Statistical Distributions involving Meijer's G-Function of matrix Argument in the complex case

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Abstract:

The aim of this paper is to investigate the probability distributions involving Meijer's g Functions of matrix argument in the complex case. Many known or new Result of probability distributions have been discussed. All the matrices used here either symmetric positive definite or hermit ions positive definite.

I Introduction: The G-Function

The G-functions as an inverse matrix transform in the following form given by Saxena and Mathai (1971).

$$\int_{z>0} |\tilde{z}|^{\delta-m} G_{r,s}^{p,q}\left[\tilde{z}\middle|\begin{matrix} a_1 \ldots\ldots a_r \\ b_1 \ldots\ldots b_r\end{matrix}\right] \overline{dz} = \frac{\prod_{j=1}^{p} \tilde{\Gamma}_m (b_j+\delta) \prod_{j=1}^{q} \tilde{\Gamma}_m (m-a_j-\delta)}{\prod_{j=P+1}^{s} \tilde{\Gamma}_m (m-a_j-\delta) \prod_{j=q+1}^{r} \tilde{\Gamma}_m (b_j+\delta)} \qquad 1.1$$

For p < q or p = q, q ≥ 1, z is a complex matrix and $\tilde{z} > 0, Re\left(b_j+\delta\right) > m-1, (j=1,\ldots,p) and\ Re\left(a_j+\delta\right) < m-1 (j=1,\ldots,q)$. the gamma products are such that the poles of $\prod_{j=1}^{p} \tilde{\Gamma}_m (b_j+\delta)$ and those of $\prod_{j=1}^{p} \tilde{\Gamma}_m (m-a_j-\delta)$ are separated.

II. The Distribution

In the multivariate Laplace type integral

$$I = \int_{x>0} etr(-\tilde{B}\tilde{X}) |\det \tilde{X}|^{a-m} \phi(\tilde{X}) d\tilde{X}$$

taking $\phi(\tilde{X}) = G_{r,s}^{p,q}\left[\tilde{R}\tilde{X}\middle|\begin{matrix} a_1 \ldots\ldots a_r \\ b_1 \ldots\ldots b_r\end{matrix}\right]$

The integral reduces to

$$I = \left|\det(\tilde{B})\right|^{-a} G_{r+1,s}^{p,q+1}\left[\tilde{R}\tilde{B}^{-1}\middle|\begin{matrix} m-a, a_1 \ldots\ldots a_r \\ b_1 \ldots . b_s\end{matrix}\right] \qquad 2.1$$

For $Re(-a+\min b_j) > m (j=1,2,3,\ldots,p)$ and $\tilde{B}$ is hermitian positive definite matrix and $\tilde{R}$ is an arbitrary complex symmetric m X m matrix.

The result (2.1) is a direct consequence of the result Mathai and Saxena (1971, 1978).

[Notation $etr\left(-\tilde{B}\tilde{R}\right) = \exp\left[tr(X)\right]$ for exponential to the power $tr(X)$]

Thus the function

$$f(\tilde{x}) = f\left(\tilde{x}; a, a_1, \ldots, b_1, \ldots\ldots b_s;\ \tilde{B}, \tilde{R}\right) = \frac{etr\left(-\tilde{B}\tilde{X}\right) \det x|^{a-m} G_{r,s}^{p,q}\left[\tilde{R}\tilde{X}\middle|\begin{matrix} a_1 \ldots\ldots a_r \\ b_1 \ldots\ldots b_s\end{matrix}\right]}{|\tilde{B}|^{-a} G_{r+1,s}^{p,q+1}\left[\tilde{R}\tilde{B}^{-1}\middle|\begin{matrix} m-a, a_1 \ldots\ldots a_r \\ b_1 \ldots . b_s\end{matrix}\right]}$$

where $Re(-a+\min b_j) > m-1$

$= 0$, else where

provides a probability density function (p.d.f)

2.1 Special Cases

Case (i)

Replacing $\tilde{R} = \tilde{I}$, letting $\tilde{B}$ tends to null matrix and using the result due to Mathai (1977)

$$\int_{\tilde{X}=\tilde{X}'>0} |\det \tilde{X}|^{a-m} G_{r,s}^{m,n}\left[\tilde{X}\Big|_{b_1 \dots b_s}^{a_1 \dots a_r}\right] \widetilde{dx} = \emptyset_1(a)$$

where

$$\emptyset_1(a) = \frac{\prod_{j=1}^{p} \tilde{\Gamma}_m(b_j+a) \prod_{j=1}^{q} \tilde{\Gamma}_m(m-a_j-a)}{\prod_{j=p+1}^{s} \tilde{\Gamma}_m(m-b_j-a) \prod_{j=q+1}^{r} \tilde{\Gamma}_m(a_j+a)} \qquad 2.1.1$$

Where $Re(b_j + a) > m; (j = 1, 2, \dots, m)$

In (2.1.1), we get

$$f(\tilde{X}) = [\emptyset_1(a)]^{-1} = |\det(\tilde{X})|^{a-m} G_{r,s}^{p,q}\left[\tilde{X}\Big|_{b_1 \dots b_s}^{a_1 \dots a_r}\right] \qquad 2.1.2$$

where $Re(b_j + a) > m; (j = 1, 2, \dots, m)$

$Re(a_j + a) > m; (j = 1, \dots, n), \tilde{X} = \tilde{X}' > 0 = 0$, elsewhere

Case (ii)

Putting p = 1, q = 0, r = 0, s = 1, B = I, then (2.1.1) takes the form

$$I = \int_{x>0} etr(-\tilde{X}) |\det \tilde{X}|^{a-m} G_{0,1}^{1,0}[\tilde{R}\tilde{X}|a] d\tilde{X} \qquad 2.1.3$$

We know that $G_{0,1}^{1,0}[\tilde{R}\tilde{X}|a] = |\det \tilde{R}|^{a} |\det(\tilde{X})|^{a} e^{-tr \tilde{R}\tilde{X}}$

where $\tilde{X} = \tilde{X}' > 0$ 2.1.4

Using (2.1.4) iin (2.1.3), we have

$$I = |\det R|^{a} \int_{\tilde{X}>0} e^{-tr(\tilde{R}+1)\tilde{X}} |\det(\tilde{X})|^{a+a-m} d\tilde{X} \qquad 2.1.5$$

The integral reduces to

$$= |\det R|^{a} \tilde{\Gamma}_m(a+a) |\det(\tilde{I}+\tilde{R})|^{-(a+a)}$$

(2.1.2) takes the form

$$f(\tilde{X}) = e^{-tr(1+\tilde{R})\tilde{X}} \frac{|\det(\tilde{X})|^{a+a-m}}{\tilde{\Gamma}_m(a+a) |\det(\tilde{I}+\tilde{R})|^{-(a+a)}} \qquad 2.1.6$$

$where\ Re(a+a) > m, Re(I+\tilde{R}) > 0, \tilde{X} = \tilde{X}' > 0$

$= 0, elsewhere$

Which is a gamma distribution.

Taking $(a + a) = m, (2.2.6) takes\ the\ form$

$$f(\tilde{X}) = \frac{e^{-tr(1+\tilde{R})\tilde{X}}}{\tilde{\Gamma}_m(m) |\det(\tilde{I}+\tilde{R})|^{-m}} \qquad 2.1.7$$

$where\ Re(\tilde{I}+\tilde{R}) > 0, \tilde{X} = \tilde{X}' = 0$

$= 0, elsewhere$

Taking $(a + a) = \frac{n}{2}, (\tilde{I}+\tilde{R}) = \frac{1}{2}\tilde{T}^{-1}$,(2.1.6) yields the wishart distribution with scalar matrix $\tilde{T}$ and n degree of freedom.

$$f(\tilde{X}) = \frac{e^{-tr\left(\frac{\tilde{T}^{-1}\tilde{X}}{2}\right)} |\det \tilde{X}|^{\frac{n}{2}-m}}{\tilde{\Gamma}_m\left(\frac{n}{2}\right) \left|\frac{1}{2}\tilde{T}^{-\frac{1}{2}}\right|^{-\frac{n}{2}}}$$

$$= \frac{2^{-\frac{n}{2}} |\det \tilde{X}|^{\frac{n}{2}-m} e^{-tr\left(-\frac{1}{2}\tilde{T}^{-1}\tilde{X}\right)}}{\tilde{\Gamma}_m\left(\frac{n}{2}\right) |\tilde{T}|^{\frac{n}{2}}} \qquad 2.1.8$$

$for\ \tilde{X} = \tilde{X}' > 0, T > 0, m \le n$

$= 0, elsewhere$

Case (iii)

Putting p = 1, q = 0, r = 1, s = 1, then (2.1.1) takes the form

$$\int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X}) |\det \tilde{X}|^{a-m} G_{1,1}^{1,0}\left[\tilde{R}\tilde{X}\Big|_{b}^{a}\right] d\tilde{X} \qquad 2.1.9$$

We know that

$$G_{0,1}^{1,0}\left[\tilde{R}\tilde{X}\Big|_{b}^{a}\right] = \frac{1}{\tilde{\Gamma}_m(a-b)} |\det(\tilde{X})|^{a} |\det(\tilde{I}-\tilde{X})|^{a-b-m} \qquad 2.1.10\ for\ 0 < \tilde{X} < \tilde{I}, Re(a-b) > m$$

Using (2.2.10) in (2.2.9) we have

$$\tilde{X} \frac{|\det(\tilde{R}|^{b}}{\tilde{\Gamma}_m(a-b)} \int_{0<\tilde{X}<\tilde{I}} etr(-\tilde{B}\tilde{X}) |\det(\tilde{R})|^{b+a-m} |\det(\tilde{I}-\tilde{X})|^{a-b-m} d\tilde{X}$$

2.1.11

The integral reduces to

$I = |\det \tilde{X}|^{b} \frac{\tilde{\Gamma}_m(b+a)}{\tilde{\Gamma}_m(a-b)} {}_1\tilde{F}_1(b+a; a+a; -\tilde{B})$ 2.1.12

For $Re(b+a) > m-1, Re(a+a) > m-1$

$= 0, else\ where$

The result (2.1.12) is a direct consequence of the result

$$e^{-tr(\tilde{X}\tilde{Z})} \int |det\ \tilde{X}|^{\delta - m} |\det(\tilde{I} - \tilde{X})|^{p-\delta-m} d\tilde{X}$$

$= \frac{\tilde{\Gamma}_m(\delta)\tilde{\Gamma}_m(p-\delta)}{\tilde{\Gamma}_m(P)} {}_1F_1[\delta; p; -z]$ 2.1.13

For $Re(\delta) > m-1, Re(p) > m-1, Re(p-\delta) > m-1$

Thus the function (2.1.1) takes the form

$$f(\tilde{X}) = \frac{\tilde{\Gamma}_m(a+a) etr(-\tilde{B}\tilde{X}) |\det X|^{b+a-m} |\det(\tilde{I}-\tilde{X})|^{(a-b)-m}}{\tilde{\Gamma}_m(a-b)\tilde{\Gamma}_m(b+a) |{}_1F_1|(b+a; a+a; -\tilde{B})}$$

for $R(a-) > m-1, Re(b+a) > m-1, Re\,\tilde{B}>0, \tilde{X} = \tilde{X}' = 0$

$= 0,\ elsewhere$ 2.1.14

Case (iv)

Putting p = 1, q = 1, r = 1, s = 1, a = m – a + b

Then (2.1.1) takes the form as

$\int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X}) |\det X|^{a-m} \tilde{G}^{11}_{1,1}\left[\tilde{R}\tilde{X}\Big|^{m-a+b}_{b}\right] d\tilde{X}$ 2.1.15

We know that

$\tilde{G}^{11}_{1,1}\left[RX\Big|^{m-a+b}_{b}\right] = \tilde{\Gamma}_m(a) \left|\det \tilde{R}\right|^{b} \left|\det(\tilde{X})\right|^{b} \left|\det(I+\tilde{X}\right|^{-a}$

for $Re(b, a-b) > m-1, \tilde{X} = \tilde{X}' = 0$ 2.1.16

Using (2.1.16) in (2.1.15) we have

$$= \tilde{\Gamma}_m(a) |\det(\tilde{R})|^{b} \times \int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X}) |\det(\tilde{X})|^{a+b-m} |\det(I+\tilde{R}\tilde{X}|^{-a} d\tilde{X}.$$

The integral reduces to

$I = \tilde{\Gamma}_m(a) |\det(\tilde{R})|^{b} \tilde{\Gamma}_m(a+b) |B|^{-(a+b)} {}_2F_0[a, a+b; -; -\tilde{R}\tilde{B}^{-1}]$

For $Re(\tilde{B}) > 0, Re(a+b) > m-1$ 2.1.17

$= 0; elsewhere.$

The result (2.1.17) is a direct consequence of the result

$$\int_{\tilde{X}>0} |\det X|^{a-m} e^{-tr(\tilde{B}\tilde{X})} {}_1F_0[a; -; -\tilde{R}\tilde{X}]\, d\tilde{X} = \tilde{\Gamma}_m(a) |\beta|^{-a} {}_2F_0[a; a; -; -\tilde{R}\beta^{-1}]$$

For $Re(\tilde{B}) > 0, Re(a+b) > m$

Where $|\det(\tilde{I} + \tilde{R}\tilde{X})|^{-a} = {}_1F_0[a; -; -\tilde{R}\tilde{X}$

Thus the p.d.f. (2.1.1) takes the form

$f(\tilde{X}) = \frac{etr(-\tilde{B}\tilde{X}|\det(\tilde{X})|^{a+b-m}|\det(\tilde{I}+\tilde{R}\tilde{X}|^{-a}}{\tilde{\Gamma}_m(a+b)|\beta|^{-(a+b)} {}_2F_0[a, a+b; -; -\tilde{R}\tilde{\beta}^{-1}]}$ 2.1.18

For $Re(\tilde{B}) > 0, Re(a+b) > m-1, Re(\tilde{B}) > Re(\tilde{R}), \tilde{X} = \tilde{X}' > 0$

$= 0; elsewhere.$

Replacing $-\tilde{R}$ with $\tilde{R}$ and $\tilde{B}$ with $\tilde{R}$, then (2.2.18) takes the form as

$f(\tilde{X}) = \frac{etr(-\tilde{R}\tilde{X}|\det(\tilde{X})|^{a+b-m}|\det(\tilde{I}-\tilde{R}\tilde{X}|^{-a}}{\tilde{\Gamma}_m(a+b)|\det\tilde{R}|^{-(a+b)} {}_2F_0[a, a+b; -; I]}$ 2.1.19

For $Re(\tilde{R}) > 0, Re(a+b) > m-1, \tilde{X} = \tilde{X}' > 0$

$= 0; elsewhere.$

Case (v)

Putting p = 1, q = 0, r = 0, s = 2 2.1.20

Then (2.1.1) takes the form as

$I = \int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X} |\det(\tilde{X})|^{a-m} \tilde{G}^{12}_{0,2}[\tilde{R}\tilde{X}|a, b] d\tilde{X}$ 2.1.21

We know that

$\tilde{G}^{10}_{0,2}[\tilde{R}\tilde{X}|a, b] = \frac{|det\ \tilde{X}|^{a} |det\ \tilde{X}|^{a} {}_0F_1[-, m+a-b; -; \tilde{R}\tilde{X}]}{\tilde{\Gamma}_m(m+a-b)}$ 2.1.22

For $Re(a-b) > -1, \tilde{X} = \tilde{X}' > 0$
Making use of (2.1.22) in (2.1.21) we get

$$I = \frac{|\det \tilde{R}|^{a}}{\tilde{\Gamma}_m(m+a-b)} \int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X}|\det(\tilde{X})|^{a-a+m}$$

$$\times {}_1F_1[-;m+a-b;-\tilde{R}\tilde{X}]d\tilde{X}]$$

$$= \frac{|\det \tilde{R}|^{a}\tilde{\Gamma}_m(a+a)}{\tilde{\Gamma}_m(m+a-b)} \times \left[|\tilde{\beta}|^{-(a+a)} {}_1F_1(a+a;m+a-b;-\tilde{R}\tilde{B}^{-1})\right] \quad 2.1.23$$

$for\ Re\ \tilde{B} > 0, Re(a+a) > m-1$
Thus the p.d.f. (2.2.1) takes the form

$$f(\tilde{X}) = \frac{etr(-\tilde{B}\tilde{X})|\det \tilde{X}|^{a+a-m}{}_0F_1[-;m+a-b;-\tilde{R}\tilde{X}]}{\tilde{\Gamma}_m(a+a)|\det \tilde{R}|^{-(a+a)}{}_1F_1[a+a;m+a-b;I]} \quad 2.1.24$$

For $Re\ \tilde{B} > 0, Re(a) > m-1, Re\ \tilde{R} > Re(\tilde{B}), \tilde{X} = \tilde{X}' > 0$
$= 0; elsewhere.$

Case (vi)

Putting p = 1, q = 1, r = 1, s = 2, then (2.1.1) takes the form

$$\int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X})\,|\det \tilde{X}|^{a-m} \tilde{G}_{1,2}^{12}\left[\tilde{R}\tilde{X}\Big|_{b',c}^{a}\right] d\tilde{X} \quad 2.1.25$$

We know that

$$\tilde{G}_{1,2}^{12}\left[\tilde{R}\tilde{X}\Big|_{b',c}^{a}\right] = \tilde{\Gamma}_m(m-a+b)|\det \tilde{X}|^{a}|\det \tilde{X}|^{b} \quad 2.1.26$$

$$\times {}_1F_1[m-a+b;m+b-c;-\tilde{R}\tilde{X} \quad 2.1.27$$

For $Re(b-c, b-a) > -1$
Using (2.1.26) in (2.1.27) we get

$$= \tilde{\Gamma}_m(m-a+b)|\det \tilde{R}|^{\beta} \int_{\tilde{X}>0} etr(-\tilde{B}\tilde{X})|\det \tilde{X}|^{a+\beta-m} \times {}_1F_1(m-a+b;m+b-c;-\tilde{R}\tilde{X} \quad 2.1.28$$

$$= \tilde{\Gamma}_m(m-a+b)|\det \tilde{R}|^{\beta}\tilde{\Gamma}_m(a+\beta)|\det \tilde{R}|^{(a+\beta)}$$

$$\times {}_2F_1[m-a+b;a+\beta;m+b-c;-\tilde{R}\tilde{B}^{-1}] \quad 2.1.29$$

For $Re(\tilde{B}) > 0, Re(a+\beta) > m-1$
The result (2.2.29) is a direct consequence of the result

$$\int_{\tilde{X}>0} |\det \tilde{R}|^{a-m} e^{-tr(\tilde{B}\tilde{X})}\,{}_1F_1[a;b;-\tilde{R}\tilde{X}]d\tilde{X}$$

$$= \tilde{\Gamma}_m(a)|\det(\tilde{B})|^{-\beta}{}_2F_1[a;a;b;-\tilde{R}\tilde{B}^{-1}] \quad 2.1.30$$

For $Re(\tilde{B}) > 0, Re(a) > m-1, \tilde{X} = \tilde{X}' > 0\ Re(\tilde{B}) > Re[\tilde{R}]$
Then the p.d.f. (2.1.30) takes the form as

$$f(\tilde{X}) = \frac{etr(-\tilde{B}\tilde{X})|\det \tilde{X}|^{a+\beta-m}{}_0F_1[*]}{\tilde{\Gamma}_m(a+\beta)|\det \tilde{X}|^{a+\beta-m}{}_2F_1[**]} \quad 2.1.31$$

Where ${}_1F_1[*] = {}_1F_1[m-a+b, m+b-c; -\tilde{R}\tilde{X}]$
${}_2F_1[**] = {}_2F_1[m-a+b, m+\beta; m+b-c; -\tilde{R}\tilde{\beta}^{-1}]$
For $Re(\tilde{B}) > 0, Re(a+\beta) > m-1, Re(\tilde{B}) > Re(\tilde{R}), \tilde{X} = \tilde{X}' > 0$
$= 0; elsewhere,$
Replacing $\tilde{B}$ with - $\tilde{R}$, (2.1.31) takes the form

$$f(\tilde{X}) = \frac{etr(-\tilde{R}\tilde{X})|\det \tilde{X}|^{a+\beta-m}{}_1F_1[+]}{\tilde{\Gamma}_m(a+\beta)|\det \tilde{R}|^{(a+\beta)}{}_2F_1[++]} \quad 2.1.32$$

Where ${}_1F_1[+] = {}_1F_1[m-a+b, m+b-c; -\tilde{\beta}\tilde{X}]$
${}_2F_1[++] = {}_2F_1[m-a+b, a+\beta; m+b-c; I]$
For $Re(\tilde{R}) > 0, Re(a+\beta) > m-1, Re(\tilde{R}) > Re(\tilde{B}), \tilde{X} = \tilde{X}' = 0$
We know that

$${}_2F_1[a,b,c;I] = \frac{\tilde{\Gamma}_m(c)\tilde{\Gamma}_m(c-a-b)}{\tilde{\Gamma}_m(c-a)\tilde{\Gamma}_m(c-b)} \quad 2.1.33$$

Making use of (2.2.33) in (2.2.32), we get

$$f(\tilde{R}) = \prod_m etr(\tilde{R}\tilde{X})|\det \tilde{X}|^{a+\beta-m}{}_1F_1[+] \quad 2.1.34$$

Where

$$\prod_m = \frac{\tilde{\Gamma}_m(a-c)\tilde{\Gamma}_m(m+b-c-a-\beta)}{\tilde{\Gamma}_m(m+b-c)\tilde{\Gamma}_m(a-c-a-\beta)\tilde{\Gamma}_m(a+\beta)\left|\det \tilde{R}\right|^{-(a+\beta)}}$$

${}_1F_1[+] = {}_1F_1[m-a+b; m+b-c; \tilde{B}\tilde{X}]$

We know that the Kummer transformation as

${}_1F_1[a; b; \tilde{B}\tilde{X}] = etr(\tilde{B}\tilde{X})\,{}_1F_1[b-a; b; -\tilde{B}\tilde{X}]$ 2.1.35

Making use of (2.2.35) in (2.2.34) we get

$f(\tilde{X}) = \prod_m etr\,[(\tilde{R}+\tilde{B})\tilde{X}\left|\det \tilde{X}\right|^{a+b-m}\,{}_1F_1[\#]$ 2.1.36

When $\prod_m$ *same as above*

${}_1F_1[\#] = {}_1F_1[a-c; m+b-c; \tilde{B}\tilde{X}]$

Where $Re(a+\beta) > m-1, Re(m+b-c\_ > m-1,$

$Re(a-c-\beta) > m-1, \tilde{X} = \tilde{X}' > 0\; Re(\tilde{R}+\tilde{B}) > 0$

Case (vii)

Putting p = 1, q = 2, r = 2, s = 2, a = -c1, b = -c2, c = a-m, a = -b

The (2.1.1) takes the form

$\int_{\tilde{X}>0} etr\;(-\tilde{B}\tilde{X})\left|\det \tilde{X}\right|^{a-m} G^{12}_{2,2}\left[\tilde{R}\tilde{X}\Big|^{a-c_2,}_{b-m,}-b\right]$ 2.1.37

Who know that? $G^{12}_{2,2}\left[\tilde{R}\tilde{X}\Big|^{1-c_2,}_{a-m,}-b\right]$

$= \frac{\tilde{\Gamma}_m(a+c_1)\tilde{\Gamma}_m(a+c_2)}{\tilde{\Gamma}_m(a+b)}\left|\det \tilde{X}\right|^{a-m} \times {}_2F_1[a+c_1, a+c_2; a+b; -\tilde{R}\tilde{X}]$

2.1.38

For $Re(a+c_1, a+c_2, a+b) > m = 1, \tilde{X} = \tilde{X}' > 0$

Making use of (2.2.38) in (2.2.37), we get

$= \frac{\tilde{\Gamma}_m(a+c_1)\tilde{\Gamma}_m(a+c_2)\tilde{\Gamma}_m(a+a-m)}{\tilde{\Gamma}_m(a+b)|\beta|^{a+a-m}} \times {}_2F_1[a+c_1, a+c_2; a+b; -\tilde{R}\tilde{X}]d\tilde{X}$

2.1.39

The result (2.1.39) is a direct consequence of the result

$\int_{\tilde{X}>0} etr\,(-\tilde{B}\tilde{X})\left|\tilde{X}\right|^{a-m}\,{}_2F_1[a_1, a_2, a; b; -\tilde{R}\tilde{B}^{-1}] = \tilde{\Gamma}_m(a)\left|\tilde{B}\right|^{-a}$

${}_3F_1[a_1, a_2, a; b; -\tilde{R}\tilde{B}^{-1}]$ 2.1.40

Thus the p.d.f. (2.1.2) takes the following form

$f(\tilde{X}) = \frac{etr(-\tilde{B}\tilde{X})|\det \tilde{X}|^{a+a-m-m}\,{}_2F_1[-]}{\tilde{\Gamma}_m(a+a-m)\,|\det \tilde{B}|^{(a+a-m)}\,{}_3F_1[-]}$ 2.1.41

Where ${}_2F_1[-] = {}_2F_1[a+c_1, a+c_2; a+b; -\tilde{R}\tilde{X}]$

${}_2F_1[-] = {}_3F_1[a+c_1, a+c_2; a+a-m; a+b; -\tilde{R}\tilde{B}^{-1}]$

For $Re(\tilde{B}) > 0, Re(a-a-m) > m-1, Re(\tilde{B}) > Re(\tilde{R}), \tilde{X} = \tilde{X}' > 0$

$= 0; elsewhere.$

Replacing $-\tilde{R}$ with $\tilde{R}$ and $\tilde{X}$ with $\tilde{R}^{-1}$, (2.2.40) reduces to

$f(\tilde{X}) = \frac{etr\,(-\tilde{B}\tilde{R})|\det \tilde{R}|^{-(a+a+m+1)}\,{}_2F_1[+]}{\tilde{\Gamma}_m(a+a-m)\,|\det \tilde{B}|^{(a+a+m)}\,{}_3F_1[++]}$ 2.1.42

Where ${}_2F_1[+] = {}_2F_1[a+c_1, a+c_2; a+c; I]$

${}_3F_1[++] = {}_3F_1[a+c_1, a+c_2; a+a-m; a+b; -\tilde{R}\tilde{B}^{-1}]$

Making use of (2.1.33) in (2.1.42), we get

$$f(\tilde{X}) = \frac{\prod_m etr\,[(-\tilde{B}\tilde{R}^{-1})\left|\det \tilde{R}\right|^{-a-a-(m+1)}\,{}_2F_1[+]}{\left|\det \tilde{B}\right|^{-(a+a+m)}\,\blacksquare\,{}_3F_1[+]}$$

Here,

$$\prod_m = \frac{\tilde{\Gamma}_m(a+b)\tilde{\Gamma}_m(b-a-c_1-c_2)}{\tilde{\Gamma}_m(a-b-c_1)\tilde{\Gamma}_m(b-c_2)\tilde{\Gamma}_m(a+a-m)}$$

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