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Analyzing disease Dynamics in Complex networks using Laplacian spectrum

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Abstract:
The Laplacian spectrum of a graph refers to the spectrum of its Laplacian matrix, which is a matrix derived from the graph's adjacency matrix. This spectrum carries important information about the graph's structure and properties. Here are several key applications of the Laplacian spectrum in graph theory and related fields. The spread of diseases through social or biological networks can be studied using the Laplacian spectrum. Eigenvalues and eigenvectors of the Laplacian matrix help in understanding the dynamics of disease transmission and designing effective intervention strategies. The eigenvalues and eigenvectors of the Laplacian matrix can help identify key nodes (individuals or locations) within a social or biological network that play crucial roles in disease transmission. High eigenvector centrality indicates nodes that are influential in spreading the disease and are therefore important targets for intervention strategies such as vaccination or quarantine. The Laplacian spectrum provides insights into the resilience of the network to disease outbreaks. Networks with higher algebraic connectivity (higher values of the second smallest eigenvalue) are more tightly connected, potentially leading to faster disease spread. Understanding these properties helps in assessing the risk of widespread outbreaks and planning mitigation measures.

Keywords: *Laplacian spectrum, the spread of diseases through social or biological networks.*

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I. Introduction and preliminaries

Let G be a graph with n vertices and m edges. The distance between two vertices u and v in G, denoted $d_G(u,v)$, refers to the length of the shortest path connecting them. The diameter of a graph G, d(G), is defined as the maximum distance between any pair of vertices in G. If G is disconnected, the diameter is $d(G)=\infty$.

Let V(G) and E(G) represent the vertex set and edge set of the graph G, respectively. The neighborhood of a vertex v, denoted $N_G(v)$, is the set of vertices adjacent to v, and the degree of vertex v, $d_G(v)$, is the number of vertices adjacent to it, i.e., $d_G(v)=|N_G(v)|$.

The adjacency matrix of G, denoted A(G), is a square matrix where the entry Auv=1 if there is an edge between vertices u and v, and Auv=0 otherwise. The degree matrix D(G) is a diagonal matrix where the diagonal entry Dvv represents the degree of vertex v. The Laplacian matrix of G, denoted L(G), is defined as:

$L(G)=D(G)-A(G)$.

This matrix has at least one eigenvalue equal to zero, which usually corresponds to the constant eigenvector (the all-ones vector). It is unique and positive semi-definite.

The eigenvalues of the Laplacian matrix L(G) can be ordered as:

$\mu_1(G)\geq\mu_2(G)\geq \ldots\ldots\ldots\ldots\ldots\ldots\geq\mu_{n-1}(G)\geq\mu_n(G)=0$

where $\mu_1(G)$ is the Laplacian spectral radius and $\mu_{n-1}(G)$ is the algebraic connectivity of G. The algebraic connectivity, $\mu_{n-1}(G)$, is the second smallest eigenvalue of L(G), and it provides a measure of how well-connected the graph is.

For the complement graph G_c of G, where two vertices are adjacent if and only if they are not adjacent in G, the eigenvalues of the Laplacian matrix $L(G_c)$ are related to those of L(G). Specifically, the eigenvalues of $L(G_c)$ are given by:

$n-\mu_1(G), n-\mu_2(G), \ldots\ldots, n-\mu_n(G)$

These eigenvalues are arranged as:

$n-\mu_n(G)=0 \leq n-\mu_{n-1}(G) \leq \cdots \leq n-\mu_1(G)$

The Laplacian spread of a graph G is defined as the difference between the largest eigenvalue

$\mu_1(G)$ and the second-largest eigenvalue $\mu_{n-1}(G)$:

Laplacian spread$=\mu_1(G)-\mu_{n-1}(G)$

A larger Laplacian spread indicates a more sparsely linked or organised network. This measure represents the spread or dispersion of the eigenvalues of the Laplacian matrix.

Clearly, $\mu_1(G)-\mu_{n-1}(G)=\mu_1(G)+\mu_1(G^c)-n=n-[\mu_{n-1}(G)+\mu_{n-1}(G^c)]=$
$\mu_1(G^c)-\mu_{n-1}(G^c)$.

II. Upper bounds of Laplacian spread

Lemma 2.1:

Let G be a graph with $n\geq 2$ vertices. The largest eigenvalue of G, denoted $\mu_1(G)$, satisfies

$\mu_1(G)\leq n$. Equality holds iff the complement graph G^c has infinite diameter (i.e., $d(G^c)=\infty$).

Theorem 2.2: If the diameter of G, denoted d(G), is infinite, then the difference between the largest and the second-largest Laplacian eigenvalues of G, $\mu_1(G)-\mu_{n-1}(G)$, is bounded above by $n-1$. Equality occurs iff G is the disjoint union of an isolated vertex K_1 and a graph H with $n-1$ vertices, where the complement of H, H^c, has infinite diameter.

Proof:
Since $d(G)=\infty$, by Lemma 2.1, we know that $\mu_1(G^c)=n$, which implies that

$\mu_{n-1}(G)=n-\mu_1(G^c)=0$. Let the components of G be $G_1, G_2, \ldots, G_s$, where $s\geq 2$. Then, the largest eigenvalue of G is bounded as follows:

$\mu_1(G)=\mu_1(G_i)\leq|V(G_i)|\leq n-1$.

By Lemma 2.1, we have $\mu_1(G)=n-1$ if and only if $G=K_1\cup H$, where $\mu_1(H)=|V(H)|=n-1$, which implies that $d(H^c)=\infty$. Furthermore, when $d(G)=1$, we have $G=K_n$, and clearly,
$\mu_1(K_n)-\mu_{n-1}(K_n)=n-n=0$.nd v $\in V(G_{i+1})$ for $i=1,2,\ldots,s-1$. Specifically, $G_1\nabla G_2$ denotesthejoin ofgraphs $G_1$and G_2.

Lemma 2.3:

Let G be a non-empty graph with vertex set $\{v_1,v_2,\ldots,v_n\}$. Then, the largest Laplacian eigenvalue of G, $\mu_1(G)$, satisfies:

$\mu_1(G)\leq d_G(v_i)+d_G(v_j)-|N_G(v_i)\cap N_G(v_j)|$,

where $N_G(v_i)$ and $N_G(v_j)$ are the neighborhoods of vertices v_i and v_j, respectively.

Let $G_1,G_2,\ldots,G_s$(with $s\geq 2$) be pairwise disjoint graphs. The graph $G_1\nabla G_2\nabla\cdots\nabla G_s$ is obtained by adding edges between all vertices $u\in V(G_i)$ and $v\in V(G_{i+1})$ for $i=1,2,\ldots,s-1$. Specifically,

$G_1\nabla G_2$ denotes the join of graphs G_1 and G_2.

Theorem 2.4:

If d(G)=2, then:

$\mu_1(G)-\mu_{n-1}(G)\leq n-1$,

with equality iff $G=K_1\nabla H$, where H is a disconnected graph on $n-1$ vertices.

Proof:

If $d(G^c)=\infty$, by Theorem 2.2, we have:

$\mu_1(G)-\mu_{n-1}(G)=\mu_1(Gc)-\mu_{n-1}(G^c)\leq n-1$,

and equality holds iff $G^c=K_1\cup H^c$, where $|V(H^c)|=n-1$ and $d(H^{*c})=\infty$. This implies that

$\mu_1(G)-\mu_{n-1}(G)=n-1$ if and only if $G=K_1\nabla H$, where $|V(H)|=n-1$ and $d(H)=\infty$.

Now, assume that G^c is connected. By Lemma 2.3, we have:

$\mu_1(G^c)\leq d_G{}^c(v_i)+d_G{}^c(v_j)-|N_G{}^c(v_i)\cap N_G{}^c(v_j)| = |N_G{}^c(v_i)\cup N_G{}^c(v_j)|$.

Since d(G)=2, for any pair of non-adjacent vertices vi and vj in G, they share common neighbors, meaning $|N_G(v_i)\cap N_G(v_j)|\geq 1$. This implies that for any pair of adjacent vertices v_i andv_j in G^c, $|N_G{}^c(v_i)\cup N_G{}^c(v_j)|\leq n-1$. Therefore, $\mu_1(G^c)\leq n-1$. Additionally, since G^c is connected, by

Lemma 2.1, $\mu_1(G)<n$, which gives:

$\mu_1(G)-\mu_{n-1}(G)=\mu_1(G)+\mu_1(G^c)-n<n-1$.
Theorem 2.4:If $d(G)=2$, then $\mu_1(G)-\mu_{n-1}(G)\leq n-1$ with equality if and only if $G\cong K_1\nabla H$, where H is a disconnected graph on $n-1$ vertices.
Proof: If $d(G^c)=\infty$, then by Theorem 2.2, $\mu_1(G)-\mu_{n-1}(G)=\mu_1(G^c)-\mu_{n-1}(G^c)\leq n-1$. Moreover, the equality holdsif and only if $G^c\cong K_1\cup H^*$, where $|V(H^*)|=n-1$ and $d(H^{*c})=\infty$. This implies that $\mu_1(G)-\mu_{n-1}(G)=n-1$ if and only if $G\cong K_1\nabla H$, where $|V(H)|=n-1$ and $d(H)=\infty$.
Now assume that G^c is connected.
By Lemma 2.3, we have $\mu_1(G^c)\leq \underbrace{max}_{vivj\epsilon E(G)}\{d_{G^c}(v_i)+d_{G^c}(v_j)-|N_{G^c}(v_i)\cap N_{G^c}(vj)|\}$
$=\underbrace{max}_{vivj\epsilon E(G)}\{|N_{G^c}(vi)\cup N_{G^c}(vj)|\}$.
Since $d(G)=2$, each pair of non-adjacent vertices v_i, v_j of G have common neighbors, that is,

$| N_G(v_i) \cap N_G(v_j)| \geq 1$. This implies that each pair of adjacent vertices v_i, v_j of G^c have $| N_{G^c}(v_i) \cup N_{G^c}(v_j)| \leq n - 1$. Therefore, $\mu_1(G^c) \leq n - 1$. Besides, since G^c is connected, by Lemma 2.1, $\mu_1(G) < n$. Thus $\mu_1(G) - \mu_{n-1}(G) = \mu_1(G) + \mu_1(G^c) - n < n - 1$.

Theorem 2.4 implies that $\mu_1(G) - \mu_{n-1}(G) < n - 1$ if $d(G) = 2$ and G^c is connected.

Theorem 2.5: If G is connected and $d(G) \geq 4$, then $\mu_1(G) - \mu_{n-1}(G) < n - d + 3 - \frac{4}{nd}$.

Proof: It is known that $\mu_{n-1}(G) \geq \frac{4}{nd}$ for any connected graph G with diameter $d(G) \geq 1$.

Also we have $\mu_1(G) < n - d + 3$. Thus the inequality holds.

Lemma 2.6: If $d(G) = 3$, then $\mu_1(G) \leq \mu_1(K_1 \nabla K_{\frac{n-2}{2}} \nabla K_{\frac{n-2}{2}} \nabla K_1)$, with equality if and only if $G \cong K_1 \nabla G_1 \nabla G_2 \nabla K_1$ for two disjoint graphs G_1, G_2 with $| V(G_1)| = \frac{n-2}{2}$ and $| V(G_2)| = \frac{n-2}{2}$.

Theorem 2.7: If $d(G) = 3$, then $\mu_1(G) - \mu_{n-1}(G) \leq n - \frac{16}{n+4+\sqrt{(n+4)^2-32}}$ with equality if and only if $G \cong P_4$, namely a path of order 4.

Proof: Denote by $S_{a,b}$ the graph obtained from K_2 by attaching a pendant edges to a vertex and b pendant edges to the other. That is, $S_{a,b} \cong aK_1 \nabla K_1 \nabla K_1 \nabla bK_1$.

Claim 1: For any positive integers a , b with $a + b = n - 2$, $\mu_{n-1}(S_{a,b}) \geq \frac{8}{n+4+\sqrt{(n+4)^2-32}}$, with equality if and only if a = b. In fact, by direct calculation,

$\det(\mu I_n - L(S_{a,b})) = \mu(\mu - 1)^{n-4} f(ab, \mu)$,

Where $f(ab, \mu) = \mu^3 - (n + 2)\mu^2 + (2n + 1 + ab)\mu - n$. Let $\mu^*(ab)$ be the minimum real root of $f(ab, \mu)$. Note that $f(ab, 1) = ab > 0$ and f is a polynomial of degree 3 on μ, thus $\mu^*(ab) < 1$ and hence $\mu_{n-1}(S_{a,b}) = \mu^*(ab)$. Now assume that $a \neq b$, then $ab < \frac{(n-2)^2}{4}$. Note that $f(ab, \mu)$ is increasing with ab.Thus $f(\frac{(n-2)^2}{4}, \mu^*(ab)) > f(ab, \mu^*(ab)) = 0$.

Similar as above $\mu^*(\frac{(n-2)^2}{4}) < \mu^*(ab)$.

Furthermore $f(\frac{(n-2)^2}{4}, \mu) = (\mu - \frac{n}{2})(-\frac{n+4}{2}(\mu+2))$.

Thus $\mu^*(\frac{(n-2)^2}{4}) = \mu_{n-1}\left(\frac{n-2}{2}, \frac{n-2}{2}\right) = \frac{8}{n+4+\sqrt{(n+4)^2-32}}$ and the claim holds.

Now observe that $S_{a,b}$ is the complement graph of $K_1 \nabla K_a \nabla K_b \nabla K_1$. Thus by Lemma 2.6 and Claim 1, for any graph G with $d(G) = 3$,

$\mu_1(G) \leq \mu_1(K_1 \nabla K_{\frac{n-2}{2}} \nabla K_{\frac{n-2}{2}} \nabla K_1) = n - \mu_{n-1}(\,) \leq n - \frac{8}{n+4+\sqrt{(n+4)^2-32}}$ with equalities if and only if $G \cong K_1 \nabla G_1 \nabla G_2 \nabla K_1$ and $| V(G_1)| = | V(G_2)| = \frac{n-2}{2}$.

Since $d(G) = 3$, we know that $2 \leq d(G^c) \leq 3$. If $d(G^c) = 2$, by Theorem 2.4,

$\mu_1(G) - \mu_{n-1}(G) = \mu_1(G^c) - \mu_{n-1}(G^c) < n - 1$. Now let $d(G^c) = 3$. By the inequality above,

$\mu_1(G^c) \leq n - \frac{8}{n+4+\sqrt{(n+4)^2-32}}$ with equality if and only if $G^c \cong K_1 \nabla G_3 \nabla G_4 \nabla K_1$ and $| V(G_3)| = | V(G_4)| = \frac{n-2}{2}$.

Thus $\mu_1(G) - \mu_{n-1}(G) = \mu_1(G) + \mu_1(G^c) - n \leq n - \frac{16}{n+4+\sqrt{(n+4)^2-32}}$

Since $(K_1 \nabla G_1 \nabla G_2 \nabla K_1)^c \cong G_1^c \nabla K_1 \nabla K_1 \nabla G_2^c$, the equality holds iff $G \cong P_4$.

III. Some classes of graphs with diameter 3

Lemma 3.1: Let G be a connected graph with maximum degree Δ. For a vertex v of G, let $m_G(v) = \Sigma_{u \epsilon NG(v)} d_G(u)/d_G(v)$. Then (i) $\mu_1(G) \leq \max\{d_G(v) + m_G(v) | v \epsilon V(G)\}$.

(ii) $\mu_1(G) \geq \Delta + 1$, with equality if and only if $\Delta = n - 1$.

Lemma 3.2: Let $S = \{v_1, v_2, \ldots, v_s\}$ ($s \geq 2$) be a vertex subset of a connected graph G such that $N_G(v_1) = N_G(v_2) = \cdots = N_G(v_s)$. Let G^* be the graph obtained from G by adding any t ($0 \leq t \leq \frac{s(s-1)}{2}$) edges among the vertices in S. Then $\mu_1(G^*) = \mu_1(G)$.

A connected graph G is said to be unicyclic if m = n and bicyclic if m = n + 1.

Lemma 3.3: (i) Among all unicyclic graphs on n vertices with diameter 3, $K_1 \nabla 2K_1 \nabla K_1 \nabla (n-4) K_1$ is the unique graph with maximal Laplacian spectral radius.

(ii) Among all bicyclic graphs on n (≥ 7) vertices with $\Delta < n - 1$, $K_1 \nabla 3K_1 \nabla K_1 \nabla (n-5) K_1$ is the unique graph with maximal Laplacian spectral radius.

Theorem 3.4: If $G \cong G_1 \nabla G_2 \nabla G_3 \nabla G_4$ for disjoint graphs G_i with $\Sigma_{1 \leq i \leq 4} | V(G\,i)| = n$,

Then $\mu_1(G) - \mu_{n-1}(G) < n - 1$.
Proof: Note that $G = G_3 \nabla G_1 \nabla G_4 \nabla G_2$. For convenience, let $|V(G_i)| = n_i$, $H_1 \cong n_1K_1 \nabla n_2K_1 \nabla n_3K_1 \nabla n_4K_1$ and $H_2 \cong n_3K_1 \nabla n_1K_1 \nabla n_4K_1 \nabla n_2K_1$.
Then by Lemma 3.2, $\mu_1(G) + \mu_1(G^c) = \mu_1(H_1) + \mu_1(H_2)$.
Let u_i be a vertex of G_i $(1 \le i \le 4)$. Then
$d_{H_1}(u_1) + m_{H_1}(u_1) = n_2 + (n_1 + n_3) \le n - 1$,
$d_{H_1}(u_2) + m_{H_1}(u_2) = n_1 + n_3 + n_1\frac{n_{2+n_3(n_2+n_4)}}{n_1+n_3} = n - \frac{n_1n_4}{n_1+n_3}$
$d_{H_1}(u_3) + m_{H_1}(u_3) = n_2 + n_4 + n_3\frac{n_{4+n_2(n_1+n_3)}}{n_2+n_4} = n - \frac{n_1n_4}{n_2+n_4}$
$d_{H_1}(u_4) + m_{H_1}(u_4) = n_3 + (n_2 + n_4) \le n - 1$.
Similarly, we have
$d_{H_2}(u_3) + m_{H_2}(u_3) \le n - 1$,
$d_{H_2}(u_4) + m_{H_2}(u_4) = n - \frac{n_2n_3}{n_1+n_2}$
$d_{H_2}(u_1) + m_{H_2}(u_1) = n - \frac{n_2n_3}{n_3+n_4}$
$d_{H_2}(u_2) + m_{H_2}(u_2) \le n - 1$.
If $G \cong P_4$, then by Theorem 2.7, $\mu_1(P_4) - \mu_{n-1}(P_4) < 3$. Now let $G \ncong P_4$. Note that $x + y > 1$ for any $x \epsilon \{1, \frac{n_1n_4}{n_1+n_3}, \frac{n_1n_4}{n_2+n_4}\}$ and $y \epsilon \{1, \frac{n_2n_3}{n_1+n_2}, \frac{n_2n_3}{n_3+n_4}\}$ by Lemma 3.1, we have
$\mu_1(H_1) + \mu_1(H_2) \le \underbrace{max}_{1 \le i \le 4}\{d_{H_1}(u_i) + m_{H_1}(u_i)\} + \underbrace{max}_{1 \le i \le 4}\{d_{H_2}(u_i) + m_{H_2}(u_i)\} < 2n - 1$.
And hence $\mu_1(G) - \mu_{n-1}(G) = \mu_1(G) + \mu_1(G^C) - n < n - 1$.
Theorem 3.5: Let a, b be positive integers with $a \le b$ and $a + b = n - 2$. If $d(G) = 3$ and $\mu_1(G) \le \mu_1(K_1 \nabla a_1 \nabla K_1 \nabla bK_1)$, then $\mu_1(G) - \mu_{n-1}(G) < n - 1$.
Proof: For convenience, denote by H_a the graph $K_1 \nabla aK_1 \nabla K_1 \nabla bK_1$.

Claim 2: For positive integers a, b with $a \le b$ and $a + b = n - 2$, $\mu_1(H_a) \le n + \frac{n+\sqrt{n^2-4n+8}}{2}$
With equality if and only if a = b.
Let X be an eigenvector of $L(H_a)$ corresponding to $\mu_1(H_a)$. Clearly, both $X|_{V(aK_1)}$ and $X|_{V(bK_1)}$ are constant vectors. Let x_a (resp. x_b) be the component of X corresponding to vertices in $V_{(aK_1)}$ (resp. $V_{(bK_1)}$). Let x_1 (resp. x_2) be the component of X corresponding to the vertex of degree a (resp. a + b).
Since $L(H_a)X = \mu_1(H_a)X$, we have
$(\mu_1(H_a) - 2)x_a = -x_1 - x_2$,
$(\mu_1(H_a) - a)x_1 = -ax_a$,
$(\mu_1(H_a) - a - b)x_2 = -ax_a - bx_b$,
$(\mu_1(H_a) - 1)x_b = -x_2$.
By above equalities and simplifying, we conclude that $\mu_1(H_a)$ is the maximum real root of the following equation.
$g(a, \mu) \triangleq \mu(\mu - 2)(\mu - n + 1) - a(\mu^2 - n\mu + n) = 0$.
By Lemma 3.1, $\mu_1(H_a) > \Delta + 1 = n - 1$ for any a. Now assume that $\mu > n - 1$. Then $\mu^2 - n\mu + n > 0$ and hence $g(a, \mu)$ is decreasing with a. Moreover, for any a and $\mu > n - 1$, $g(a, \mu)$ is increasing with μ, since $g'(a, \mu) > 0$. Thus for all $\mu \ge \mu_1(H_{\frac{n-2}{2}})$ and $a < \frac{n-2}{2} \le b$,
$g(a, \mu) \ge g(a, \mu_1(H_{\frac{n-2}{2}})) > g(\frac{n-2}{2}, \mu_1(H_{\frac{n-2}{2}})) = 0$.
This implies that $\mu_1(H_a) < \mu_1(H_{\frac{n-2}{2}})$ for $a < \frac{n-2}{2} \le b$. Furthermore,
$G(\frac{n-2}{2}, \mu) = (\mu - \frac{n}{2})(\mu^2 - n\mu + n - 2)$
Thus $\mu_1(H_{\frac{n-2}{2}}) = n + \frac{n+\sqrt{n^2-4n+8}}{2}$ and the claim holds.
Since $d(G) = 3$, we know that $2 \le d(G^c) \le 3$. If $d(G^c) = 2$, by Theorem 2.4,
$\mu_1(G) - \mu_{n-1}(G) = \mu_1(Gc) - \mu n - 1(Gc) < n - 1$. Now assume that $d(G^c) = 3$.
By (1) and Claim 2,
$\mu_1(G) - \mu_{n-1}(G) = \mu_1(G) + \mu_1(G^c) - n \le \mu_1(H_a) - \frac{8}{n+4+\sqrt{(n+4)^2-32}}$
Remark 3.6: Note that $a \le b$ is an important condition of Theorem 3.5. However, for $(n, a, b) = (5, 2, 1)$ and $(n, a, b) = (7, 3, 2)$, direct calculations show that
$\mu_1(H_a) - \frac{8}{n+4+\sqrt{(n+4)^2-32}} < n - 1$.
This implies that Theorem 3.5 also holds for these two trivial cases.
Theorem 3.7: If $d(G) = 3$ and $n - 1 \le m \le n + 1$, then $\mu_1(G) - \mu_{n-1}(G) < n - 1$.

Proof: (i) $m = n - 1$. Now G is a tree. Note that any tree with diameter 3 is isomorphic to $aK_1\nabla K_1\nabla K_1\nabla bK_1$ for some pair of positive integers a, b. According to Theorem 3.4, $\mu_1(G) - \mu_{n-1}(G) < n - 1$.

(ii) $m = n$. Then G is a unicyclic graph. Since $d(G) = 3$, by Lemma 3.3, $\mu_1(G) \le \mu_1(K_1\nabla 2K_1\nabla K_1\nabla(n-4)K_1)$.

Thus by Theorem 3.5, if $n \ge 6$, $\mu_1(G) - \mu_{n-1}(G) < n - 1$.

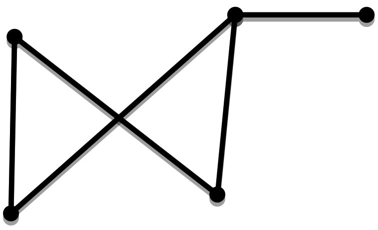
Now it remains the case $n = 5$. By Remark 3.6, we also have the inequality.

(iii) $m = n + 1$. Then G is a bicyclic graph. Since $d(G) = 3$, $\Delta < n - 1$. By Lemma 3.3, if $n \ge 7$, then $\mu_1(G) \le \mu_1(K_1\nabla 3K_1\nabla K_1\nabla(n-5)K_1)$. Thus by Theorem 3.5 and Remark 3.6, $\mu_1(G) - \mu_{n-1}(G) < n - 1$ for $n \ge 7$. If $n = 5$, then $G \cong K_1\nabla K_2\nabla K_1\nabla K_1$. Thus by Theorem 3.4, the inequality holds. Now it remains the case $n = 6$. There are twelve bicyclic graphs on 6 vertices with diameter 3 (see Fig. 1).

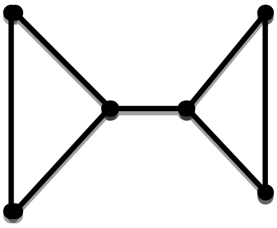
By Theorem 3.4, $\mu_1(B_i) - \mu_{n-1}(B_i) < n - 1$ for $2 \le i \le 7$. And by Matlab, we can find $\mu_1(B_i) - \mu_{n-1}(B_i) < n - 1$ for other B_i.

Theorems 2.4, 2.5 and 3.7 imply the following result, which simultaneously determines the unique tree, unicyclic graph and bicyclic graph with maximal Laplacian spread.

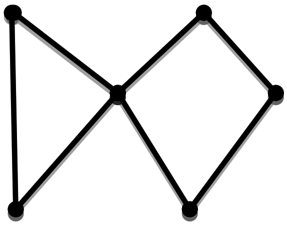
Theorem 3.8: Let G be a connected graph on n ($n \ge 5$) vertices and m ($n - 1 \le m \le n + 1$) edges. Then $\mu_1(G) - \mu_{n-1}(G) \le n - 1$, with equality if and only if G is obtained from $K_1\nabla(n-1)K_1$ by adding $m - n + 1$ edges.



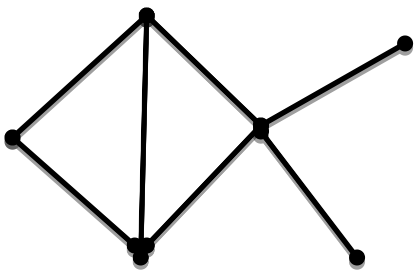
B_1



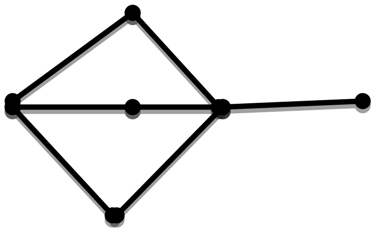
B_2



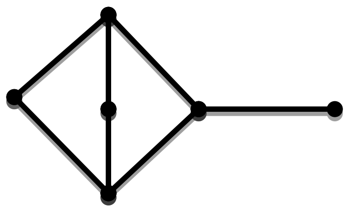
B_3



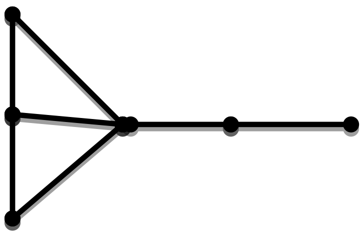
B_4



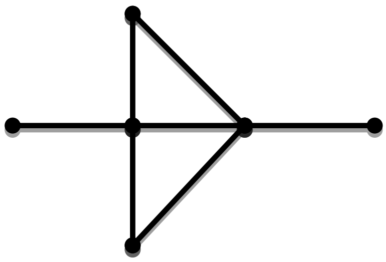
B_5



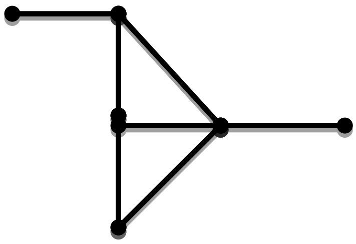
B_6



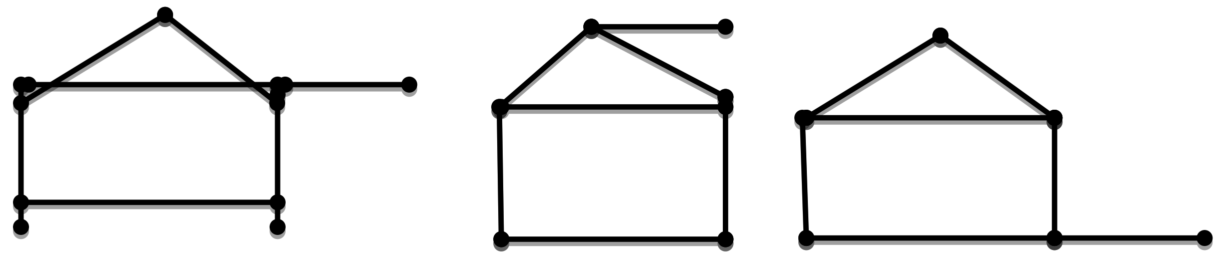
B_7



B_8



B_9



B_{10} B_{11} B_{12}

Fig. 1. Bicyclic graphs on 6 vertices with diameter 3.

Theorems 3.4, 3.5 and 3.7 give some classes of graphs with diameter 3 and Laplacian spread less than $n - 1$. Since Theorems 2.2, 2.4 and 2.5 shows that $\mu_1(G) - \mu_{n-1}(G) \leq n - 1$ as long as $d(G) \neq 3$, we may present a conjecture as follows.
Conjecture 3.9: For any graph G, $\mu_1(G) - \mu_{n-1}(G) \leq n - 1$, with equality if and only if G or G^c is isomorphic to the join of an isolated vertex and a disconnected graph on $n - 1$ vertices.
The study of the spread and control of diseases within populations can benefit from the applicationof the Laplacian spectrum in several ways beyond understanding disease transmission dynamics. Here are additional applications:

Identifying Key Nodes for Intervention:
The eigenvalues and eigenvectors of the Laplacian matrix can help identify key nodes (individuals or locations) within a social or biological network that play crucial roles in disease transmission. High eigenvector centrality indicates nodes that are influential in spreading the disease and are therefore important targets for intervention strategies such as vaccination or quarantine.

Assessing Network Resilience:
The Laplacian spectrum provides insights into the resilience of the network to disease outbreaks. Networks with higher algebraic connectivity (higher values of the second smallest eigenvalue)are more tightly connected, potentially leading to faster disease spread. Understanding these properties helps in assessing the risk of widespread outbreaks and planning mitigation measures.
Modeling Control Strategies:
Eigenvalues of the Laplacian matrix can be used in mathematical models to simulate the effectiveness of different disease control strategies. For example, they can inform the design of optimal vaccination campaigns by targeting individuals based on their network centrality or connectivity.
Evaluating Community Structure:
Spectral clustering techniques based on the Laplacian matrix can reveal underlying community structures within the network. These communities often have distinct patterns of disease transmission, which can guide tailored intervention strategies that account for local interactions and behaviors.
Comparing Transmission Dynamics:
By comparing the Laplacian spectra of different networks (e.g., networks in different geographic regions or with different demographic compositions), epidemiologists can gain insights into variations in disease transmission dynamics. This understanding is crucial for developing region-specific or population-specific interventions.

Optimizing Surveillance Efforts:
The Laplacian spectrum helps in optimizing surveillance efforts by identifying critical nodes where monitoring for disease outbreaks should be intensified. Nodes with high network centrality or specific spectral properties may serve as early indicators of potential outbreaks.
Studying Co-infections and Multi-pathogen Dynamics:
In situations involving co-infections or multiple pathogens circulating within a population, the Laplacian spectrum can assist in modeling and analyzing complex interaction patterns between different diseases. This can lead to a better understanding of how co-infections affect disease dynamics and influence intervention strategies.

IV. Conclusion

Spectral graph theory provides a powerful mathematical framework for analyzing networks, leveraging concepts from linear algebra (eigenvalues, eigenvectors and matrix decompositions) to understand complex network structures and dynamics. These applications demonstrate its versatility in fields ranging from biology and epidemiology to computer science and optimization, offering insights into network behavior and facilitating the design of efficient algorithms and interventions.

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Declarations

- Consent to participate -Not Applicable
- Consent for publication -Not Applicable
- Author's contributions:
  Author 1 - Review and Editing the Manuscript
  Author 2 - Wrote the Manuscript, Prepared the content, give some idea to improve the quality of the Article
- Compliance with Ethical Standards:
  This article does not contain any studies with human participants or animals performed by any of the authors.
- Conflict of interest/Competing interests:
  Author 1 declares that he has no conflict of interest.
  Author 2 declares that she has no conflict of interest.
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