave transformation $u(x,t) = U(\xi)$, with wave variable $\xi = (x + \lambda t)$, Eq. (49) takes the form of an ordinary differential equation

$$(\lambda^2 \tau - 1)U'' + (p\, U^s + \lambda)\, U' = 0 \quad (50)$$

Seeking the method in (12), Eq. (50) becomes

$$(\lambda^2 \tau - 1)\, \alpha\, \beta\mu^2\, [(\beta - 1)\ \tan^{\beta - 2}(\mu\xi) + 2\beta\ \tan^{\beta}(\mu\xi) + (\beta + 1)\tan^{\beta + 2}(\mu\xi)] + (\,p\alpha^s \tan^{s\beta}(\mu\xi) + \lambda)\quad \alpha\, \beta\, \mu\ [\tan^{\beta - 1}(\mu\xi) + \tan^{\beta + 1}(\mu\xi)] = 0 \quad (51)$$

Equating the exponents and the coefficients of each pair of the tan function, we obtain

$$(\lambda^2 \tau - 1)\, \mu\, [(\beta - 1)\ \tan^{\beta - 2}(\mu\xi) + 2\beta\ \tan^{\beta}(\mu\xi) + (\beta + 1)\tan^{\beta + 2}(\mu\xi)] + p\, \alpha^s \tan^{s\beta + \beta - 1}(\mu\xi) + p\, \alpha^s \tan^{s\beta + \beta + 1}(\mu\xi) + \lambda \tan^{\beta - 1}(\mu\xi) + \lambda\ \tan^{\beta + 1}(\mu\xi) = 0 \quad (52)$$

Then:

$s\beta + \beta + 1 = \beta + 2$, so that $\beta = \frac{1}{s}$ (53)

Substitute (53) into (52) give

$$\alpha = \left[\frac{2\,(1-\lambda^2\tau)}{s\,p}\ \mu\right]^{\frac{1}{s}} \quad (54)$$

Then by substituting Eq.(54) into Eq.(12), the exact solution of equation (49) can be written in the form

$$u(x,t) = \left[\frac{2\,(1-\lambda^2\tau)}{s\,p}\ \mu\right]^{\frac{1}{s}}\ \tan^{\frac{1}{s}}(\mu(x + \lambda t)) \quad (55)$$

For $\mu = \lambda = p = s = 1,\ \tau = .5$, Eq.(55) becomes

$$u(x,t) = \tan(x + t) \quad (56)$$

IV. CONCLUSION

In this paper, the tan-cot function method has been successfully implemented to establish new solitary wave solutions for various types of nonlinear PDEs. We can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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