Solution Of Matrix Game In Intuitionistic Fuzzy Environment

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Abstract:
In this paper, an intuitionistic fuzzy matrix game has been considered and its solution method has been proposed using defuzzification method. Score functions have been defined to construct the problem and numerical example has been given in support of the solution method.
Key words: Intuitionistic fuzzy number, score function, matrix game.

1 Introduction

In modern era there are lot of situations in the society where there is a conflicting interest situation and such situation is handled by game theory. But there are lot of cases where the information given are not in precise manner and in such situation we apply fuzzy mathematics to get a solution.Fuzziness in matrix games can appear in so many ways but two classes of fuzziness are very common. These two classes of fuzzy matrix games are referred as matrix games with fuzzy goal [1] and matrix games with fuzzy pay off [2]. But there such situation may exist where the players can estimate the approximate pay-off values with some degree but with a hesitation. These situations are overcome by applying intuitionistic fuzzy (IF) numbers in game theory.. Atanassov[3] first introduced the concept of IF-set where he explained an element of an IF-set in respect of degree of belongingness, degree of non-belongingness and degree of hesitancy. This degree of hesitancy is nothing but the uncertainty in taking a decision by a decision maker(DM).Atanassov [4] first described a game using the IF-set. Li and Nan [5] considered the matrix games with pay-offs as IF-sets. Nayak and Pal [6] considered a bi-matrix game where they used IF-set. In this paper, We have considered a matrix game where the elements of the pay-off matrix are all intuitionistic fuzzy numbers. We have applied score function method to defuzzify such matrix. Two theorems given, establish the reason behind such defuzzification. An example establishes the theory on strong ground. The paper is organized as follow: In section 2 a basic definition of intuitionistic fuzzy set and intuitionistic fuzzy number are given.In section 3 score function is defined and some properties are given thereafter. In section 4 intuitionistic matrix game has been defined.In section 5 Numerical example is given. In section 6 conclusion has been drawn.

2 Intuitionistic Fuzzy Sets

Here we are going to discuss some basic preliminaries, notations and definitions of Intuitionistic fuzzy sets (IFS), in particular the works of Atanassov [3, 7].

Definition 1 Let $X=\{x_1,x_2,\ldots,x_n\}$ be a finite universal set. An Atanassov's intuitionistic fuzzy set (IFS) in a given universal set X is an expression A given by

$$A=\{\langle x_i,\mu_A(x_i),\nu_A(x_i)\rangle : x_i\in X\} \tag{1}$$

where the functions

$$\mu_A : X\to[0,1];\ \ x_i\in X\to\mu_A(x_i)\in[0,1]$$

and

$$\nu_A : X\to[0,1];\ x_i\in X\to\nu_A(x_i)\in[0,1]$$

define the degree of membership and the degree of non-membership of an element $x_i\in X$ to the set $A\subseteq X$, respectively, such that they satisfy the following condition : for every $x_i\in X$

$$0\le\mu_A(x)+\nu_A(x)\le 1.$$

Let

$$\pi_A(x_i) = 1 - \mu_A(x) - \nu_A(x)$$

which is called the Atanassov's [7] intuitionistic index of an element x_i in the set A. It is the degree of indeterminacy membership of the element x_i to the set A. Obviously,

$$0 \le \pi_A(x_i) \le 1$$

If an Atanassov's IFS C in X has only an element, then C is written as follows

$$C = \{\langle x_k, \mu_C(x_k), \nu_C(x_k)\rangle\}$$

which is usually denoted by $C = \{\langle \mu_C(x_k), \nu_C(x_k)\rangle\}$ for short.

Definition 2 Let A and B be two Atanassov's IFSs in the set X. $A \subset B$ iff $\mu_A(x_i) \le \mu_B(x_i)$ *and* $\nu_A(x_i) \ge \nu_B(x_i)$; *for any* $x_i \in X$.

Definition 3 Let A and B be two Atanassov's IFSs in the set X. $A = B$ iff $\mu_A(x_i) = \mu_B(x_i)$ *and* $\nu_A(x_i) = \nu_B(x_i)$; *for any* $x_i \in X$. Namely, $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 4 Let A and B be two Atanassov's IFSs in the set X. The intersection of A and B is defined as follows :

$$A \cap B = \{\langle x_i, \min(\mu_A(x_i), \mu_B(x_i)), \max(\nu_A(x_i), \nu_B(x_i))\rangle \mid x_i \in X\}.$$

Definition 5 (**Intuitionistic Fuzzy Number**): Intuitionistic fuzzy number was introduced by Seikh et al.[8]. An

intuitionistic fuzzy number A

1. an intuitionistic fuzzy subset of the real line
2. normal i.e. there exists $x_0 \in \Re$ such that $\mu_A(x_0) = 1$ (so $\nu_A(x_0) = 0$)
3. convex for the membership function μ_A i.e.

$$\mu_A(\lambda x_1 + (1-\lambda x_2)) \ge \min\left\{\mu_A(x_1), \mu_A(x_2)\right\}; \forall x_1, x_2 \in \Re, \lambda \in [0,1]$$

4. concave for the non-membership function ν_A i.e.

$$\nu_A(\lambda x_1 + (1-\lambda x_2)) \le \max\left\{\nu_A(x_1), \nu_A(x_2)\right\}; \forall x_1, x_2 \in \Re, \lambda \in [0,1].$$

In our discussion we consider an intuitionistic fuzzy number A as (μ_{ij}, ν_{ij}) and the addition and scalar multiplication operations are given as

1. $(\mu_{ij}, \nu_{ij}) + (\mu'_{ij}, \nu'_{ij}) = (\mu_{ij} + \mu'_{ij} - \mu_{ij}\mu'_{ij}, \nu_{ij}\nu'_{ij})$ (2)
2. $k(\mu_{ij}, \nu_{ij}) = (k\mu_{ij}, k\nu_{ij})$ for $0 \le k < 1$ (3)

3 Defuzzification by Score Function

Chen and Tan[9] first defined a score function S as deviation of a membership function μ from non-membership function ν as

$$S_{ij} = \mu_{ij} - \nu_{ij} \qquad (4)$$

Here bigger the value of S represents bigger IFN but when S of two IFN are same then this definition does not work. So, analyzing the deficiency of this score function Hong and Chi[10] have given a precise function as

$$H_{ij} = \mu_{ij} + \nu_{ij} \qquad (5)$$

.

Here also bigger the value of H gives bigger IFN.Now these two scoring functions defined above have fundamental deficiency that they do not involve the uncertainty function π and this seems to be very unrealistic. Liu[11] analyzing the hesitancy degree π modified the definition as

$$S1_{ij} = (\mu_{ij} - \nu_{ij})(1 + \pi_{ij}) \qquad (6)$$

But this is a non-linear function and hence we will introduce a score function,called linearizing score function which is defined as

$$D\{f(\mu_{ij}), g(\nu_{ij})\} = L\{f(\mu_{ij})\} \qquad (7)$$

Where $f(\mu_{ij})$ and $g(\nu_{ij})$ are functions of μ_{ij} and ν_{ij} and the function L gives the linear part of the function f. The arithmetic operations on the function D are given as

1. $D(t_1 + t_2) = D(t_1) + D(t_2)$
2. $kD(t_1) = D(kt_1) \quad where \quad 0 \le k < 1$ (8)

The inequality relation is given as

$$D(t_1) \le D(t_2) \quad iff \quad t_1 \le t_2 \qquad (9)$$

.

4 Intuitionistic matrix game

Let $A_i (i = 1,2,\cdots,m)$ and $B_j (j = 1,2,\cdots,n)$ be pure strategies for players (or DMs) A and B, respectively. If player A adopts the pure strategy A_i (i.e., the row i) and player B adopts pure strategy B_j (i.e., the column j), then the pay-off for player A is expressed with the intuitionistic number (μ_{ij}, ν_{ij}). The intuitionistic pay-off matrix of a matrix game is concisely expressed in the matrix form as

$$B_1 \qquad B_2 \qquad \cdots \qquad B_n$$

$$G = \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{pmatrix} (\mu_{11},\nu_{11}) & (\mu_{12},\nu_{12}) & \cdots(\mu_{1n},\nu_{1n}] \\ (\mu_{21},\nu_{21}) & (\mu_{22},\nu_{22}) & \cdots(\mu_{2n},\nu_{2n}) \\ \vdots & \vdots & \vdots \\ (\mu_{m1},\nu_{m1}) & (\mu_{m2},\nu_{m2}) & \cdots(\mu_{mn},\nu_{mn}) \end{pmatrix} \qquad (10)$$

which is said to be intuitionistic matrix game .

4.1 Pure strategy

Pure strategy is a decision making rule in which one particular course of action is selected. For fuzzy games the min-max principle is described by Nishizaki [2]. The course of the fuzzy game is determined by the desire of A to maximize his gain and that of restrict his loss to a minimum. Now for IF game,

$$max-min = \bigvee_i \{\bigwedge_j (\mu_{ij},\nu_{ij})\} = \bigwedge_j \{\bigvee_i (\mu_{ij},\nu_{ij})\}. \qquad (11)$$

Based on TIFN order, for such games, we define the concepts of $\min-\max$ equilibrium strategies.

Definition 6 (Saddle Point) : The concept of saddle point in classical form is introduced by Neumann [12]. The $(k,r)th$ position of the pay-off matrix will be called a saddle point, if and only if,

$$(\mu_{kr},\nu_{kr}) = \bigvee_i \{\bigwedge_j (\mu_{ij},\nu_{ij})\} = \bigwedge_j \{\bigvee_i (\mu_{ij},\nu_{ij})\}. \qquad (12)$$

We call the position (k,r) of entry a saddle point, the entry itself (μ_{kr},ν_{kr}) ,the value of the game (denoted by V) and the pair of pure strategies leading to it are optimal pure strategies.

Definition 7 (IF expected pay-off): If the mixed strategies $x = (x_1, x_2, \cdots, x_m)$ and $y = (y_1, y_2, \cdots, y_n)$ are proposed by players A and B respectively, then the expected pay-off of the player A by player B is defined by

$$E(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{m}(\mu_{ij},\nu_{ij})x_i y_j. \qquad (13)$$

Addition and other composition rules on IF set which we have discussed in section 21 are used in this definition of expected pay-off (13). In such a situation,player A chooses x so as to maximize his expectation and player B chooses y so as to minimize player A's maximum expectation and mathematically we write

$$\min_y \max_x E(x,y) = E(x^*, y^*) = \max_x \min_y E(x,y) \qquad (14)$$

where (x^*, y^*) is called strategic saddle point of the game and $V = E(x^*, y^*)$ is value of the game.

Theorem 1 *If a pay-off matrix with elements as IFN has saddle point (k,r) and t_{kr} be the value of the game then the pay-off matrix obtained after defuzzification with the help of score function D has also saddle point (k,r) and $D(t_{kr})$ is the value of the game.*

Proof: If (k,r) be the saddle point of the pay-off matrix and t_{kr} be the value of the game then

$$t_{kr} = \bigvee_i \{\bigwedge_j t_{ij}\} = \bigwedge_j \{\bigvee_i t_{ij}\}$$

Now using the equations (6) and (1) we get

$$D(t_{kr}) = D(\bigvee_i \{\bigwedge_j t_{ij}\}) = D(\bigwedge_j \{\bigvee_i t_{ij}\})$$

$$\Rightarrow D(t_{kr}) = \bigvee_i D(\{\bigwedge_j t_{ij}\}) = \bigwedge_j D(\{\bigvee_i t_{ij}\})$$

$$\Rightarrow D(t_{kr}) = \bigvee_i \{\bigwedge_j D(t_{ij}\}) = \bigwedge_j \{\bigvee_i D(t_{ij}\})$$

Therefore, (k,r) is also the saddle point of the defuzzified pay-off matrix. $D(t_{kr})$ is the value of the game. Hence the theorem.

Theorem 2 *If* (x^*, y^*) *be the strategic solution of the pay-off matrix with mixed strategies then* (x^*, y^*) *is also the solution of the pay-off matrix after defuzzification by score function* D.

Proof: Let (x^*, y^*) be the solution of the pay-off matrix then

$$\min_y \max_x E(x,y) = E(x^*, y^*) = \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} x^*_{ij} y^*_{ij} = \max_x \min_y E(x,y)$$

$$\Rightarrow D(\min_y \max_x E(x,y)) = D(E(x^*, y^*)) = D(\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} x^*_{ij} y^*_{ij}) = D(\max_x \min_y E(x,y))$$

$$\Rightarrow \min_y \max_x D(E(x,y)) = D(E(x^*, y^*)) = \sum_{i=1}^{m}\sum_{j=1}^{n} D(t_{ij}) x^*_{ij} y^*_{ij} = E(D) = \max_x \min_y D(E(x,y)).$$

Therefore, (x^*, y^*) is also a strategic solution of the defuzzified pay-off matrix and value of the game is $V(x^*, y^*) = \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} x^*_{ij} y^*_{ij}$. Hence the theorem.

5 Numerical example

Let us consider a pay-off matrix with intuitionistic fuzzy elements as

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} (0.5,0.25) & (0.45,0.20), \\ (0.4,0.3) & (0.3,0.1) \end{pmatrix}. \end{array}$$

When this matrix gets defuzzified with the given score function then we get the matrix as

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 0.5 & 0.45, \\ 0.4 & 0.3 \end{pmatrix}. \end{array}$$

Since $\bigvee_i \{\bigwedge_j (a_{ij}, b_{ij})\} = 0.45 = \bigwedge_j \{\bigvee_i (a_{ij}, b_{ij})\}$ the saddle point is $(2,1)$ and value of the game is 0.1625.

Hence, saddle point of the original pay-off matrix is $(2,1)$ and value of the game is (0.45,0.2).

Now let us consider the pay-off matrix as

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} (0.4,0.3) & (0.45,0.20), \\ (0.5,0.25) & (0.3,0.1) \end{pmatrix}. \end{array}$$

When we defuzzify this matrix using the score function then we get the matrix as

$$\begin{array}{ccc} & B_1 & B_2 \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \multicolumn{2}{c}{\begin{pmatrix} 0.4 & 0.45, \\ 0.5 & 0.3 \end{pmatrix}} \end{array}.$$

Since $\bigvee_i \{\bigwedge_j (a_{ij}, b_{ij})\} = 0.4 \neq 0.45 = \bigwedge_j \{\bigvee_i (a_{ij}, b_{ij})\}$,saddle point does not exits. Since this is a mixed strategy game its solution is obtained as the strategic solution $x^* = (0.8,0.2), y^* = (0.8,0.2)$ and value of the game is $(0.388,0.012)$.

6 Conclusion

In this paper, a matrix game has been considered with pay-off elements as intuitionistic fuzzy numbers. The intuitionistic fuzzy number is considered as a membership and a non-membership function which actually represents the acceptance and rejection degree of a decision maker. The matrix is defuzzified with the help of a score function. It gives a strategic solution and value of the game as an intuitionistic fuzzy number. The example given, establishes the theory on strong ground. It has strong impact on modern socio economic structure where conflicting interests exist. There is a scope to apply other score function for such defuzzification.

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