IJCER ONLINE PEER REVIEWED JOURNAL International Journal of Computational Engeneering Research

International Journal Of Computational Engineering Research (ijceronline.com) Vol. 2 Issue. 7

Some Dynamical Behaviors In Lorenz Model

Dr. Nabajyoti Das

Assistant Professor, Department of Mathematics, Jawaharlal Nehru College, Boko -781123
District: Kamrup, State: Assam, Country: India

Abstract

Lorenz's discovery of chaotic behaviors in nonlinear differential equations is one of the most exciting discoveries in the field of nonlinear dynamical systems. The chief aim of this paper is to develop a eigenvaluel theory so that a continous system undergoes a Hopf bifurcation, and to investigate dynamic behaviors on the **Lorenz model:**

$$\frac{dx}{dt} = -kx + ky, \frac{dy}{dt} = -xz + px - y, \ \frac{dz}{dt} = xy - qz,$$

where k, p, q are adjustable parameters.

Key Words: Nonlinear differential equation, Hopf bifurcation, Dynamic behavior, Eigenvalue theory, Chaotic behavior

2010 AMS Classification: 37 G 15, 37 G 35, 37 C 45

1. Introduction

The mathematics of differential equantions is not elementary. It is one of the great achievements made possible by calculus. Lorenz's discovery of strange attractor attractor was made in the numerical study of a set of differential equations which he had refined from mathematical models used for testing weather prediction. Although the topic of differential equations is some 300 years old, nobody would have though it possible that differential equations could behave as chaotically as Lorenz found in his experiments [2 Front]. In case of one-dimensional maps, the lack of hyperbolicity is usually a signal for the occurrence of bifurcations. For higher dimensional systems, these types of bifurcations also occur, but there are other possible bifurcations of periodic points as well. The most typical of these is the Hopf bifurcation. In the theory of bifurcations, a Hopf bifurcation refers to the local birth and death of a periodic solution as a pair of complex conjugate eigenvalues of the linearization around the fixed point which crosses the imaginary axis of the complex plane as the parameter varies. Under reasonably generic assumptions about the dynamical system, we can expect to see a small amplitude limit cycle branching from the fixed point [3-8].

We first highlight some related concepts for completeness of our exploration.

1.1 Limit cycles

A cyclic or periodic solution of a nonlinear dynamical system corresponds to a closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called **cycles**, and if trajectories in the neighborhood to the cycle are attracted toward it, we call the cycle a **limit cycle** [2, 5].

1.2 The Hopf bifurcation theorem:

In this discussion we will restrict our discussion on second-order systems of nonlinear ordinary differential equations, although almost all the results and discussions given below can be extended to general nth-order systems.

We consider the system $\frac{d\bar{\mathbf{x}}}{dt} = \xi(\bar{\mathbf{x}}; R), \quad \bar{\mathbf{x}} \in \Re^2$ (1.1)

where R denotes a real parameter on an interval I. We assume that the system is well defined, with certain smoothness on the nonlinear vector field ξ, and has a unique solution for each given initial value $\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}$ for each fixed $R \in I$. We also assume that the system has an equilibrium point $\underline{\mathbf{x}}^*(R)$ and that the associated Jacobian

$J = \left.\frac{\partial \xi}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*}$ has a single pair of complex conjugate eigenvalues $\eta(\mathrm{R}), \overline{\eta(\mathrm{R})} = \mathrm{Re}\,\eta \pm \mathrm{Im}\,\eta.$ Now suppose that this pair of eigenvalues has the largest real part of all the eigenvalues and is such that in a small neighborhood of a bifurcation value R_c, (i) $\mathrm{Re}\,\eta < 0$ if $R < R_c$, (ii) $\mathrm{Re}\,\eta = 0, \mathrm{Im}\,\eta \neq 0$ if $R = R_c$ and (iii) $\mathrm{Re}\,\eta > 0$ if $R > R_c$. Then, in a small neighborhood of $R_c, R > R_c$, the steady state is unstable by growing oscillations and, at least, a small amplitude limit cycle periodic solution exists about the equilibrium point. The appearance of periodic solutions (all depend on the particular nonlinear function ξ) out of an equilibrium state is called Hopf bifurcation. When the parameter R is continuously varied near the criticality R_c, periodic solutions may emerge for $R < R_c$ (this case is referred to as **supercritical bifurcation**) or for $R > R_c$ (which is referred to as **subcritical bifurcation**) [5-7]. Armed with these concepts, we now concentrate to our main study and investigation.

1.3 The principal investigation

We consider a two-dimensional system $\dot{\mathbf{x}} = \xi(\mathbf{x}\,; R),\ R \in \Re,\ \mathbf{x} = (x, y) \in \Re^2$ where ξ depends smoothly on the real variable parameter R such that for each R near the origin $(0, 0)$ there is an equilibrium point $\mathbf{x}^*(R)$ with the Jacobian matrix $D\xi_{\underline{\mathbf{x}}}(\mathbf{x}^*(b), b)$ having a complex conjugate pair of eigenvalues which cross the imaginary axis as the parameter b passes through $(0, 0)$. Using complex coordinate $z = x + iy$, the system can be expressed in the variable z as $\dot{z} = \eta z + A_1 z^2 + B_1 z\bar{z} + C_1 \bar{z}^2 + M_1 z^2 \bar{z} + \ldots$ (1.2)

where A_1, B_1, C_1, M_1 are complex constants. By making a suitable change of variables the system can be transformed to a normal form:

$$\dot{w} = w(\eta + L|w|^2) + o(|w|^4), \tag{1.3}$$

where w, L are both complex numbers. We write $L = D + iE;\ D, E \in \Re.$ The behavior of the system (1.3) is most conveniently studied using polar coordinate $w = re^{i\theta}$. From this we obtain, $\dot{w} = e^{i\theta}\dot{r} + ire^{i\theta}\dot{\theta}.$ Hence $\dot{r} = r^{-1}\,\mathrm{Re}(\overline{w}\dot{w})$ and $\dot{\theta} = r^{-2}\,\mathrm{Im}(\overline{w}\dot{w})$ and then (1.3) implies

$$\dot{r} = Dr^3 + o(r^4),\ \dot{\theta} = \psi + o(r^2) \tag{1.4}$$

Supercritical and subcritical Hopf bifurcation occur according as $D < 0$ and $D > 0$ respectively. If $D = 0$, considering high order terms we can draw the same conclusion. Now to determine k, we apply the transformation $w = z + \delta z^2 + \rho z\bar{z} + \theta \bar{z}^2$. We have

$$\begin{aligned}\dot{w} &= \dot{z} + 2\delta\, z\dot{z} + \rho\, \bar{z}\dot{z} + \rho z\dot{\bar{z}} + 2\theta\, \bar{z}\, \dot{\bar{z}} \\ &= \eta z + A_1 z^2 + B_1 z\bar{z} + C_1 \bar{z}^2 + M_1 z^2 \bar{z} + 2\delta z(\eta z + B_1 z\bar{z}) + \\ &\quad \rho\, \bar{z}(\eta z + A_1 z^2) + \rho\, z(\overline{\eta z} + \overline{B}_1 z\bar{z}) + 2\theta\, \bar{z}(\overline{\eta z} + \overline{C}_1 z^2)\end{aligned}$$

keeping only terms upto second order, where cubic terms are neglected other than $z^2\bar{z}$. We eliminate the quadratic terms by putting

$$\delta = -A_1/\eta = iA_1/\psi,\ \rho = \text{-i}B_1/\psi,\ \theta = -iC_1/3\psi.$$

Then we obtain

$$\dot{w} = \eta w + \left(M_1 + iA_1 B_1/\psi - i|B_1|^2/\psi - 2i|C_1|^2/3\psi\right)w^2\overline{w},$$

We conclude that $L = M_1 + iA_1 B_1/\psi - i|B_1|^2/\psi - 2i|C_1|^2/3\psi$.

and

$$D = \mathrm{Re}\left(M_1 + iA_1B_1/\psi\right)$$
$$= \mathrm{Re}(M_1) - \psi^{-1}\,\mathrm{Im}(A_1B_1).$$

1.4. Extension to three order differential equations

Let us assume that we have a three-dimensional system:

$$\dot{\mathbf{x}} = \xi(\underline{\mathbf{x}}), \quad \mathbf{x} = (x, y, z)^T, (x, y, z) \in \Re^3$$

which has an equilibrium point for which there is one negative eigenvalue and an imaginary pair. The behavior of the system near the equilibrium point can be analyzed by a reduction of the system to a two-dimensional one, as follows. First we choose coordinates so that the equilibrium point is the origin and so that the linearised system is

$$\dot{v} = \rho v, \qquad \dot{z} = \lambda z$$

where v is a real variable and z is complex, and $\rho < 0,\ \lambda = i\sigma.$

We can now express the system as

$$\dot{v} = \rho v + \alpha vz + \overline{\alpha} v\overline{z} + \gamma z^2 + \delta z\overline{z} + \overline{\gamma}\,\overline{z}^2 + ...$$
$$\dot{z} = \lambda z + pvz + qv\overline{z} + rz^2 + sz\overline{z} + t\,\overline{z}^2 + dz^2\overline{z} + ...$$

If the equation for *v* were of the form $\dot{v} = \rho v + vf(v, z)$ then the plane *v = 0* would be invariant, in the sense that solutions starting on this plane stay on it, and we could restrict attention to the behavior on this plane. What we do below is to find a change of variables which converts the system into one which is sufficiently close to this form. We try the change of variables

$$v = w + az^2 + bz\overline{z} + \overline{a}\overline{z}^2, \text{ where } b \text{ is real.}$$

We obtain

$$\dot{w} = \rho w + \rho az^2 + \rho bz\overline{z} + \rho\overline{a}\overline{z}^2 + \alpha wz + \overline{\alpha} w\overline{z} + \gamma\, z^2 + \delta z\overline{z} + \overline{\gamma}\,\overline{z}^2 - 2a\lambda z^2 - 2\overline{a}\overline{\lambda}\overline{z}^2,$$

neglecting terms of order 3 and higher. Then if we choose

$$a = \gamma \div (2i\sigma - \rho)$$

and $$b = -\delta \div \rho$$

We have $$\dot{w} = \rho w + \alpha wz + \overline{\alpha} w\overline{z} + ...$$

which is of the desired form (as far as of second-order, which turns out to be sufficient). Putting $w = 0$, in the equation for $\dot{z}$, and retaining only terms of order second and those involving $z^2\overline{z}$, we obtain

$$\dot{z} = \lambda z + rz^2 + sz\overline{z} + t\overline{z}^2 + \left(\frac{-p\delta}{\rho} + \frac{q\gamma}{2i\sigma - \rho} + d\right)z^2\overline{z}$$

and using the two-dimensional theory we obtain

$$D = \text{Real part of}\left(\frac{-p\delta}{\rho} + \frac{q\gamma}{2i\sigma - \rho} + d + \frac{irs}{\sigma}\right).$$

Supercritical and subcritical Hopf bifurcation occur according as $D < 0$ and $D > 0$ respectively. If $D = 0,$ considering high order terms we can draw the same conclusions.

1.5 Our main study

For our main study we consider the **Lorenz model:**

$$\frac{dx}{dt} = -kx + ky, \ \frac{dy}{dt} = -xz + px - y, \ \frac{dz}{dt} = xy - qz. \qquad (1.5)$$

For our purpose, the parameters are fixed as in the Lorenz model as given below [8]:

$$k = 10, \quad p = 28, \quad q = \frac{8}{3}$$

With these parameter values the equilibrium points $(x_i^*, y_i^*, z_i^*), i = 1,2,3$ of the system (1.5) are given by setting the left-hand sides zero and solving the resulting system of equations, to get

$$(x_1^* = 0,\ y_1^* = 0,\ z_1^* = 0),$$

$$or\ (x_2^* = -8.485281374238571,\ y_2^* = -8.485281374238571,\ z_2^* = 27.000000000000004),$$

$$or\ (x_3^* = 8.485281374238571,\ y_3^* = 8.485281374238571,\ z_3^* = 27.000000000000004).$$

Out of these equilibrium

points $(x_3^* = 8.485281374238571,\ y_3^* = 8.485281374238571,\ z_3^* = 27.000000000000004)$

is suitable for our purpose.

et us take a linear transformation which moves the equilibrium point to the origin. We take $u = x - x_3^*,\ \mathrm{v} = \mathrm{y} - y_3^*$

and $w = \mathrm{z} - z_3^*$. Then the system (1.5) becomes

$$\frac{du}{dt} = -k(u + x_3^*) + k(v + y_3^*) = 0u - 10v + 10w \qquad (1.6)$$

$$\frac{dv}{dt} = -(u + x_3^*)(w + z_3^*) + r(u + x_3^*) - (v + y_3^*) \qquad (1.7)$$

$$= -5.68434 \times 10^{-14} v + u(1 - w) - 8.48528w$$

$$\frac{dw}{dt} = (u + x^*)(v + y_3^*) - q(w + z_3^*) \qquad (1.8)$$

$$= -5.68434 \times 10^{-14} v + u(1 - w) - 8.48528w$$

The matrix of linearized system is then of the form

$$G = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -8.485281374238571 \\ 8.485281374238571 & 8.485281374238571 & -2.6666666666666665 \end{bmatrix}$$

The eigenvalues $\rho, \lambda_1, \lambda_2$ of M are

$$\rho = -13.854577914596039,$$

$$\lambda_1 = 0.09395562396468556 + 10.194505220927851i,$$

$$\lambda_2 = 0.09395562396468556 - 10.194505220927851i$$

Let us take $H = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$

as the diagonal matrix. Then we

obtain $G^{-1}H = \begin{bmatrix} 0.718386 & -0.00173992 - 0.188787i & 0.00553639 - 0.600717i \\ -0.667072 & -0.00173992 - 0.188787i & 0.00553639 - 0.600717i \\ 0.163278 & -0.0110728 - 1.20143i & 0 + 0i \end{bmatrix}$

In order to make the linearized system into a diagonal form, we make the coordinate change by $G^{-1}HW$, where U is the column matrix, $W = [f, g, h]^T$.

Now

$$G^{-1}HW=\begin{bmatrix}(0.718386+0i)f & -(0.00173992+0.188787i)g & (0.00553639-0.600717i)h\\ (-0.667072+0i)f & -(0.00173992+0.188787i)g & (0.00553639-0.600717i)h\\ (0.163278+0i)f & -(0.0110728+1.20143i)g & (0+0i)h\end{bmatrix}$$

Puttin g

$$\begin{aligned}u&=(0.718386+0i)f-(0.00173992+0.188787i)g+(0.00553639-0.600717i)h,\\ v&=(-0.667072+0i)f-(0.00173992+0.188787i)g+(0.00553639-0.600717i)h,\\ w&=(0.163278+0i)f-(0.0110728+1.20143i)g+(0+0.i)h\end{aligned}$$

in equations (1.6) and (1.7), we get

$$\begin{aligned}\frac{du}{dt}&=10((-0.667072+0i)f-(0.00173992+0.188787i)g+(0.00553639-0.600717i)h)\\ &\quad+10((0.163278+0i)f-(0.0110728+1.20143i)g+(0+0.i)h)\end{aligned}$$

$$\begin{aligned}\frac{dv}{dt}&=-5.68434\times10^{-14}((-0.667072+0i)f-(0.00173992+0.188787i)g+(0.00553639-\\ &\quad 0.600717i)h)+((0.718386+0i)f-(0.00173992+0.188787i)g+(0.00553639-\\ &\quad 0.600717i)h)(1-((0.163278+0i)f-(0.0110728+1.20143i)g))-8.48528((0.163278\\ &\quad +0i)f-(0.0110728+1.20143i)g)\end{aligned}$$

Finally, under the stated transformation (as described in General theory) the system becomes

$$\frac{du}{dt}=-7.18386f+0fg+0fh+0g^2+0gh+(-0.0553639+6.00717i)h^2+\ldots \tag{1.9}$$

$$\begin{aligned}\frac{dv}{dt}&=(0.099556+10.1945i)g+(0.00823861+0.893918i)fg+(-0.000903969-0.0980837i)fh\\ &\quad+(0.226796-0.0041808i)g^2+0.721783gh+0h^2+0g^2h+\ldots\end{aligned} \tag{1.10}$$

From above, we obtain

ρ = Coefficient of f in (1.9) = -7.18386,

p = Coefficient of fg in (1.10) = $0.00823861+0.893918i$,

δ = Coefficient of gh in (1.9) = 0

q = Coefficient of fh in (1.10) = $-0.000903969-0.0980837i$,

γ = Coefficient of g^2 in (1.9) = 0

d = Coefficient of g^2h in (1.10) = 0,

r = Coefficient of g^2 in (1.10) = $0.226796-0.0041808i$,

s = Coefficient of gh in (1.10) = 0.721783,

σ =Imaginary part of eigenvalues = $-13.854578\ldots$.

Using the above values we can calculate the value of k as

$$\begin{aligned}D&=\text{Real part of}\left(\frac{-p\delta}{\rho}+\frac{q\gamma}{2i\sigma-\rho}+d+\frac{irs}{\sigma}\right).\\ &\approx-0.000217807\end{aligned}$$

Hence, we have a supercritical Hopf bifurcation. Of course, this bifurcation is stuied in a **rigorous manner**. Similarly, we can study the Hopf bifurcation of a given system for different values of the parameters.

[For all numerical results, used in this paper, the Computer package "**MATHEMATICA"** is used]

2. Conclusion:

We think, our method is quite suitabale for obtaining hopf bifurcation for any order nonlinear differential equation, if hopf bifurcation exists.

3. Acknowledgement

I am grateful to **Dr. Tarini Kumar Dutta**, Senior Professor of the Department of Mathematics, Gauhati University, Assam: India, for a valuable suggestion for the improvement of this research paper.

4. References:

[1] Das, N and Dutta, T. K., *Determination of supercritical and subcriti hopf bifurcation on a two-dimensional chaotic model*, International Journal of Advance Scintific Research and Technology, Issue2, Vol. 1, February, 2012.

[2] Hilborn, Robert C., *Chaos and Nonlinear Dynamics*, Oxford University Press, 1994

[3] Hopf, E., *Abzweigung einer periodischen Losung von einer stationaren Losung eines Differential systems*, Ber. Verh. Sachs. Akad. Wiss. Leipsig Math.-Nat. 94(1942), 3-22, Translation to English with commentary by L. Howard and N. Kopell,in[81;163-205]

[4] Marsden, J. E. and McCracken, M., *The Hopf Bifurcation and Its Applications*, Springer-Verlag, New York, 1976

[5] Moiola, J. L. and Chen, G., *Hopf Bifurcation Analysis: a frequency domain approach*, World Scientific, 1996

[6] Murray, J. D., *Mathematical Biology I: An Introduction*, Third Edition (2002), Springer

[7] Roose, D. and Hlavacek, V., *A Direct Method for the computation of Hopf bifurcation points*, SIAM J. APPL. MATH., Vol. 45, No. 6, December 1985

[8] Peitgen H.O., Jurgens H. and Saupe D., “Chaos and Fractal”, New Frontiers of Science, Springer Verlag, 1992