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Application of Ellipse for Horizontal Alignment

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Abstract:

In highway design, horizontal curves provide directional transition for roadways. Three categories of horizontal curves are simple circular curves, compound circular curves, and spiral circular curves. Compound and spiral curves, as alternatives to a simple circular curve, are often more costly since they are longer in length and require additional right-of-way; with cost differences amplified at higher design speeds. This study presents calculations associated with using a single elliptical arc in lieu of compound or spiral curves in situations where the use of simple circular curves is not prudent due to driver safety and comfort considerations. The study presents an approach to analytically determine the most suitable substitute elliptical curve for a given design speed and intersection angle. Computational algorithms are also provided to stakeout the elliptical curve. These include algorithms to determine the best fit elliptical arc with the minimum arc length and minimum right-of-way; and algorithms to compute chord lengths and deflection angles and the associated station numbers for points along the elliptical curve. These algorithms are applied to an example problem in which elliptical results are compared to the equivalent circular curve and spiral-circular curve results.

Keywords: Horizontal alignment, elliptical curve, circular curve, spiral curve

I. Introduction

In highway design, a change in the direction of the roadway is achieved by a circular or a compound circular curve connecting the two straight sections of the roadway known as tangents. A common horizontal alignment treatment is a compound curve. It consists of a circular curve and two transition curves, one at each end of the circular curve. The transition curves are either circles of larger radii or spiral curves. In some cases no transition curves are needed when the design speeds or degrees of curvature are fairly low. In such cases, the horizontal alignment could be a single circular curve.

The most important factor in designing horizontal curves is the design speed. When a vehicle negotiates a horizontal curve, it experiences a lateral force known as the centrifugal force. This force, which is due to the change in the direction of the velocity vector, pushes the vehicle outward from the center of curvature. The vehicle is also subjected to an inward radial force, the centripetal force. In fact, the centripetal force is always directed orthogonal to the velocity vector, towards the instantaneous center of curvature. At high speeds, the centripetal force acting inward may not be large enough to balance the centrifugal force acting outward. To mitigate this problem, a lateral roadway angle, known as the superelevation angle e(or banking angle) is provided (Garber and Hoel, 2002, p. 70). To keep these forces in balance, the minimum required radius is then given by the following equation:

$$R = \frac{v^2}{g(e + f_{side})}. \tag{1}$$

Where R is the minimum radius,   is the design speed, e is the superelevation angle in radians, f_{side} is the coefficient of side friction, and **g** is the acceleration of gravity.

1.1 Spiral Transitions

On simple circular curves, as the vehicle enters the horizontal curve with a velocity,   the centrifugal force jumps from zero on the tangent section to mv^2/R on the curve. A transition curve such as a larger radius circle or a spiral helps moderate this sudden increase in force, thus making the alignment smoother and safer. Spiral is a particularly good transition curve as its radius decreases gradually along its length (the curvature changes linearly in length), from an infinite radius (zero curvature) at the tangent to spiral to the design radius at the spiral to circular point. The minimum length of spiral recommended by AASHTO for a horizontal curve of radius is given by:

$$l_s = \frac{3.15\ \mathrm{V}^3}{\mathrm{R.C}} \tag{2}$$

Where

l_s = minimum length of transition spiral (ft)
V = design speed (mph)
R = radius of curvature (ft)
C = rate of change of centripetal acceleration ($\mathrm{ft/sec^3}$).

The use of transition curves such as the spiral, although yielding smoother alignments, often results in longer roadway lengths and greater right-of-way requirements. In addition the stake-out computations are considerably more involved than using simple circular curves.

II. Approach

2.1 Use of Ellipses as Horizontal Curves

In this section, the application of ellipses as horizontal alignment curves is examined. This includes a general discussion of properties of ellipse followed by a procedure for finding an appropriate elliptical curve that could provide a smooth and safe transition from the PC to PT. The associated chord length and deflection angle calculations for an elliptical arc are also presented.

Geometrically, an ellipse is the set of points in a plane for which the sum of distances from two points F_1 and F_2 is constant (See Figure 2). These two fixed points are called the **foci**. One of the Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at points $(-c, 0)$ and $(c, 0)$ so that the origin, which is called the **center** of ellipse, is halfway between F_1 and F_2 (Figure 1). Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Let us also suppose that $\mathrm{P(x, y)}$ is any point on the ellipse. According to the definition of the ellipse, we will have:

$$|\mathrm{PF_1}| + |\mathrm{PF_2}| = 2a \tag{3}$$

that is,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a\ . \tag{4}$$

Or,

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)\ . \tag{5}$$

In the triangle F_1F_2P (Figure 1), it can be seen that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes

$$b^2x^2 + a^2y^2 = a^2b^2\ . \tag{6}$$

Or by dividing both sides by a^2b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ . \tag{7}$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts, we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation 7 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$ and $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

In mathematics, there is a parameter for every conic section called **eccentricity** (Larson et al., 2010, p. 701). Eccentricity defines how much the conic section deviates from being a circle. As a conic section, ellipse has its own eccentricity τ which is calculated as,

$$\tau = \frac{c}{a}, \tag{8}$$

in which:

$\tau =$ eccentricity,
$a =$ length of major axis,
$c = \sqrt{a^2 - b^2}$.

In most mathematics literature, the eccentricity is denoted by **e** or $\boldsymbol{\varepsilon}$. In this text, we use τ to denote the eccentricity in order to avoid confusion with the superelevation angle, e.

2.2 Circular Curve, Design Speed, and Superelevation

As discussed earlier, the relation between the radius of the circular curve, the design speed, and the superelevation is governed by Eq. 1. Therefore, the desired elliptical curve should as a minimum satisfy the minimum radius required by AASHTO, as per Eq. 1. This establishes one of the constraints for finding an appropriate elliptical curve. Before considering this and other constraints, however, we should determine what constitutes a “radius” for an ellipse. To achieve this, we would utilize the polar coordinate system.

In the polar coordinate system, there are two common equations to describe an ellipse depending on where the origin of the polar coordinates is assumed to be. If, as shown in Figure 2, the origin is placed at the center of the ellipse and the angular coordinate θ is measured from the major axis, then the ellipse’s equation will be:

$$r(\theta) = \frac{ab}{\sqrt{\left(b\cos(\theta)\right)^2 + \left(a\sin(\theta)\right)^2}}. \tag{9}$$

On the other hand, if the origin of polar coordinates is located at a focus (Figure 3) and the angular coordinate θ is still measured from the major axis, then the ellipse’s equation will be:

$$r(\theta) = \frac{a(1-\tau^2)}{1 \pm \tau\cos(\theta)} \tag{10}$$

Where the sign in the denominator is negative if the reference direction is from $\theta = 0$ towards the center.

From the astronomical point of view, Kepler’s laws established that the orbits of planets in a solar system are ellipses with a sun at one focus. Thus, the ellipse’s polar Eq. 10, in which the origin of the polar coordinates is assumed at one focus, will be helpful to obtain the desired elliptical arc.

Figure 3 shows that the minimum radius of the desired ellipse with respect to the focus F_2 is $a - c$. Since $= a\tau$, then we have: $a - c = a(1-\tau)$. On the other hand, the minimum radius should not be smaller than the minimum radius R_{min} recommended by AASHTO (Eq. 1). Thus, we have:

$$a(1-\tau) \geq R_{min}\ . \tag{11}$$

Since we are looking for an elliptical curve to connect the PC to PT, it would be an arc of an ellipse that satisfies the equality below as a constraint:

$$a(1-\tau) = \mathrm{R}_{\min} . \tag{12}$$

Therefore, we should first find an appropriate ellipse and then identify the desired arc to be used as a highway curve. In our design problem, as in most highway horizontal alignment problems, the known parameters are the location of PI, the angle , and the design speed, V_d. Based on the known design speed, we can determine a value for R_{min} . With R_{min} known,

we now need to identify an equivalent elliptical arc. In Eq. 12, we have two unknown variables $(a \text{ and } \tau)$ relating to the ellipse. Using numeric methods, we can find all pairs of (a, τ) which satisfy our constraint by inserting acceptable values for τ and solving the equation for a. The eccentricity of ellipse, τ ranges from 0 to 1. To make a finite set of values for τ, we should consider only one or two decimal points depending on the level of accuracy required.

By having the major axis a and the eccentricity τ, the equivalent ellipse can be easily identified as,

$$b = a\sqrt{1-\tau^2}. \tag{13}$$

Other than the minimum radius requirement, another design constraint requires that the arc of the ellipse be tangent to the lines connecting the PI to the PT and the PC. A third constraint is an aesthetic consideration. According to AASHTO, symmetric designs enhance the aesthetics of highway curves. Therefore, a symmetric arc of the ellipse is desirable to meet the aesthetics requirement. Note that since the desired arc should be symmetrical and be of minimum possible length, the arc must be symmetric with respect to the ellipse's major axis (and not the minor axis).

Let us assume a hypothetical ellipse in the Cartesian coordinate system with the center at the origin and the focus on the y-axis (Fig. 4). Suppose that the desired arc is the smallest arc between points A = PT = (x_1, y_1) and B = PC = (x_2, y_2). Since the arc is symmetric with respect to the major axis, we have $x_1 = -x_2$, and $y_1 = y_2$.

Let us also assume that the slope of the tangent line at points and are m_1 and m_2, respectively. So, $m_1 = -m_2$.

As shown in Figure 4, the long chord for the desired arc of the ellipse and the tangent lines form an isosceles triangle. Therefore,

$$m_1 = \tan\left(180 - \frac{\Delta}{2}\right), \tag{14}$$

and

$$m_2 = -\tan\left(180 - \frac{\Delta}{2}\right). \tag{15}$$

On the other hand, the equation of the ellipse in the Cartesian coordinate system is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \tag{16}$$

By taking the derivative of Eq. 16, the slope of the tangent line at any point on the ellipse is obtained, namely,

$$\frac{dy}{dx} = \frac{-a^2 x}{b^2 y}. \tag{17}$$

Eq. 17 can be re-written as:

$$y = \pm\frac{a}{b}\sqrt{b^2 - x^2} . \tag{18}$$

By combining Eqs.17 and 18, we obtain:

$$\frac{dy}{dx} = \frac{-ax}{b\sqrt{b^2 - x^2}} . \tag{19}$$

Therefore,

$$m_1 = \frac{-ax_1}{b\sqrt{b^2 - {x_1}^2}} , \tag{20}$$

and

$$x_1 = \frac{m_1 . b^2}{\sqrt{a^2 + {m_1}^2 . b^2}} , \tag{21}$$

$$y_1 = \frac{a}{b}\sqrt{b^2 - x_1^2}\,. \tag{22}$$

By the same token, the location of point $B = (x_2, y_2)$ is determined to be:

$$x_2 = -x_1 = \frac{-m_1.b^2}{\sqrt{a^2 + m_1{}^2.b^2}} \tag{23}$$

and

$$y_2 = y_1. \tag{24}$$

Consequently, any desired arc of the ellipse can be found by having the slope (direction) of the tangent lines and the intersection angle between them.

2.3 Length of the Arc of an Ellipse

To minimize the right-of-way, a minimum-length arc is desired that meets all three constraints discussed earlier. In the previous section, a method was introduced to identify an arc that only satisfies the tangent lines constraint. After this, the resulting arc should be checked to ensure it is the minimum-length arc.

To find the length of an ellipse arc, the polar coordinate system is again useful. The length of an ellipse arc between 0 and θ can be found from the following integration (Larson et al., 2010, p. 704):

$$E(\theta, \tau) = a\int_0^{\theta} \sqrt{1-(\tau)^2 sin^2 t}\;\; dt \tag{25}$$

in which,

a: is the length of the major axis of the ellipse;
τ: is the eccentricity of the ellipse; and
$E(\theta, \tau)$: is the length of the arc of an ellipse with eccentricity of τ, between 0 and θ.

To be able to use this integration, we need to know the coordinates of points A and B in the polar coordinate system. Since $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the polar coordinates of points A and B can be obtained to be:

$$\theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right),\; r_1 = \sqrt{x_1^2 + y_1^2}\,; \tag{26}$$

and

$$\theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right),\; r_2 = \sqrt{x_2^2 + y_2^2}\,. \tag{27}$$

Now, let us define the length of the arc as:

$$l_{a,\tau}(\theta_1, \theta_2) = E(\theta_2, \tau) - E(\theta_1, \tau) \tag{28}$$

in which,

θ_1 = the angle at which the ellipse arc starts;
θ_2 = the angle at which the ellipse arc ends; and
$l_{a,\tau}(\theta_1, \theta_2)$ = is the length of the arc starting at angle θ_1 and ending at θ_2 on an ellipse with major axis a and the eccentricity of τ.

2.4 Area of an Elliptical Sector and the Right-Of-Way

As discussed earlier, there are two ellipse sectors commonly used. One is defined with respect to the center of the ellipse and the other with respect to a focus. The ellipse's equation with respect to the focus is used here to calculate the area of the sector (Figure 5), as follows:

$$A_{E,\tau}(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta)\; d\theta \; = \int_{\theta_1}^{\theta_2} \frac{a^2(1-\tau^2)^2}{2[1 \pm \tau\cos\theta]^2}\, d\theta\,. \tag{29}$$

Let us assume that the right-of-way width is 100 ft., 50 ft. on each side of the centerline. In Figure 6, the shaded strip shows the right-of-way associated with an elliptical arc. The right-of-way can be accordingly calculated as:

$$ROW = \int_{\theta_1}^{\theta_2} \frac{1}{2}(r(\theta) + 50)^2 d\theta \; - \; \int_{\theta_1}^{\theta_2} \frac{1}{2}(r(\theta) - 50)^2 d\theta \,. \tag{30}$$

2.5 Chord Length and Deflection Angle Calculations

Assuming that the desired arc of an ellipse connecting the PC to PT is identified, the next step is the calculation of the chord lengths and deflection angles. In Figure 7, the chord length l_c and the deflection angle δ are schematically shown. Using the polar equation with respect to the center of the ellipse, we know that:

$$r(\theta) = \frac{ab}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}} \,. \tag{31}$$

As shown in Figure 7, OB, BD, and DO form the triangle OBD. The length of OB and DO can be calculated by inserting θ_1 and θ_2 in the polar equation of the ellipse. Let us suppose that the deflection angles need to be calculated in decrement of α from θ_2 to θ_1. Therefore, the angle between OB and DO is α, as shown in Figure 7. Thus, the length of chord l_c can be obtained by applying the law of cosines:

$$l_c^2 = r(\theta_2)^2 + r(\theta_1)^2 - 2\, r(\theta_2)\, r(\theta_1)\, \cos(\alpha), \tag{32}$$

Now, we need to find the deflection angle, δ. According to the law of sines, in the triangle OBD we have:

$$\frac{\sin(\gamma)}{r(\theta_2)} = \frac{\sin(\alpha)}{l_c} \,. \tag{33}$$

Then:

$$\gamma = \sin^{-1}\left(\frac{r(\theta_2)\sin(\alpha)}{l_c}\right) \tag{34}$$

In Figure 7, we also have:

$$\varphi = 180° - \theta_1 \tag{35}$$

On the other hand, in the triangle BDE:

$$\begin{aligned} \beta &= 180° - \gamma - \varphi \\ &= 180° - \gamma - (180° - \theta_1) = \theta_1 - \gamma \,. \end{aligned} \tag{36}$$

Then,

$$\delta = \mathrm{B}_1 - \beta = \frac{\Delta}{2} - (\theta_1 - \gamma) = \frac{\Delta}{2} - \theta_1 + \gamma. \tag{37}$$

2.6 Station Number Calculations for PC and PT

Referring to Figure 7, the locations of points A (PT) and B (PC) are known. Based on definition, the intersection of the tangent lines at points A and B will be the location of the PI. The equations of tangent lines are:

$$\text{Tangent Line at A: } y = y_1 + m_1(x - x_1), \tag{38}$$

and

$$\text{Tangent Line at B: } y = y_2 + m_2(x - x_2)\,. \tag{39}$$

By solving Eqs.38 and 39 simultaneously, the location of the PI can be determined.
Note that x_1, x_2, y_1, y_2, m_1, and m_2 are all known.

Then:

$$x^* = \frac{y_1 - y_2 + m_1 x_1 - m_2 x_2}{m_1 - m_2} \tag{40}$$

And since $m_1 = -m_2$:

$$x^* = \frac{y_1 - y_2 + m_1(x_1 + x_2)}{2m_1} \,, \tag{41}$$

and

$$y^* = y_1 + m_1(x^* - x_1). \tag{42}$$

Therefore, the length of tangent **T** is:

$$\mathrm{T} = \sqrt{(y^* - y_1)^2 + (x^* - x_1)^2}\,. \tag{43}$$

Given the tangent length T above, the length of the elliptical arc $l_{a\,elp}$ can be calculated as:

$$\begin{aligned} l_{a\,elp} &= E(\theta_2, \tau) - E(\theta_1, \tau) \\ &= a\left(\int_0^{\theta_2} \sqrt{1-(\tau)^2 sin^2 t}\ \ dt\ - \int_0^{\theta_1} \sqrt{1-(\tau)^2 sin^2 t}\ \ dt\right). \end{aligned} \tag{44}$$

Therefore, the stations numbers for the PC and the PT can be determined to be:

$$Sta.@\,\mathbf{PC} = \ Sta.@\,\mathbf{PI} - \ \mathbf{T} \tag{45}$$

and

$$Sta.@\,\mathbf{PT} = \ Sta.@\,\mathbf{PC} + \ \boldsymbol{l_{a_{elp}}}. \tag{46}$$

2.7 Finding the Minimum Arc of Ellipse from the PC to PT

As discussed earlier, the angle Δ, the design speed design, V_d, and the location of the PI are typically known. Given that, the following steps could then be followed to find the desired arc of ellipse connecting the PC to the PT:

Algorithm (A)

0. Angle Δ, design speed V_d, and location of PI are given.
1. According to the design speed V_d, the value of R_{min} for V_d is known.
2. Start with eccentricity τ of 0.1.
3. Find the major axis a by inserting the value of τ into

$$a = \frac{R_{min}}{1-\tau}. \tag{47}$$

4. Calculate the minor axis b:

$$b = a\sqrt{1-\tau^2}\,. \tag{48}$$

5. Find the slope of the tangent line at point A, m_1:

$$m_1 = \tan\left(180 - \frac{\Delta}{2}\right) \tag{49}$$

6. Find the slope of the tangent line at point B, m_2:

$$m_2 = -m_1. \tag{50}$$

7. Find the x-coordinate of point A, x_1:

$$x_1 = \frac{m_1\, b^2}{\sqrt{a^2 + {m_1}^2\, b^2}} \tag{51}$$

8. Find the y-coordinate of point A, y_1:

$$y_1 = \frac{a}{b}\sqrt{b^2 - x_1^2}\,. \tag{52}$$

9. Determine $\theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$.
10. Determine $\theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right)$.
11. Calculate the length of the elliptical arc,

$$l_{a,\tau}(\theta_1,\theta_2) = E(\theta_2,\tau) - E(\theta_1,\tau) \tag{53}$$

in which

$$E(\theta,\tau) = a\int_0^{\theta}\sqrt{1-(\tau)^2 sin^2 t}\ \ dt \tag{54}$$

12. Calculate the area of the piece of ellipse, $A_{E,\tau}(\theta_1,\theta_2)$:

$$A_{E,\tau}(\theta_1,\theta_2) = \frac{1}{2}\int_{\theta_1}^{\theta_2}\left[\left(\frac{a(1-\tau^2)}{1\pm\tau\cos(\theta)}+50\right)^2 - \left(\frac{a(1-\tau^2)}{1\pm\tau\cos(\theta)}-50\right)^2\right].d\theta. \tag{55}$$

13. Repeat the preceding steps for a new eccentricity τ with increments of 0.1 until the current τ is 0.9.
14. Compare the length of the arc and the area of the piece of ellipse gained for each value of eccentricity τ, and pick the eccentricity τ with the minimum length of arc and area. This is the desired minimum-length elliptical arc to be used.

2.8 Calculating Chords Length and Deflection Angles

After the desired elliptical arc is determined, chord lengths and deflection angles should be calculated in order to stake out the elliptical arc, as follows:

<u>Algorithm (B)</u>

0. θ_1 and θ_2 are obtained in steps 9 and 10 above
1. Degree of curvature is:

$$D = \frac{(\theta_2-\theta_1)\times 100}{l_{a,\tau}(\theta_1,\theta_2)}. \tag{56}$$

2. If $\frac{(\theta_2-\theta_1)}{D}$ is an integer, then

$$N = \frac{(\theta_2-\theta_1)}{D}\ ; \tag{57}$$

else

$$N = \left\lceil\frac{(\theta_2-\theta_1)}{D}\right\rceil. \tag{58}$$

3. Let $i = 1$.
4. $\alpha = i \times D$
5. Calculate $\theta^* = \theta_2 - \alpha$.
6. Find the length of chord by applying the equation below:

$$(l_c)_i = \sqrt{r(\theta_2)^2 + r(\theta^*)^2 - 2\,r(\theta_2)\,r(\theta^*)\cos(\alpha)} \tag{59}$$

in which

$$r(\theta) = \frac{ab}{\sqrt{(a\cos\theta)^2+(b\sin\theta)^2}}. \tag{60}$$

7. Find γ:

$$\gamma = \sin^{-1}\left(\frac{r(\theta_2)\sin(\alpha)}{l_c}\right). \tag{61}$$

8. Find deflection angle, δ_i:

$$\delta_i = \frac{\Delta}{2} - \theta^* + \gamma . \tag{62}$$

9. $i = i + 1$.
10. Go to step 4 and repeat until $i = N$.
11. Now, we have the deflection angle and the corresponding chord length for any station along the elliptical arc.

III. An Application Example and Its Results

Let us assume that it is desired to connect the PC to PT through an elliptical arc such that $\Delta = 120°$, $R_{min} = 1000$ ft, and the Sta. # at PI $= 40 + 40$. First, the arc of ellipse should be found so that it satisfies the initial constraints. Applying the algorithm (A) yields the results tabulated in Table 1. Comparing the length of the arc and the right-of-way area, the ellipse with $\tau = 0.1$ provides the minimum length and the minimum right-of-way. Therefore, the desired ellipse is an ellipse with major axis a of 1111.1 ft. and minor axis b of 1105.5 ft. Using the algorithm (B), chord lengths and deflection angles can be obtained, as shown in Table 2.

3.1 The Equivalent Circular Curve Solution

Again, let us assume that it is desired to connect the PC to PT, but this time through a circular curve such that $\Delta = 120°$, $R_{min} = 1000$ ft, and $Sta.\#\ at$ PI $= 40 + 40.0$.
Accordingly, the length of the tangent **T** is:

$$T = R\tan\left(\frac{\Delta}{2}\right) = 1000\ \tan\left(\frac{120°}{2}\right) = 1732.1 \text{ ft.} \tag{63}$$

On the other hand, the length of arc l_a is:

$$l_a = \frac{\pi}{180}\ \Delta\ R = \frac{\pi}{180} \text{ x } 120 \text{ x } 1000 = 2094.4 \text{ ft.} \tag{64}$$

Given the above, the locations of the PC and PT along with the deflection angles and chord lengths for the stations in between can be determined using the conventional circular curve relations.

Note that in highway design, the length of horizontal alignment and its associated right-of-way are two significant variables in evaluating alternative designs. According to the results obtained for the circular versus the elliptical approach, the right-of-way for circular curve connecting A to D, shown Figure 8, is:

$$ROW_{Circular} = (\overline{AB} \times 100) + \frac{\Delta\,\pi}{360} \times [(r_{\overline{BC}} + 50)^2 - (r_{\overline{BC}} - 50)^2] + (\overline{CD} \times 100) \tag{65}$$

where,

$$\begin{aligned}\overline{AB} = \overline{CD} &= \frac{LC_E - LC_C}{2} \times \frac{1}{\cos\left(\frac{\Delta}{2}\right)} \\ &= \frac{1913 - 1732}{2} \times \frac{1}{\cos\left(\frac{120}{2}\right)} = \frac{181}{2} \times 2 = 181 \text{ ft.}\end{aligned} \tag{66}$$

Then,

$$\begin{aligned}ROW_{Circular} &= (181 \times 100) + \frac{\pi}{3} \times [(1000 + 50)^2 - (1000 - 50)^2] \\ &+ (181 \times 100) = 24{,}5633 \text{ sq. ft.} = 5.64 \text{ acres.}\end{aligned} \tag{67}$$

The roadway length from A to D through circular curve can be computed as:

$$\begin{aligned}L_{ABCD} &= L_{AB} + L_{BC} + L_{CD} = \overline{AB} + l_{a\,Circular} + \overline{CD} \\ &= 181 + 2094.3 + 181 = 2{,}456.3 \text{ ft.}\end{aligned} \tag{68}$$

Table 3 is a comparison of the ROW requirements for the simple circular versus the elliptical curves depicted in Figure 8 above. As shown, the elliptical curve is 137 ft. shorter in length than the equivalent simple circular curve. Another possible advantage of the elliptical alternative, not apparent in Figure 8 or in Table 3, is that the transition from the normal crown to the fully superelevated cross-section and back can be achieved more gradually through the entire length of the elliptical arc. This provides for a smoother cross-sectional transition. However, the circular curve needs a somewhat smaller right of way, 0.17 acres less than the elliptical curve in this example.

3.2 Comparing the Spiral-Circular Curve Results to the Elliptical Curve Results

In Figure 9, both the elliptical curve and the equivalent spiral-circular curve are shown. The moderately thick curves are the spiral curves and the thin curve in the middle is the circular curve. The thick curve is the elliptical curve. The right-of-way and the length of the spiral-circular curve can be computed as follows:

$$\begin{aligned} ROW_{\text{Spiral}\ -\text{Circular}} = &(\overline{AB} \times 100) \\ &+ \frac{\Delta_C \pi}{360} \times [(r_{\overline{BC}} + 50)^2 - (r_{\overline{BC}} - 50)^2] \\ &+ \frac{2\theta_s \pi}{360} \times [(r_{\overline{B-S.C.}} + 50)^2 - (r_{\overline{B-S.C.}} - 50)^2] \\ &+ (\overline{CD} \times 100) \end{aligned} \quad (69)$$

where,

$$\overline{\mathrm{AB}} = \overline{\mathrm{CD}} = \frac{\mathrm{LC_E} - \mathrm{LC_{S.C.}}}{2} \times \frac{1}{\cos\left(\frac{\Delta}{2}\right)} = \frac{1913.2 - 1865.6}{2} \times \frac{1}{\cos\left(\frac{120}{2}\right)} = 47.6 \text{ ft.} \quad (70)$$

Then,

$$ROW_{\text{Spiral}\ -\text{Circular}} = 32{,}119 \text{ sq. ft.} = 5.33 \text{ acres}. \quad (71)$$

Length of roadway from A to D through spiral-circular curve is:

$$\begin{aligned} L_{ABCD} &= L_{AB} + L_{B-S.C.} + L_{BC} + L_{S.C.-C} + L_{CD} = \overline{\mathrm{AB}} + l_s + l_{a\,Circular} + l_s + \overline{\mathrm{CD}} \\ &= 47.6 + 263 + 1{,}831.1 + 263 + 47.6 = 2{,}452.3 \text{ ft.} \end{aligned} \quad (72)$$

A comparison of the length and ROW requirements for these two curves is also shown in Table 3 above. It can be noted in Table 3 that the elliptical curve is 133 ft. shorter than its equivalent spiral-circular curve. However, the spiral-circular curve needs a slightly smaller right of way, 0.48 acres less than the elliptical curve for this specific example.

IV. Conclusions and Recommendations

Based on the results presented, elliptical curves can be used as viable horizontal transition curves in lieu of simple circular or spiral-circular curves. A possible advantage in using elliptical curves is that elliptical curves can shorten the length of the roadway as shown in the application example while providing a smoother transition in terms of more gradual increase in centrifugal forces. Another possible advantage is that the transition from the normal crown to the fully superelevated cross-section and back to the normal crown can be achieved more gradually through the entire length of the elliptical arc. Therefore, it can also provide a smoother cross-sectional transition and one that is likely more aesthetically pleasing.

As a result, elliptical curves should be considered as an alternative design for horizontal alignments. For instance, for each specific horizontal alignment problem with a given intersection angle and design speed V_d, alternative calculations for simple circular, spiral-circular, compound circular, and elliptical curves can be conducted. Then, the results for each alternative should be compared with respect to the arc length and ROW requirements to optimize the design.

In terms of calculations, the key equation to find the elliptical arc length is an elliptic integral, known as complete elliptic integral of the second kind. This integral should be numerically estimated for each feasible ellipse satisfying the intersection angle and the design speed. Therefore, it is recommended to develop a software to find the most suitable elliptical curve for any given and V_d. Also, elliptical calculations as an alternative design to circular, circular compound, or spiral-circular alignments should be incorporated in highway design software packages such as Geopak (Bentley Systems, 2012) and Microstation (Bentley Systems, 2012). There may also be geometric and aesthetics benefits in using elliptical arcs for reverse curves; an aspect that can be investigated as an extension of this work.

Another computational aspect not addressed here is the sight distance computations associated with tall roadside objects that may interfere with driver's line of sight. In circular curves, this is typically addressed by computing the middle ordinate distance from the driving edge of the road, which establishes a buffer area on the inside of the circular curve to be kept free of potential line of sight obstructions. If an elliptical arc is used instead, equivalent calculations would be necessary. However, in lieu of conducting those computations, the equivalent circular middle ordinate will be a conservative and safe value to use.

Regarding environmental issues, using elliptical curves has the potential to reduce air pollutants as well. Elliptical curves can shorten the length of the roadway as well as provide a smoother transition from the normal crown to the fully superelevated cross-section and back. Both of these properties could reduce vehicular fuel consumption. During a roadway's design life, an elliptical curve can therefore save road users a significant amount of fuel. Less fuel consumption also typically results in less air pollution. In addition, in the case of asphalt pavements, the shorter length of the roadway will decrease solar radiation absorbed by the asphalt surface. Therefore, elliptical curves can be more environmentally beneficial as they have the potential to substantially reduce air pollution and solar radiation absorbed by the asphalt surface over the design life of the roadway. Another possible extension of this work could be a user-cost study of elliptical versus the more conventional horizontal alignments. The user cost could be quantified in terms of fuel consumption and air pollutants over the design life of a project and be utilized in evaluation of alternative designs.

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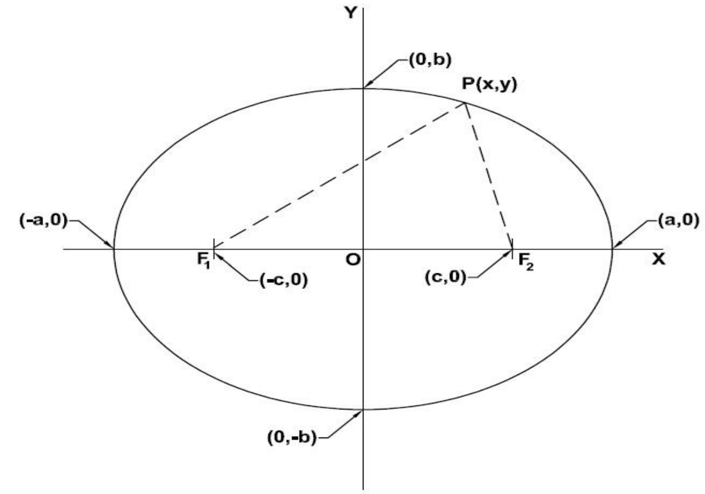

Figure 1. Cartesian components of a point on ellipse with the center at the origin.

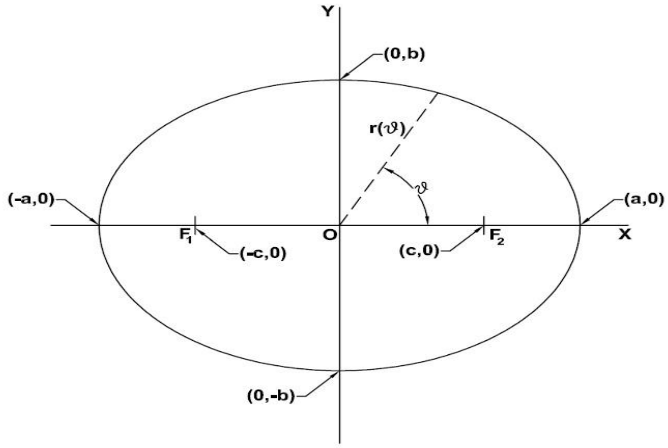

Figure 2. Polar Coordinate System with Origin at Center of the Ellipse.

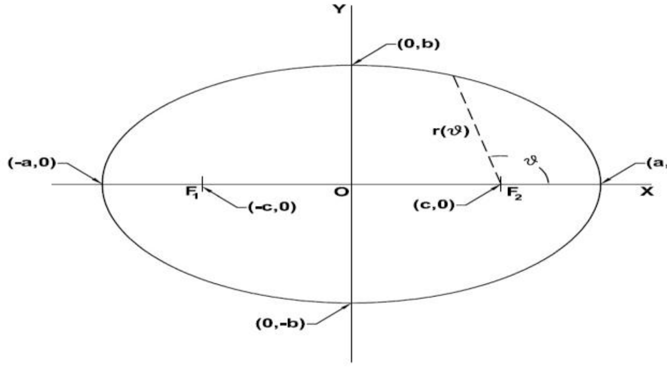

Figure 3. Polar Coordinate System with Origin at a Focus of the Ellipse.

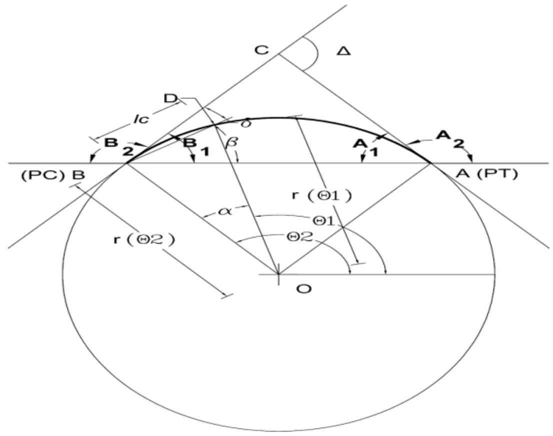

Figure 4. An arc of ellipse needed to connect PC to PT.

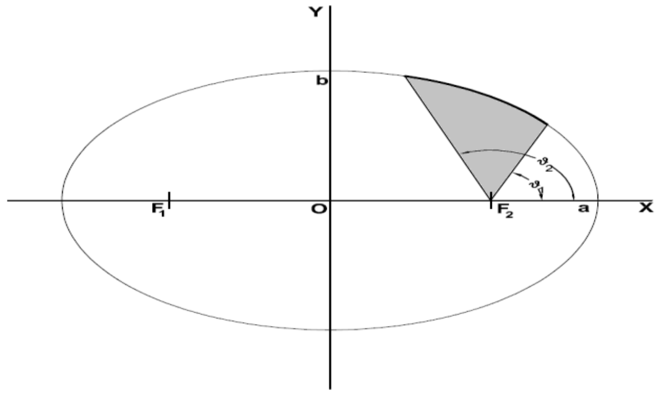

Figure 5. Sector of ellipse with respect to the focus.

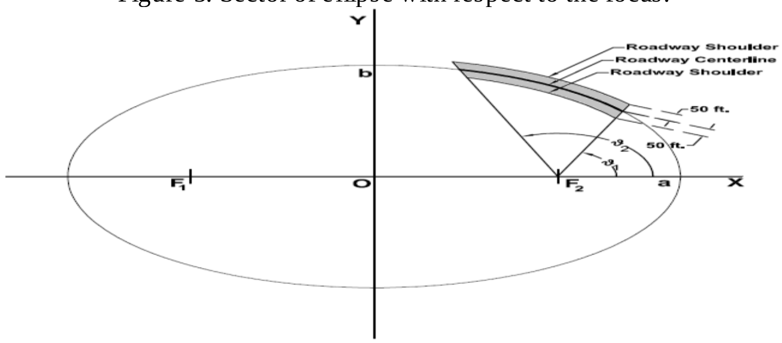

Figure 6. Elliptical Arc Right-Of-Way.

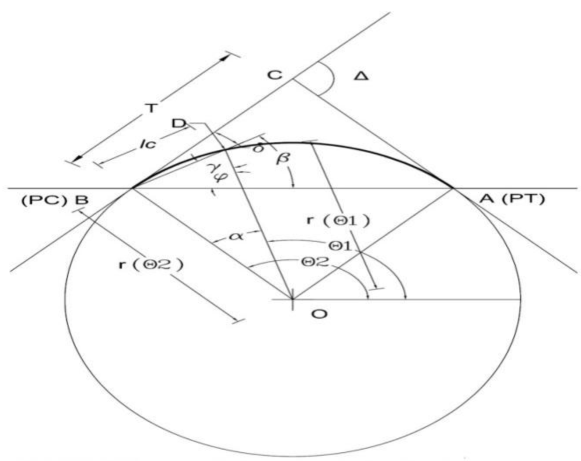

Figure 7. Diagram of an Elliptical Arc.

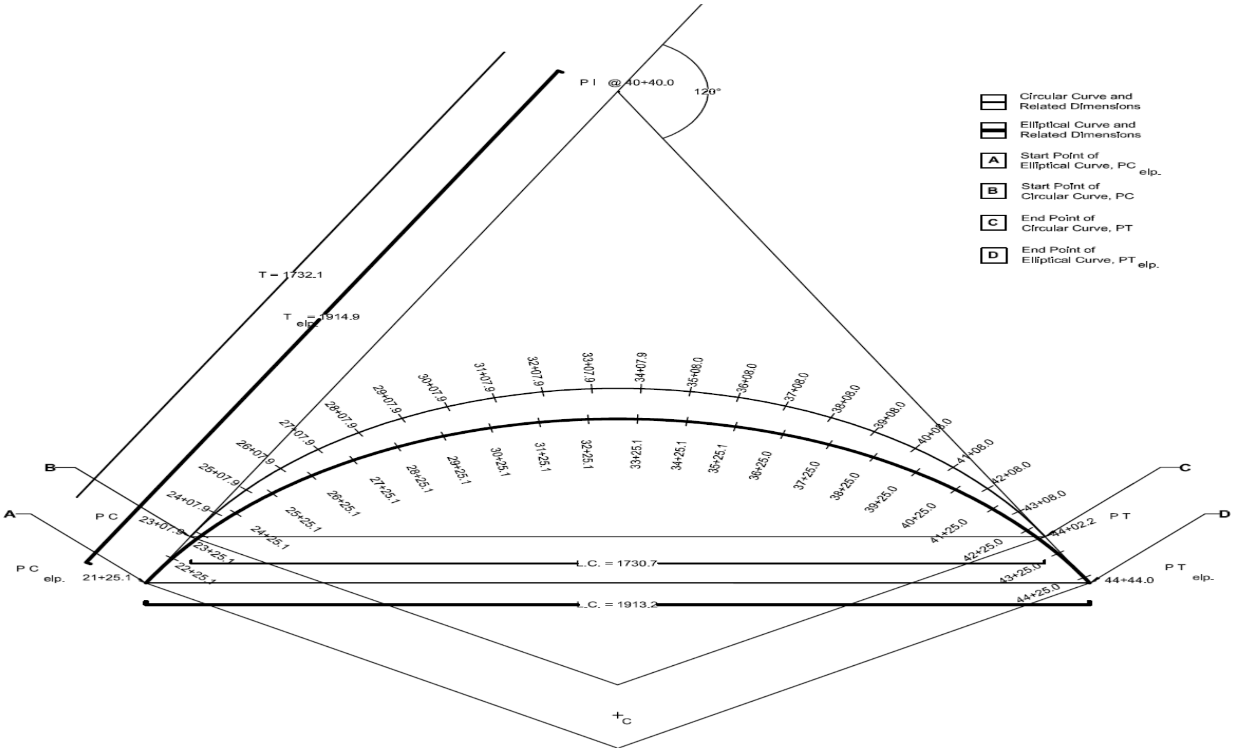

Figure 8. Final Profile: Elliptical curve vs. circular curve.

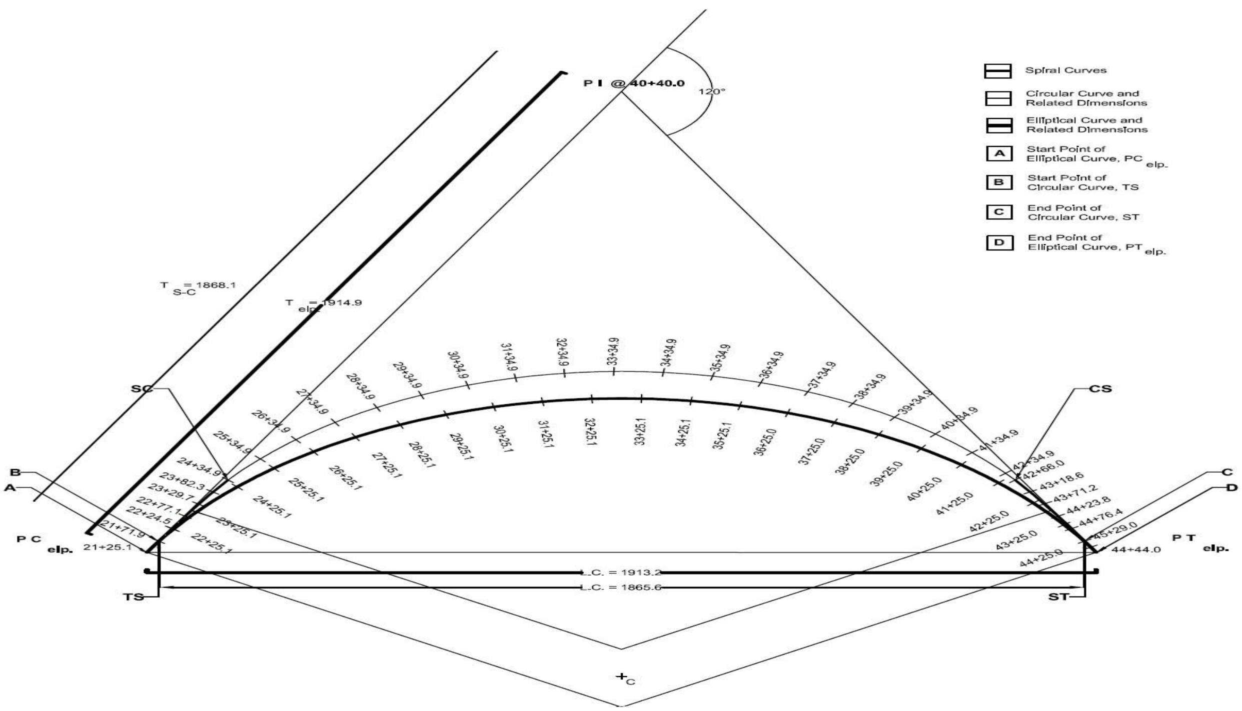

Figure 9. Final Profile. Elliptical curve vs. spiral-circular curve.

Table 1. Determining the Most Suitable Elliptical Arc

Eccentricity τ	R_{min}	a (ft.)	b (ft.)	Δ (deg.)	Slope m_1	Point A x_1 (ft.)	Point A y_1 (ft.)	Point B x_2 (ft.)	Point B y_2 (ft.)	θ_1 (deg.)	θ_2 (deg.)	$l_{a,\tau}(\theta_1,\theta_2)$ (ft.)	ROW) (Sq.ft.)
0.1	1000	1111.1	1105.5	120	-1.734	956.5	557.1	-956.5	557.1	30.22	149.78	2318.9	252,894
0.2	1000	1250.0	1224.7	120	-1.734	1055	635.2	-1055	635.2	31.06	148.94	2570.2	308,832
0.3	1000	1428.6	1362.8	120	-1.734	1166	739.3	-1166	739.3	32.37	147.63	2869.8	385,238
0.4	1000	1666.7	1527.5	120	-1.734	1293	887.3	-1293	887.3	34.46	145.54	3219.2	492,565
0.5	1000	2000.0	1732.1	120	-1.734	1442	1108	-1442	1108	37.54	142.46	3636.5	652,790
0.6	1000	2500.0	2000.0	120	-1.734	1622	1462	-1622	1462	42.04	137.96	4131.2	909,085
0.7	1000	3333.3	2380.5	120	-1.734	1852	2094	-1852	2094	48.52	131.48	4734.7	1,369,860
0.8	1000	5000.0	3000.0	120	-1.734	2163	3466	-2163	3466	58.04	121.96	5467.8	2,366,885
0.9	1000	10000.0	4358.9	120	-1.734	2628	7976	-2628	7976	71.8	108.20	6283.4	5,593,074

Table 2. Chord Lengths and Deflection Angles for Staking Out the Elliptical Arc

	Station Number	θ^* (deg.)	$r(\theta^*)$ (ft.)	Deflection Angle (deg.) δ	Chord Length (ft.) l_c	Arc Length (ft.) $l_{a,\tau}(\theta^*,\theta_2)$
Sta. # @ PC	21+25.1	149.80	1106.9	0	0.0	0.0
	22+25.1	144.64	1107.4	2.52	99.6	100.0
	23+25.1	139.49	1107.9	5.09	199.1	200.0
	24+25.1	134.33	1108.4	7.66	298.2	300.0
	25+25.1	129.17	1108.9	10.24	396.7	400.0
	26+25.1	124.02	1109.4	12.82	494.5	500.0
	27+25.1	118.86	1109.8	15.41	591.3	600.0
	28+25.1	113.70	1110.2	17.99	686.9	700.0
	29+25.1	108.55	1110.5	20.58	781.2	800.0
	30+25.1	103.39	1110.8	23.18	873.8	900.0
	31+25.1	98.23	1111.0	25.77	964.7	1000.0
	32+25.1	93.08	1111.1	28.37	1053.6	1100.0
	33+25.1	87.92	1111.1	30.96	1140.4	1200.0
	34+25.1	82.76	1111.0	33.56	1224.8	1300.0
	35+25.1	77.60	1110.9	36.16	1306.6	1400.0
	36+25.0	72.45	1110.6	38.76	1385.7	1499.9
	37+25.0	67.29	1110.3	41.36	1462.0	1599.9
	38+25.0	62.13	1109.9	43.96	1535.2	1699.9
	39+25.0	56.98	1109.4	46.55	1605.3	1799.9

	40+25.0	51.82	1109.0	49.15	1672.1	1899.9
	41+25.0	46.66	1108.5	51.74	1735.4	1999.9
	42+25.0	41.51	1108.0	54.33	1795.2	2099.9
	43+25.0	36.35	1107.5	56.92	1851.3	2199.9
	44+25.0	31.19	1107.0	59.50	1903.7	2299.9
Sta. # @ PT	44+44.0	30.22	1106.9	59.99	1913.2	2318.9

Table 3. A Comparison of Arc Length and ROW Requirements for the Three Alternative Curves

	Simple Circular Curve	Spiral-Circular Curve	Elliptical Curve
Length	2456 ft.	2452 ft.	2319 ft.
Right-Of-Way	5.64 acres	5.33 acres	5.81 acres

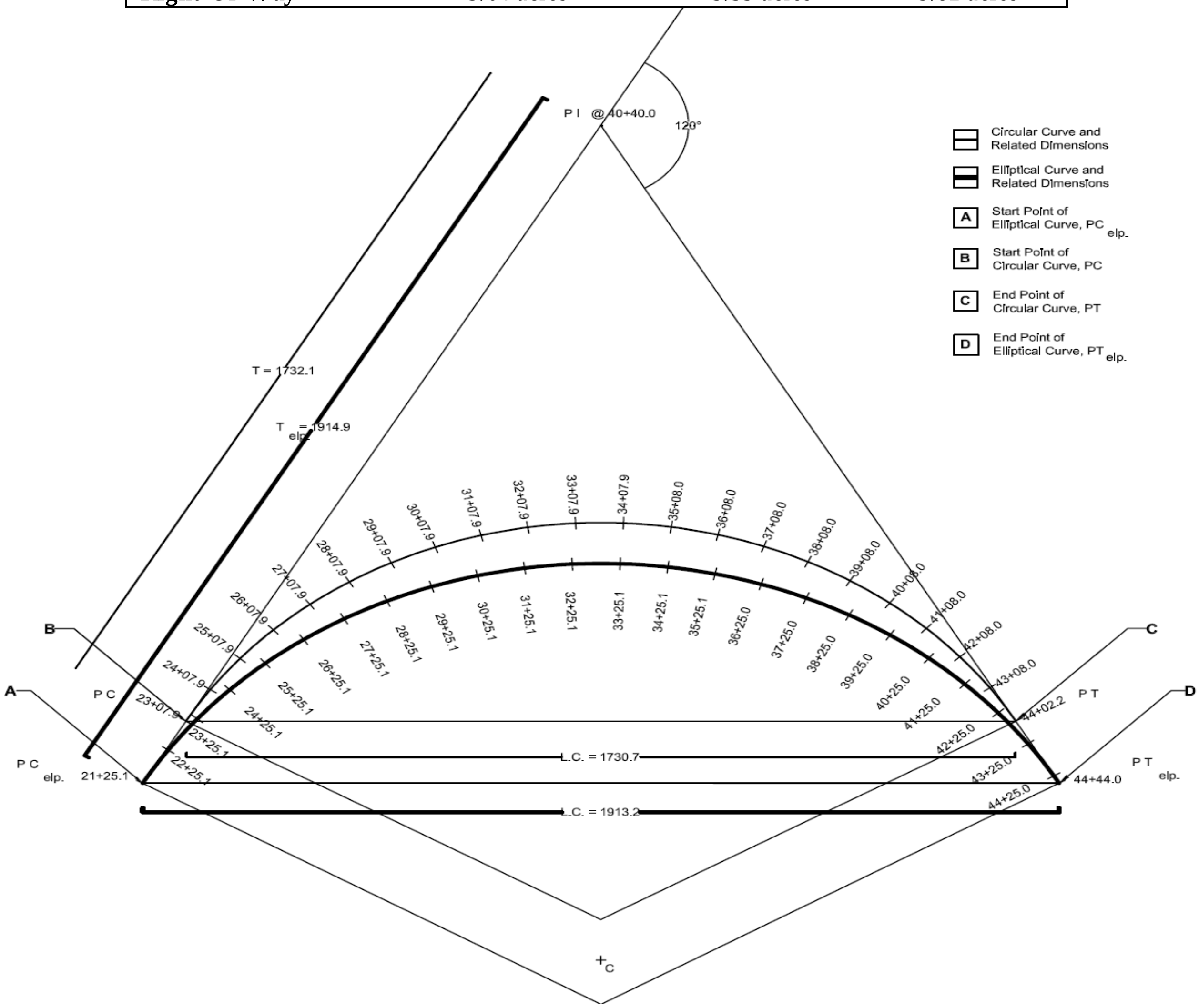

Figure 8. Final Profile: Elliptical curve vs. circular curve.

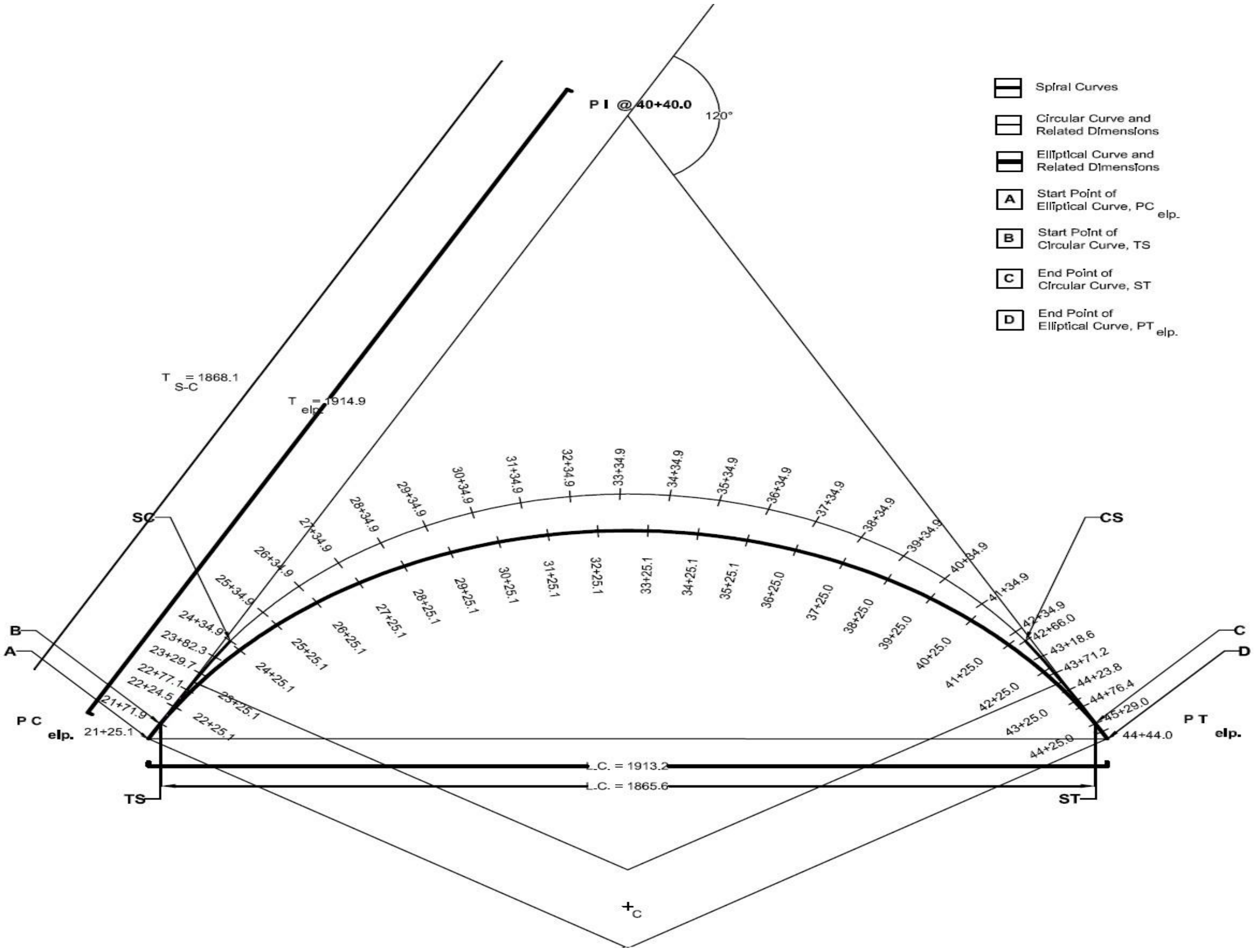

Figure 9. Final Profile. Elliptical curve vs. spiral-circular curve.